

The final stages of eccentric EMRI's

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UMONS

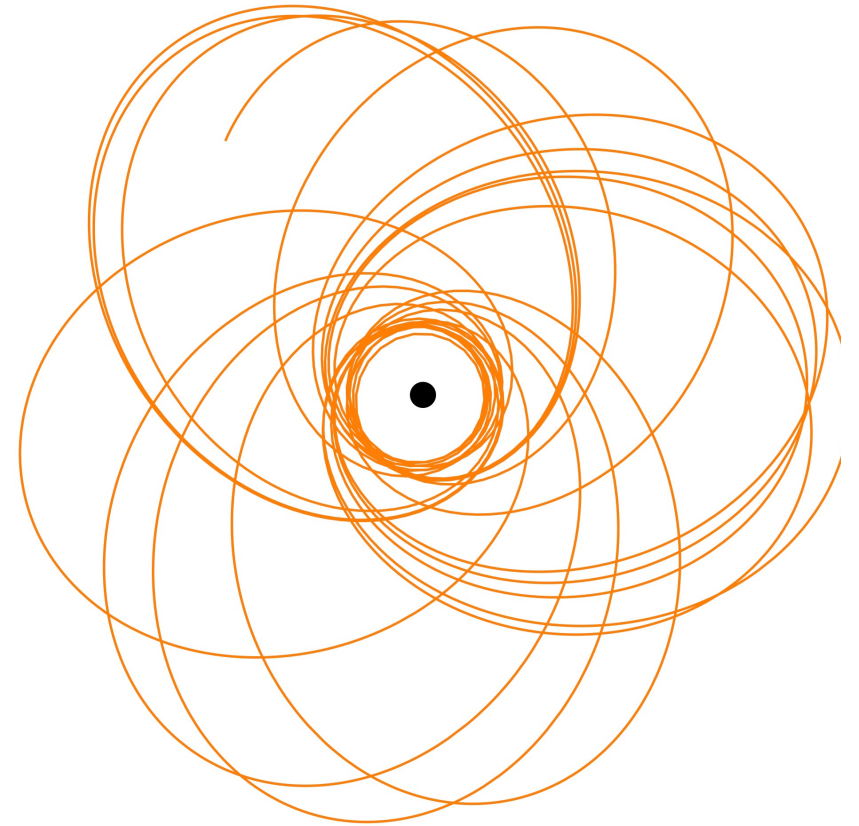
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In this talk

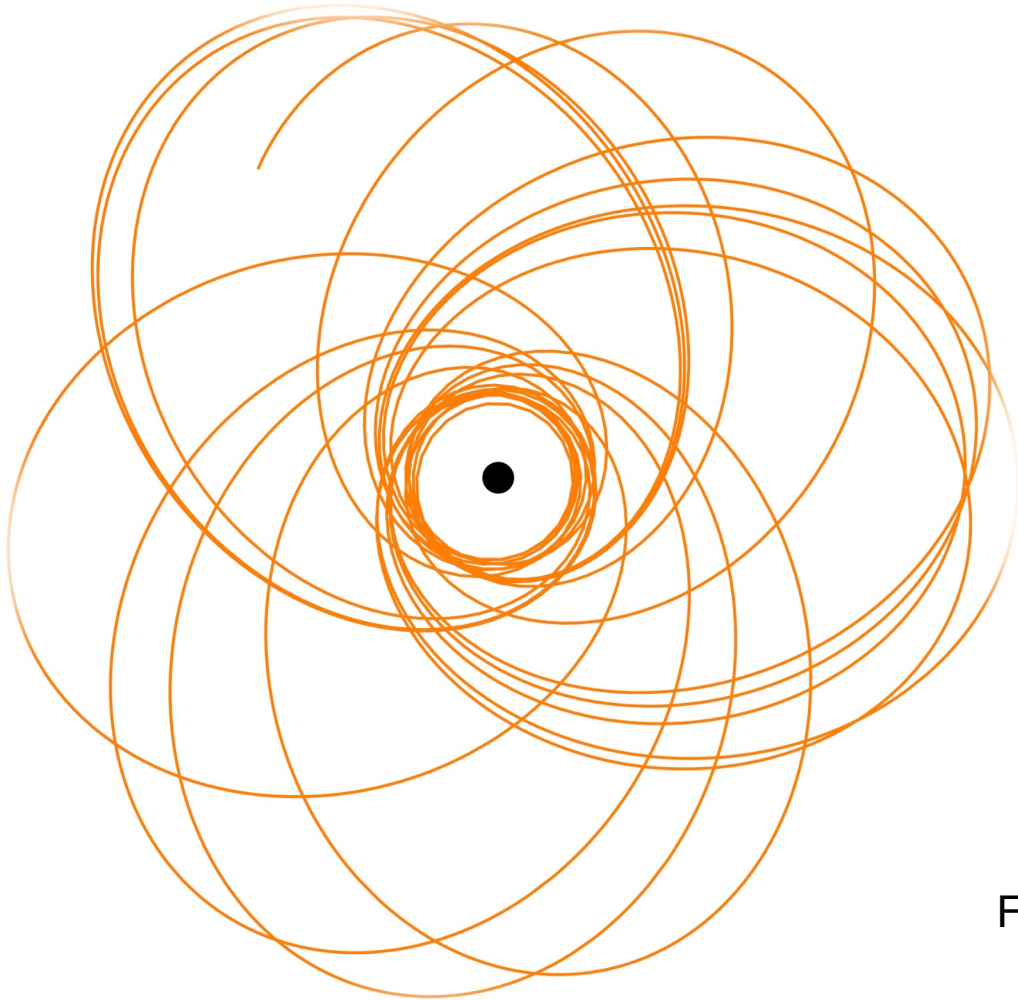
- EMRIs: Problem of the transition to merger
- Theoretical aspects of the eccentric transition to plunge
- Waveforms on the final stages of eccentric EMRIs





**Context: problem of
the transition to
plunge**

EMRIs



- EMRIs: Extreme Mass Ratio Inspirals
- Mass Ratio: $\frac{m}{M} =: \epsilon \approx 10^{-4} - 10^{-7}$ used for adiabatic expansions,
- Expected duration of the inspiral $\left(T_{\text{tot}} \approx \frac{1}{\epsilon}\right)$
- Number of cycles: $10^4 - 10^7$,
- Typical frequencies: $10^{-3} - 10^{-1} \text{ Hz}$ (LISA),
- Self-force problem.

