

MEANDERS

(P. DF + E. Guitten + O. Golincelli + J. Jacobsen)

- Preamble: the physics of folding
- Folding a strip of stamps from Lemoine to Arnold
meanders
- Alternative descriptions: I - Algebra
Temperley-Lieb Algebra ; Meander determinant
- Alternative descriptions : II - Geometry
2D quantum gravity and a model of rivers and roads
Field Theory predictions $\rightarrow$ asymptotics
- Related Problems: Hamiltonian cycles on random Eulerian
Triangulations

Preamble: THE PHYSICS OF FOLDING

- Polymer chains + interactions
 - Polymer membranes + bending rigidity
 - Phantom Folding
- } crumpling transitions

Membrane folding meets Integrable Lattice Models

EUROPHYSICS LETTERS

20 May 1994

Europhys. Lett., **26** (6), pp. 455-460 (1994)