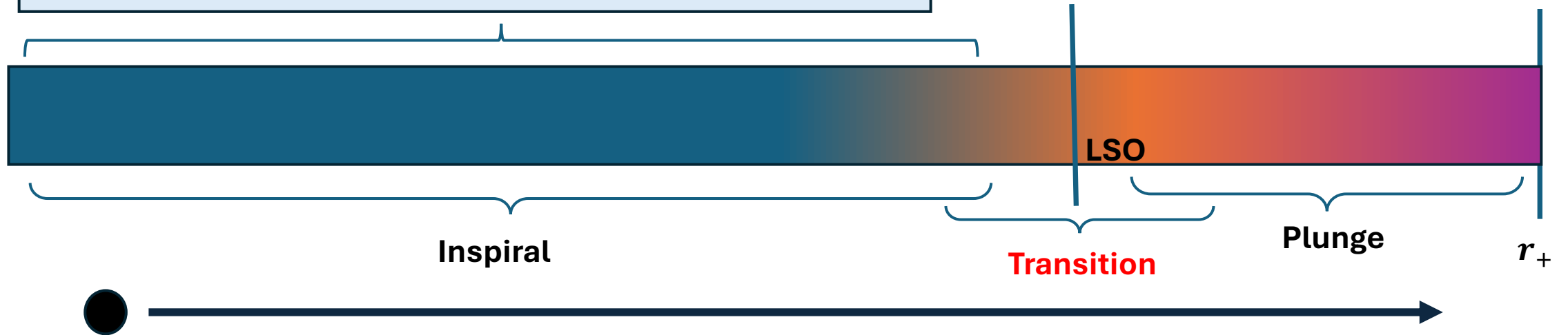
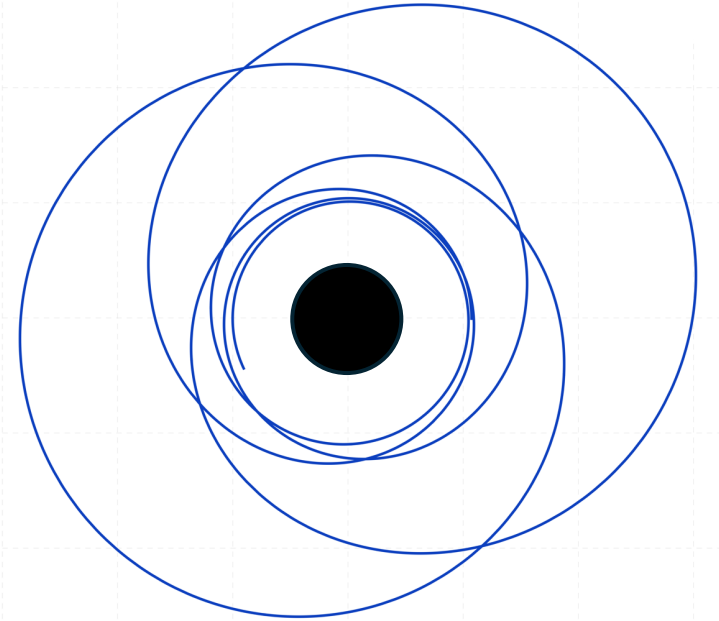
Future GW observatories will permit the **detection of EMRIs**;

- Seek strong field regime
- Test General Relativity

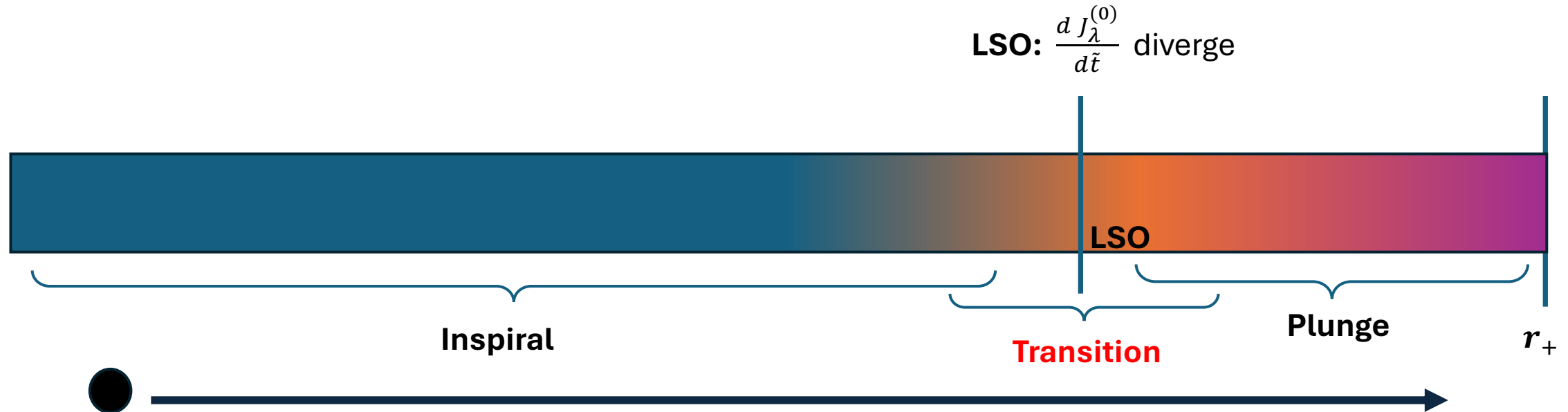
- **First Principles:** $f^\mu = \varepsilon f_{(1)}^\mu + \mathcal{O}(\varepsilon^2)$, and separation between **fast** (angles q_α) and **slow** (actions J_λ) variables,
- Valid all along the inspiral,
- **OPA equations:**

$$\Omega_\alpha^{(0)}(\tilde{t}) = \omega_\alpha [J^{(0)}(\tilde{t})]$$

$$\frac{dJ_\lambda^{(0)}}{d\tilde{t}} = \langle G_\lambda^{(1)} \rangle [J^{(0)}(\tilde{t})]$$



- There always exist a Last Stable Orbit (LSO/separatrix),
- First Principles fail at the LSO,
- We need a transition to plunge scheme to connect with the merger,
- Goal: model a full IMR trajectory and waveform.



Transition to plunge:

- In the **quasi-circular case**:

- Characterized by expansions in $O(\varepsilon^{1/5})$

- Quasicircularity condition:

$$\dot{E} = \Omega_* \dot{L} + O(\varepsilon^{6/5}),$$

- Direct matching between inspiral, transition and plunge.

- In the **eccentric case**:

- Less understood,

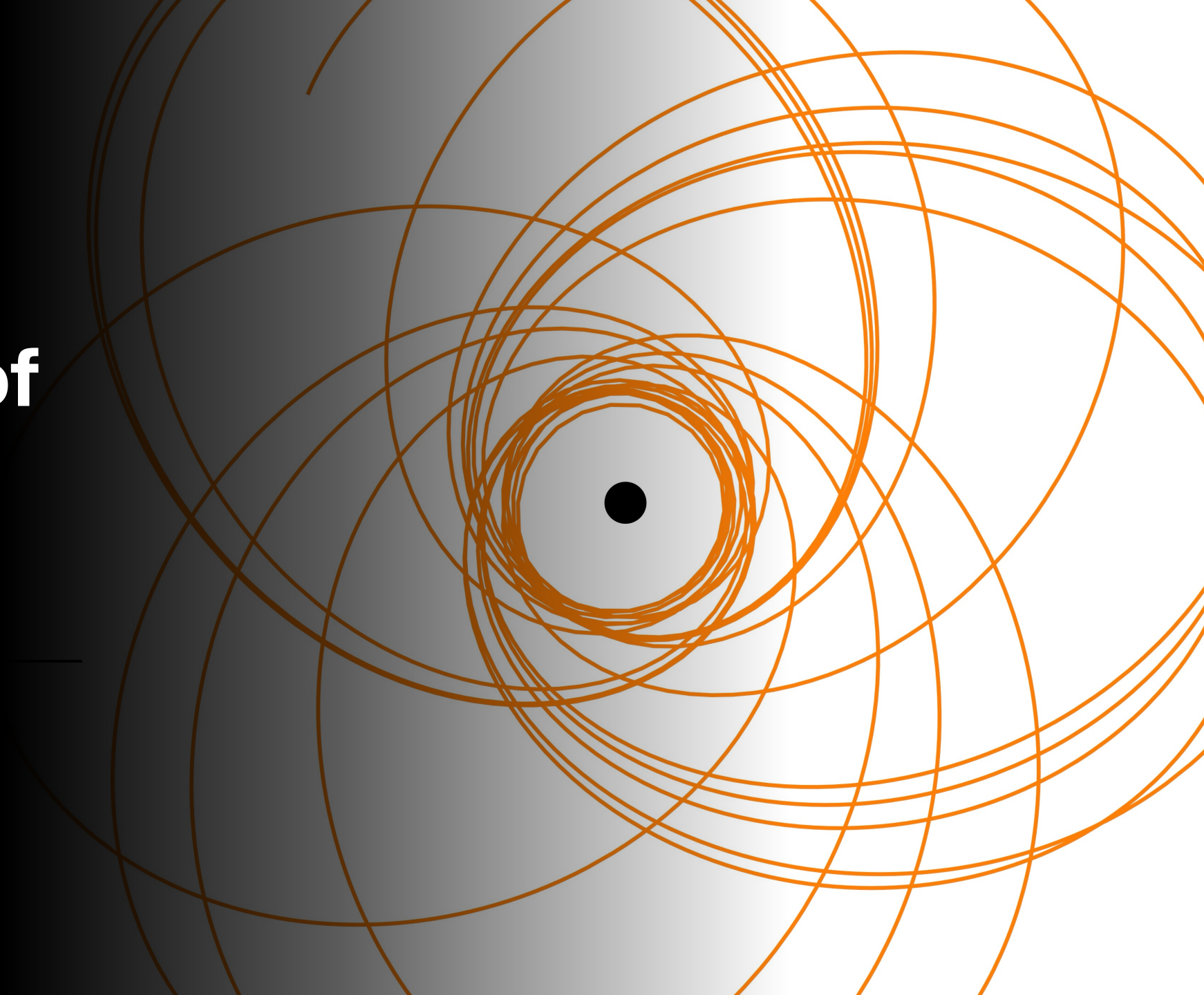
- No quasicircularity condition,

- Additional step before the transition.

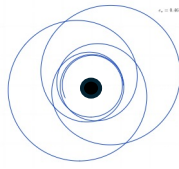




The epochs of the orbital evolution



Epoch 1: inspiral



- Inspiral = Succession of bounces between periapsis and apoapsis,
- With the forced equation:

$$\ddot{r}_p = -\frac{1}{2} \partial_r V(r_*, L) + f^r$$

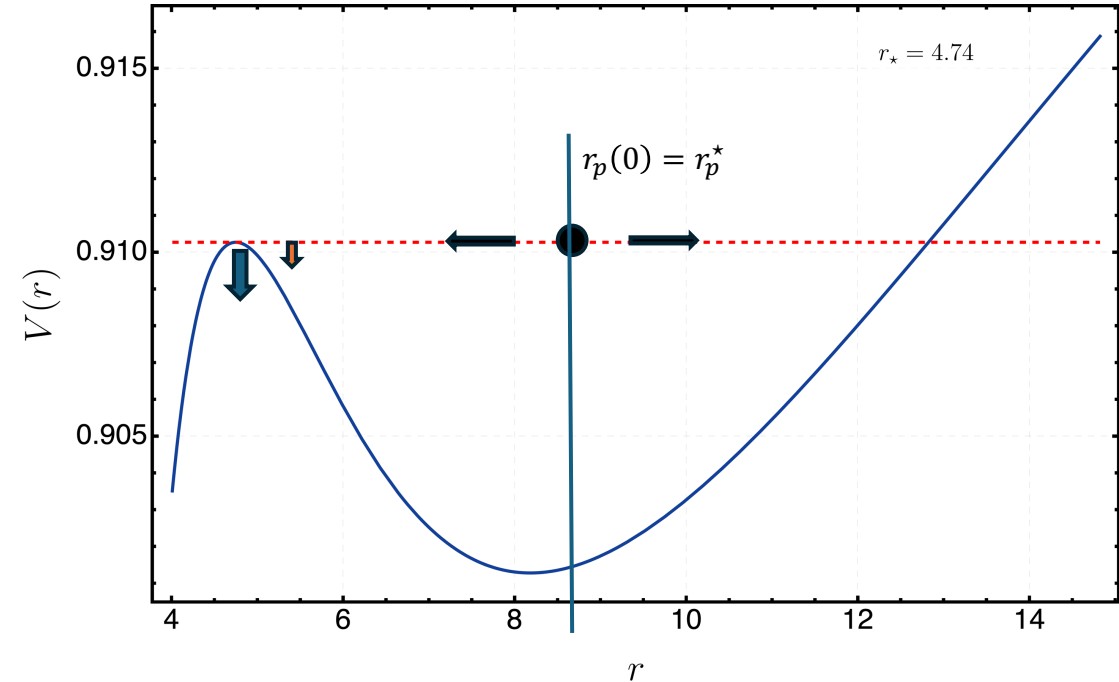
- Definition of separatrix:

$$\dot{r}_p = \ddot{r}_p = 0$$

$\leftrightarrow$

$$E^2 = V(r_*, L) \quad ; \quad \partial_r V(r_*, L) = 2 f^r,$$

- Separatrix: parametric curve in $(L, r_p, \dot{r}_p)$ phase space
- $4 < r_* < 6$: parameter along the separatrix



The secondary is still **free** to bounce once against the potential

Mino time at the separatrix crossing: $\lambda = 0$.

$$\begin{aligned} E(0) &= E_*^G(r_*) + \varepsilon \Omega_* \delta L_*^{(1)}(r_*) \\ L(0) &= L_*^G(r_*) + \varepsilon \delta L_*^{(1)}(r_*) \end{aligned}$$

Epoch 2: final approach

- Final approach:
 E and L have crossed the separatrix line, and $r_p > r_*$.
- Expansion :

$$r_p(\lambda) = r_{hom}(\lambda - \lambda_{ap}) + \varepsilon r_{<1>}(\lambda - \lambda_{ap}) + O(\varepsilon^2).$$

- Along the final approach

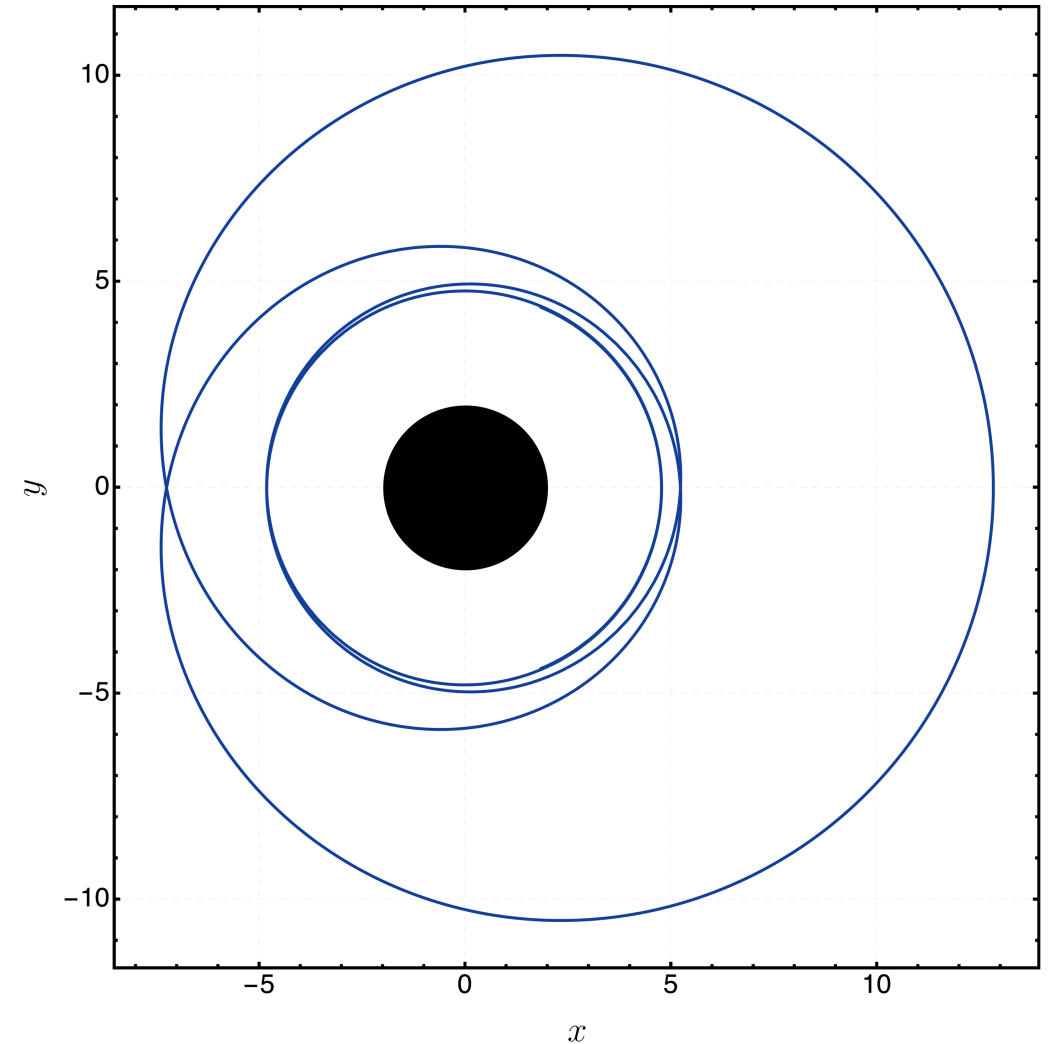
$$E = E(0) + \varepsilon \Delta E^{(1)}(\lambda) + O(\varepsilon^2)$$

$$L = L(0) + \varepsilon \Delta L^{(1)}(\lambda) + O(\varepsilon^2).$$

- Non quasi-circularity: allows to “**climb over the potential hill**”

$$\Delta E^{(1)}(\lambda) = - \int_0^\lambda d\lambda' r_p^2 F_t^{(1)} \neq \Omega(\lambda) \Delta L^{(1)}(\lambda)$$

- It exists $\lambda_*(\varepsilon) > 0$ such that $r_p(\lambda_*) = r_*$.



Epoch 3: Transition

- The leading order solution $r_{hom}(\lambda)$ **never reaches r_*** !
- We can formulate an asymptotic match expansion scheme around

$$\lambda_*^-(\varepsilon) < \lambda_*.$$

- Transition expansion = expansion in ε **and** in $\Delta\lambda = \lambda - \lambda_*^-$
- After expansion

$$\begin{aligned} E(\lambda; \varepsilon) &= E_*^G(r_*) + \varepsilon \lambda_*^-(\varepsilon) \Omega_* \mathcal{F}_L^{\text{circ}}(r_*) + \varepsilon \Delta_*^{(1)} + \varepsilon \Omega_* \alpha_L(\Delta\lambda) + o(\varepsilon), \\ L(\lambda; \varepsilon) &= L_*^G(r_*) + \varepsilon \lambda_*^-(\varepsilon) \mathcal{F}_L^{\text{circ}}(r_*) + \varepsilon \alpha_L(\Delta\lambda) + o(\varepsilon). \end{aligned}$$

$$\Delta_*^{(1)}(r_p^*, r_*) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^{\lambda_*^-} d\lambda' (\dot{E} - \Omega_* \dot{L})$$

Must be positive to start plunging
Need self-force

- Expansion of the radius : $r_p = r_* + \sqrt{\varepsilon} r_{[0]}(\Delta\lambda) + \varepsilon r_{[1]}(\lambda) + o(\varepsilon)$.
- With solution

$$r_{[0]} = r_{[0]}(0) \cosh\left(\frac{\Delta\lambda}{\Gamma_*}\right) + v_* \Gamma_* \sinh\left(\frac{\Delta\lambda}{\Gamma_*}\right)$$

Matching with the final approach iff

$$r_{[0]}(0) = 2 r_*(6 - r_*)$$

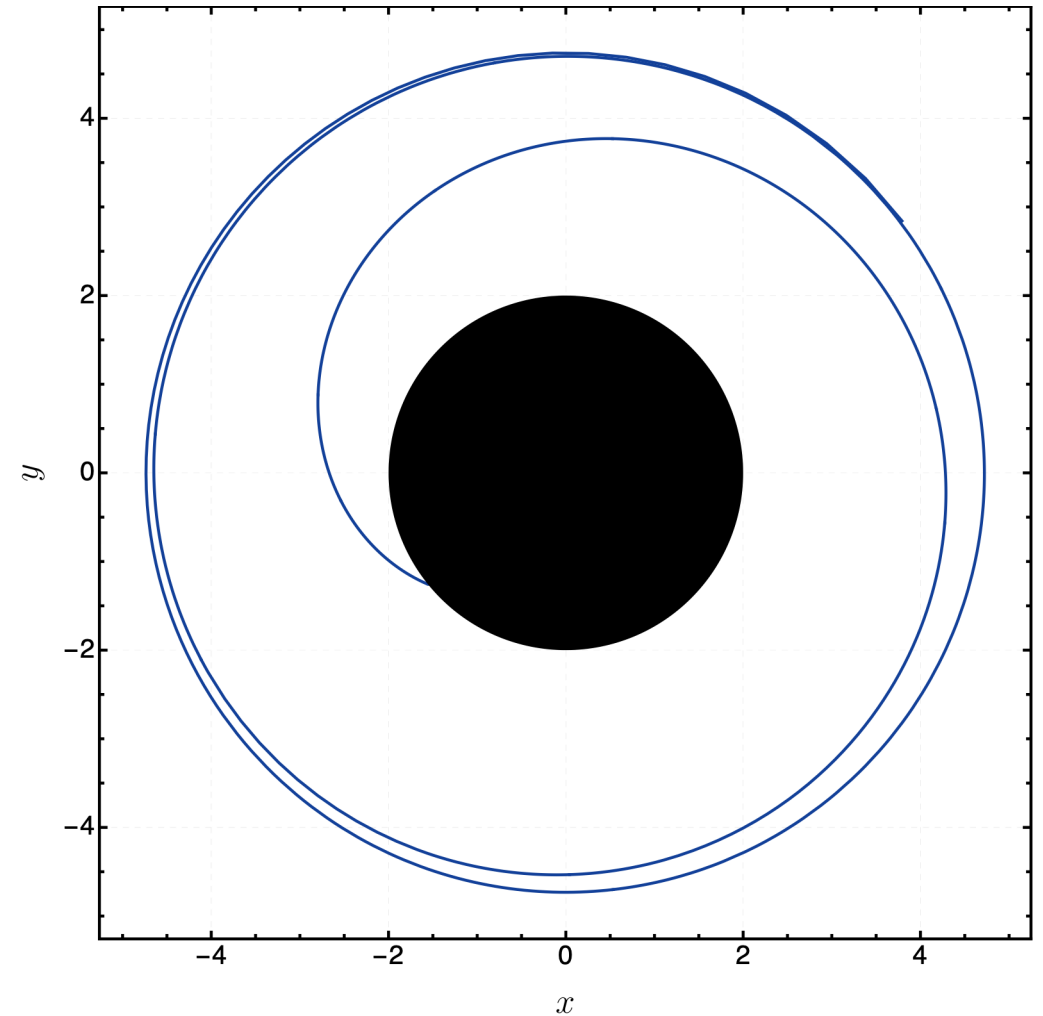
$$\lambda_*^- = \lambda_{ap} + \Gamma_* \operatorname{arccosh}\left(\frac{1}{2} \left(\frac{1}{\sqrt{\varepsilon}} + \frac{r_{[0]}(0)}{r_*}\right) - 1\right)$$

Epoch 4: plunge

- Final part: **plunge towards the horizon.**
- Starts at $\lambda = \lambda_{\star}^{+}$ and ends at the horizon at $\lambda = \lambda_H$
- Expansion :

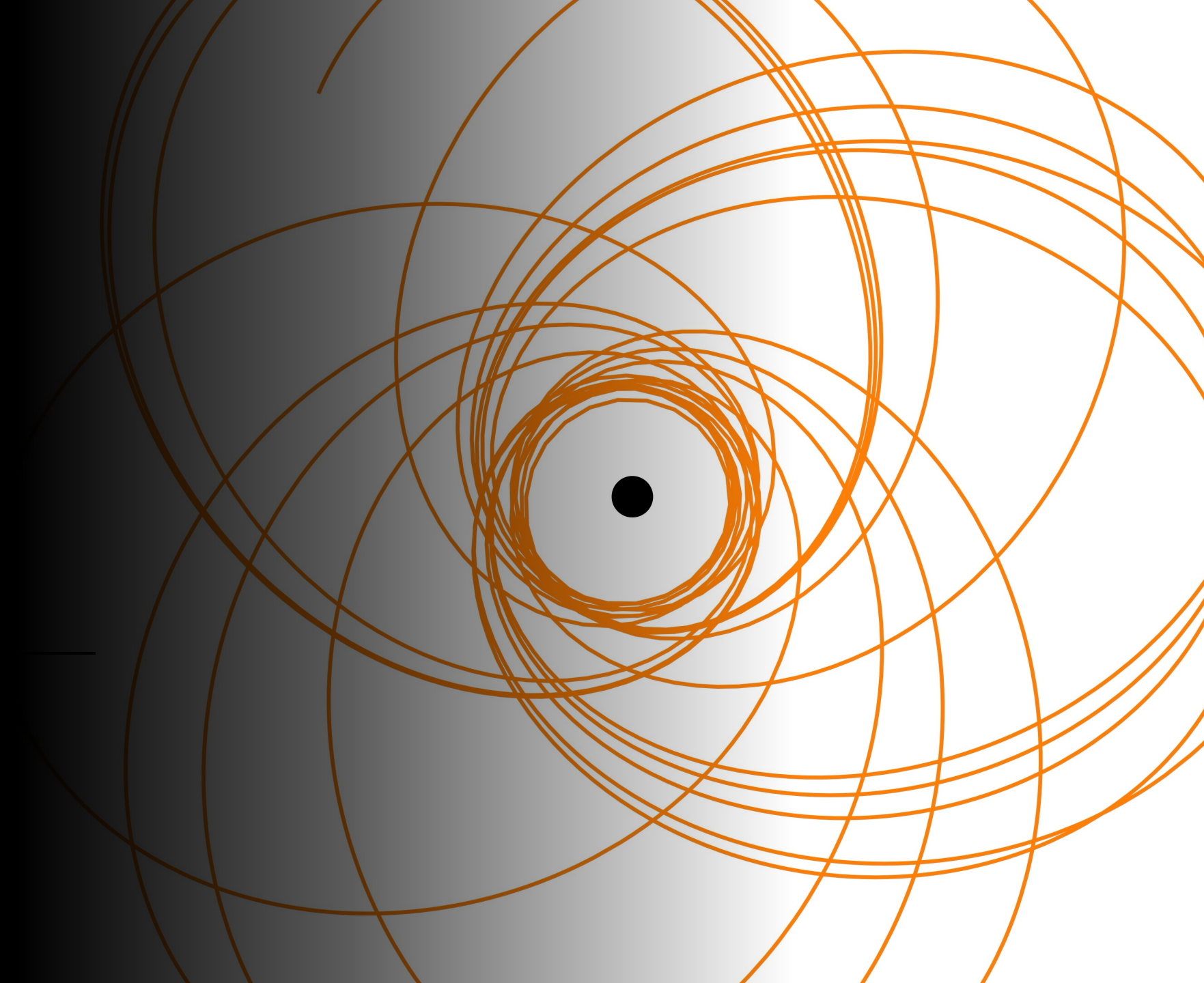
$$r_p(\lambda) = r_{crit}(\lambda - \lambda_H) + \varepsilon r_{\{1\}}(\lambda - \lambda_H) + O(\varepsilon^2)$$

- We can match the plunge with the transition at $\lambda_{\star}^{+}$
- $\lambda_{\star}^{+} - \lambda_{\star}^{-} = \frac{\Gamma_{\star}}{2} \log \left(\frac{-v_{\star} + r_{[0]}(0)/\Gamma_{\star}}{-v_{\star} - r_{[0]}(0)/\Gamma_{\star}} \right)$ is the timescale of the transition





Waveforms



- Metric perturbation sourced by a point-particle in Schwarzschild background $\Rightarrow$ RWZ equations

$$\left(\partial_x^2 - \partial_t^2 - V_\ell^{e/o} \right) \Psi_{\ell m}^{e/o}(t, r) = \tilde{S}_{\ell m}^{e/o}(t, r)$$

R-W (odd) and Zerilli (even) potentials

Source: massive particle on a geodesic

- Strategy: solve the frequency domain RWZ equation

$$\left(\partial_x^2 + \omega^2 - V_\ell^{e/o}(r) \right) \hat{\Psi}_{\ell m}^{e/o}(\omega, r) = \hat{S}_{\ell m}^{e/o}(\omega, r)$$

- **Offline** computation: Fourier modes of the waveform (slow),
- **Online** steps: inverse Fourier and get time domain $\psi_{\ell m}^{e/o}$ (fast),
- Get amplitudes, phase, fluxes,... For the homoclinic and critical plunge.

Offline computation

- Critical plunge and homoclinic: $t_p \in] - \infty, +\infty[$
- Decomposition of the source:

$$\tilde{S}_{\ell m}^{e/o} = \left(A_{\ell m}^{e/o}(r_p, s_r) \delta(r - r_p) + B_{\ell m}^{e/o}(r_p) \partial_r \delta(r - r_p) \right) e^{-im\phi_p(r_p)}$$

- Solution: Convolution of the source terms with the Green functions

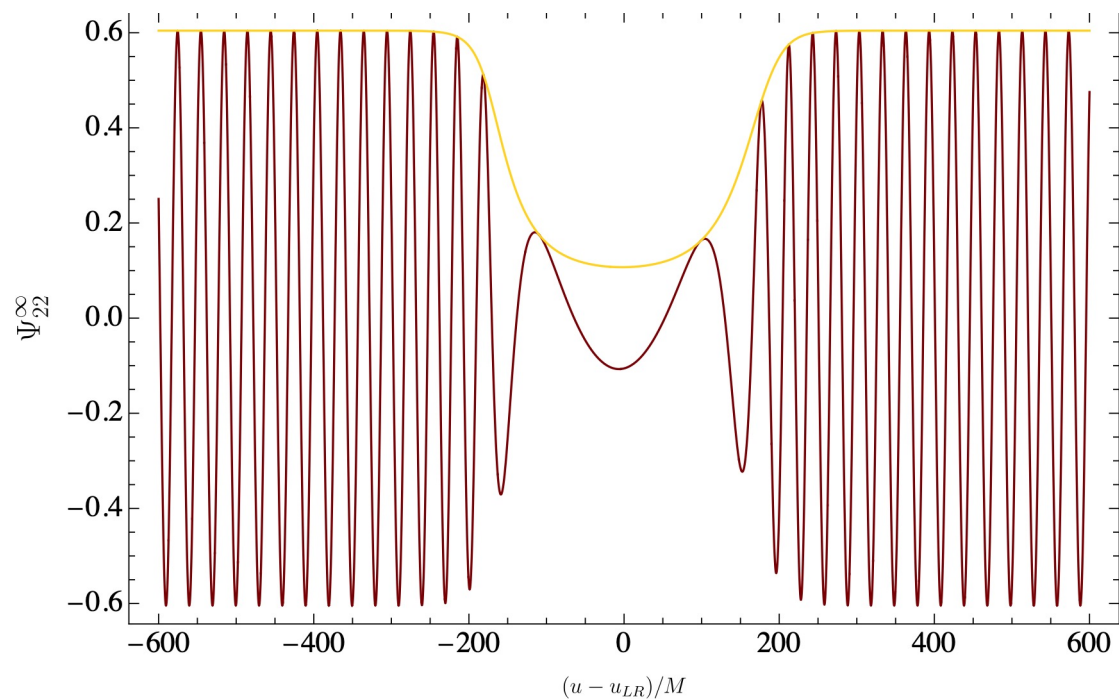
- **Solution:**

$$C_{\ell m}^{e/o \infty/\mathcal{H}}(\omega) = \int_{-\infty}^{+\infty} dt e^{i\omega t - im\phi_p(t)} K_{\ell m}^{e/o \infty/\mathcal{H}}(\omega, t)$$

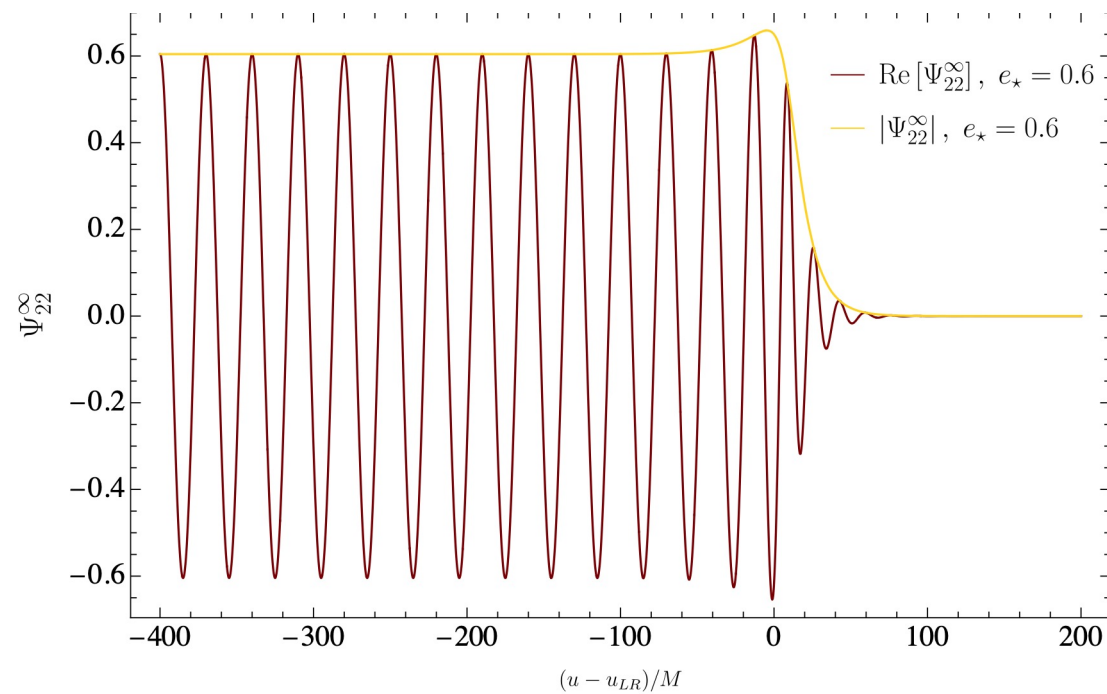
$$\Psi_{\ell m}^{e/o \infty/\mathcal{H}} = \mathcal{F}^{-1} \left(\frac{C_{\ell m}^{e/o \infty/\mathcal{H}}}{W_{\ell}(\omega)} \right)$$

Depends on A and B

Homoclinic orbit



Critical plunge

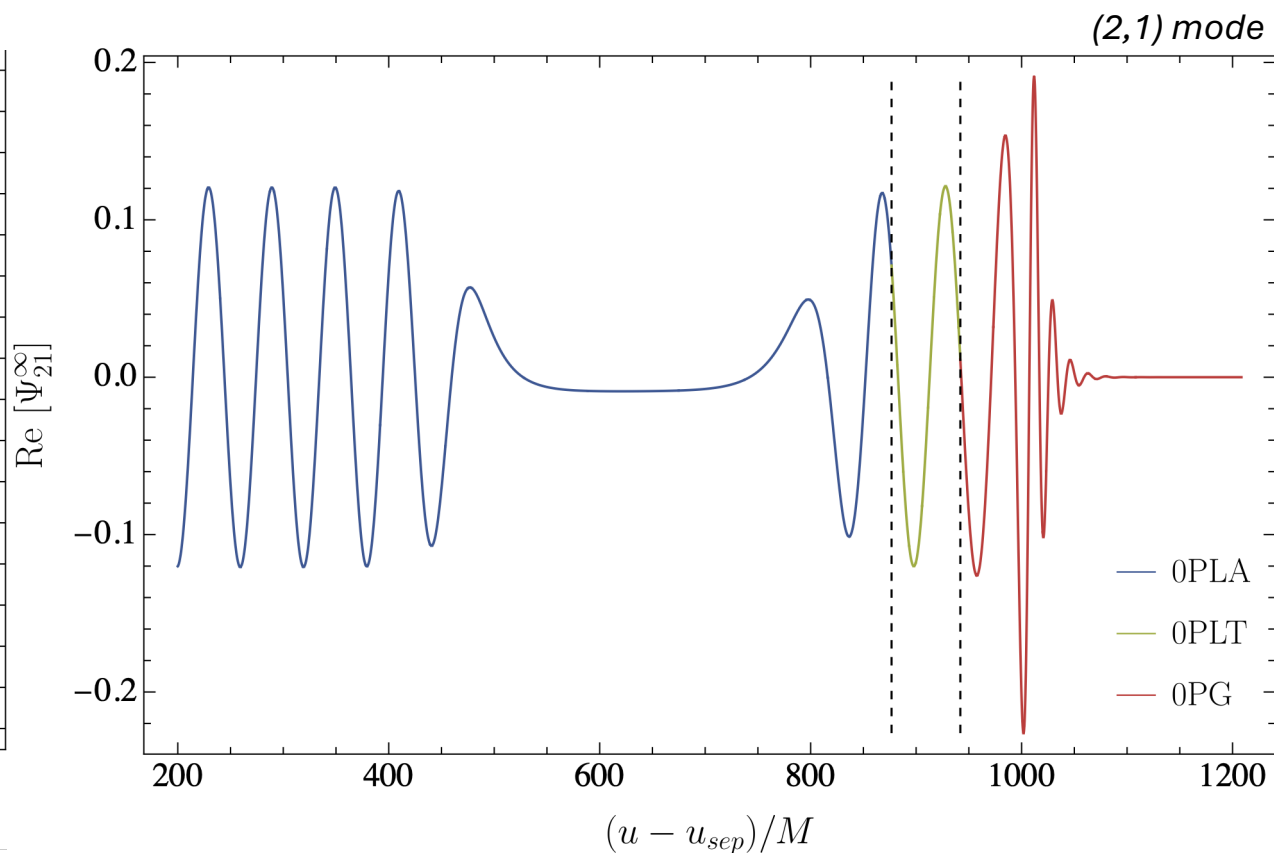
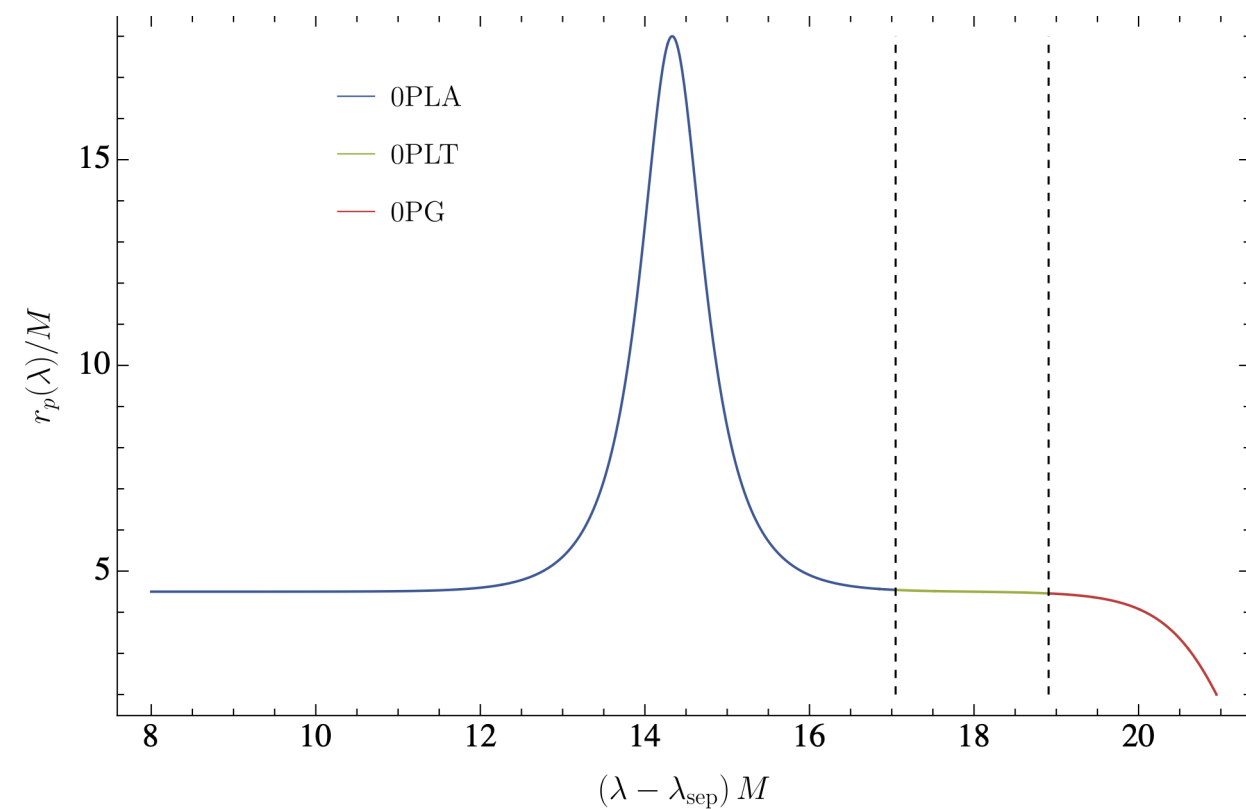


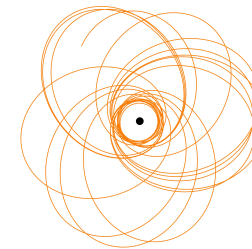
(2,2) mode
 $r_\star = 4.5$

All in one

$$\varepsilon = 10^{-5}, \quad \psi_r^* = 10^{-6}, \quad r_* = 4.5$$

Any ψ_r^* : we need self-force data





Conclusion

- Transition and plunge must be computed to finally construct the waveform model
- Eccentric EMRI: study the final approach preceding the transition regime
- The transition takes place on a timescale of order $\log(\varepsilon)$ after the separatrix crossing

Prospects

- Get self-force data on the homoclinic
- Build IMR waveforms in Schwarzschild, then in Kerr
- Go to higher order



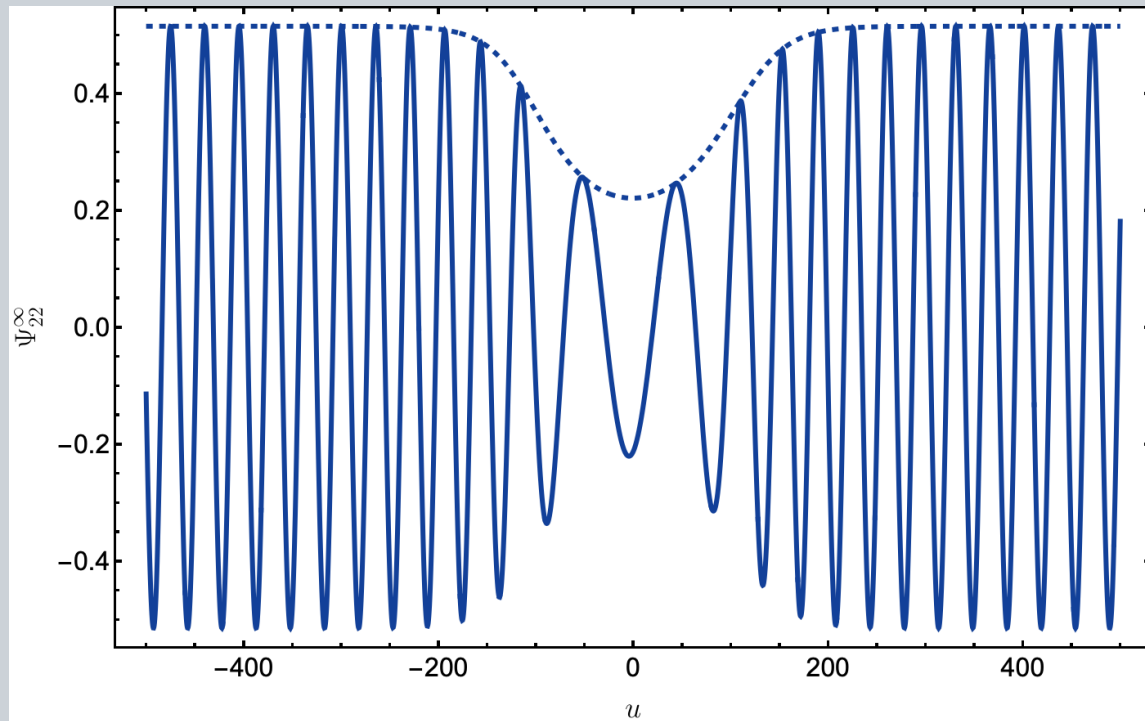
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Attention!



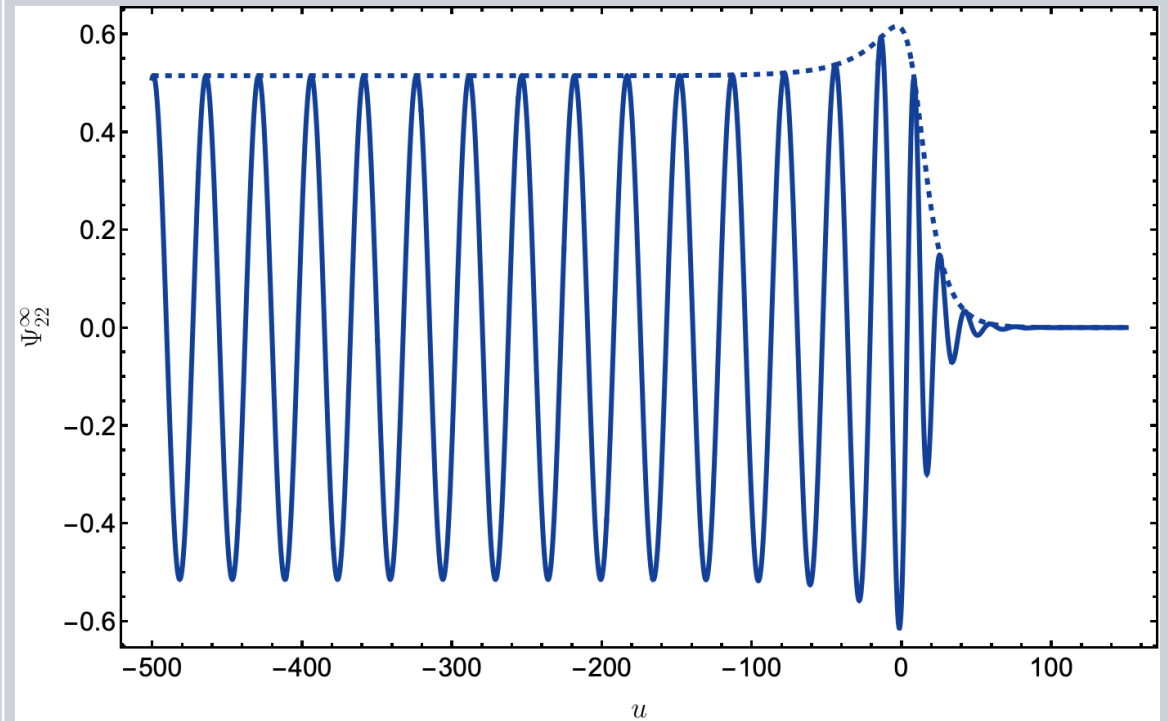
Backup slides

Online computation

Homoclinic orbit

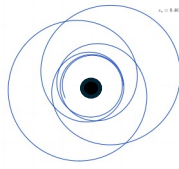


Critical plunge



(2,2) mode
 $r_\star = 5$

Epoch 1: inspiral



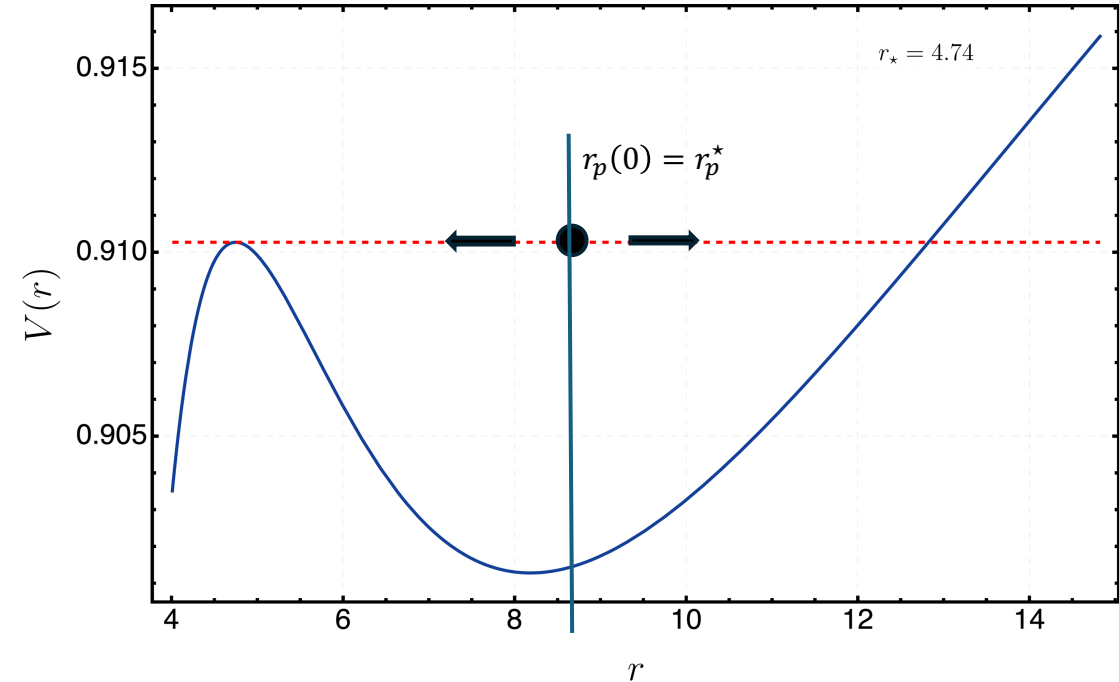
- Keplerian parametrisation:

$$E = E_K(p, e)$$

$$L = L_K(p, e)$$

- Inspiral = Succession of bounces between

$$r_{per} = \frac{p}{1+e} \text{ and } r_{ap} = \frac{p}{1-e}, \text{ with } \ddot{r}_p = -\frac{1}{2} \partial_r V(r_*, L) + f^r$$



The secondary is still **free** to bounce once against the potential

- Definition of separatrix:

$$\dot{r}_p = \ddot{r}_p = 0$$

$\Leftrightarrow$

$$E^2 = V(r_*, L) \quad ; \quad \partial_r V(r_*, L) = 2 f^r,$$

- Separatrix: parametric curve in $(L, r_p, \dot{r}_p)$ phase space
- $4 < r_* < 6$: parameter along the separatrix

Mino time at the separatrix crossing: $\lambda = 0$.

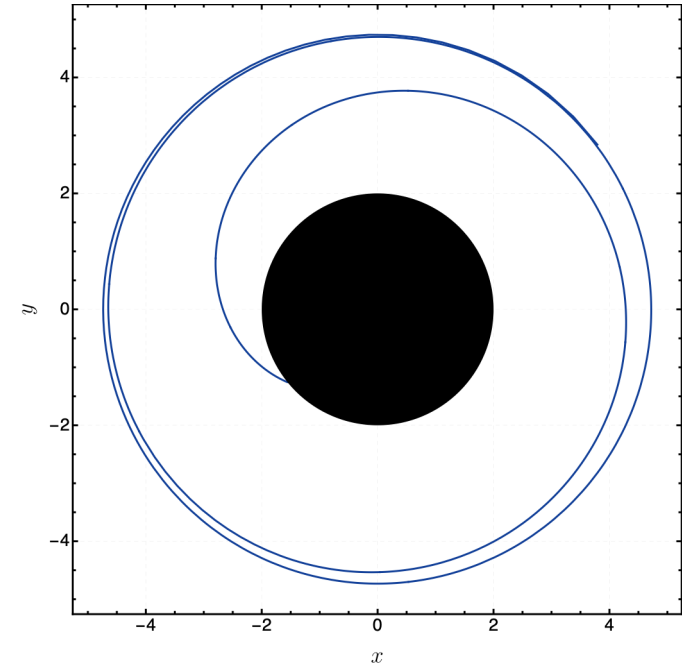
$$E(0) = E_*^G(r_*) + \varepsilon \Omega_* \delta L_*^{(1)}(r_*)$$

$$L(0) = L_*^G(r_*) + \varepsilon \delta L_*^{(1)}(r_*)$$

Epoch 4: plunge

- Final part: **plunge into the abyss.**
- Starts at $\lambda = \lambda_\star^+$ and ends at the horizon at $\lambda = \lambda_H$
- Expansion :

$$r_p(\lambda) = r_{crit}(\lambda - \lambda_H) + \varepsilon r_{\{1\}}(\lambda - \lambda_H) + O(\varepsilon^2).$$



$$r_{crit}(\lambda - \lambda_H) = r_\star - \frac{r_\star}{1 - \frac{p_\star}{4 r_\star e_\star} [1 - \cosh\left(\frac{\lambda - \lambda_H - \Delta\lambda_H}{\Gamma_\star}\right)]}$$

- $\Delta\lambda_H$ depends only on $r_\star$
- We can match the plunge with the transition: $r_{crit}(\lambda_\star^+ - \lambda_H) = r_\star + \sqrt{\varepsilon} r_{[0]}(\lambda_\star^+ - \lambda_\star^-)$
- $\lambda_\star^+ - \lambda_\star^- = \frac{\Gamma_\star}{2} \log\left(\frac{-v_\star + r_{[0]}(0)/\Gamma_\star}{-v_\star - r_{[0]}(0)/\Gamma_\star}\right)$ is the timescale of the transition

How to solve RWZ

Equation
$$\left(\partial_x^2 - \partial_t^2 - V_\ell^{e/o}\right) \Psi_{\ell m}^{e/o}(t, r) = \tilde{S}_{\ell m}^{e/o}(t, r),$$

Regularised sources
$$S_{\ell m}^{e/o} = A_{\ell m}^{e/o}(r_p, s_r) \delta(r - r_p) + B_{\ell m}^{e/o}(r_p) \partial_r \delta(r - r_p).$$

Solution
$$\Psi_{\ell m}^{e/o \infty}(r_p, s_r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t_G(r_p, s_r) - x_p)} \frac{C_{\ell m}^{e/o \infty}(\omega)}{2i\omega A_\ell^{\text{in}}(\omega)}$$

With
$$C_{\ell m}^{e/o \infty/\mathcal{H}}(\omega) = \int_{r_\star}^2 \frac{dr'}{\dot{r}_G(r', 1)} e^{i\omega t_G(r', 1) - im\phi_G(r', 1)} \times K_{\ell m}^{e/o \infty/\mathcal{H}}(\omega, r', 1)$$

- Metric perturbation sourced by a point-particle in Schwarzschild background $\Rightarrow$ RWZ equations

$$\left(\partial_x^2 - \partial_t^2 - V_\ell^{e/o} \right) \Psi_{\ell m}^{e/o}(t, r) = \tilde{S}_{\ell m}^{e/o}(t, r)$$

$$V_\ell^e(r) := \frac{f(r)}{r^2 \Lambda_\ell^2} \left[2\gamma_\ell^2 \left(\gamma_\ell + 1 + \frac{3}{r} \right) + \frac{18}{r^2} \left(\gamma_\ell + \frac{1}{r} \right) \right],$$

$$V_\ell^o(r) := \frac{f(r)}{r^2} \left[\ell(\ell + 1) - \frac{6}{r} \right],$$

$$\tilde{S}_{\ell m}^{e/o} = \left(A_{\ell m}^{e/o}(r_p, s_r) \delta(r - r_p) + B_{\ell m}^{e/o}(r_p) \partial_r \delta(r - r_p) \right) e^{-im\phi_p(r_p)}$$

Separation in several epochs

Epoch 0: inspiral

Epoch 1: final approach

Epoch 3: plunge

Epoch 2: transition

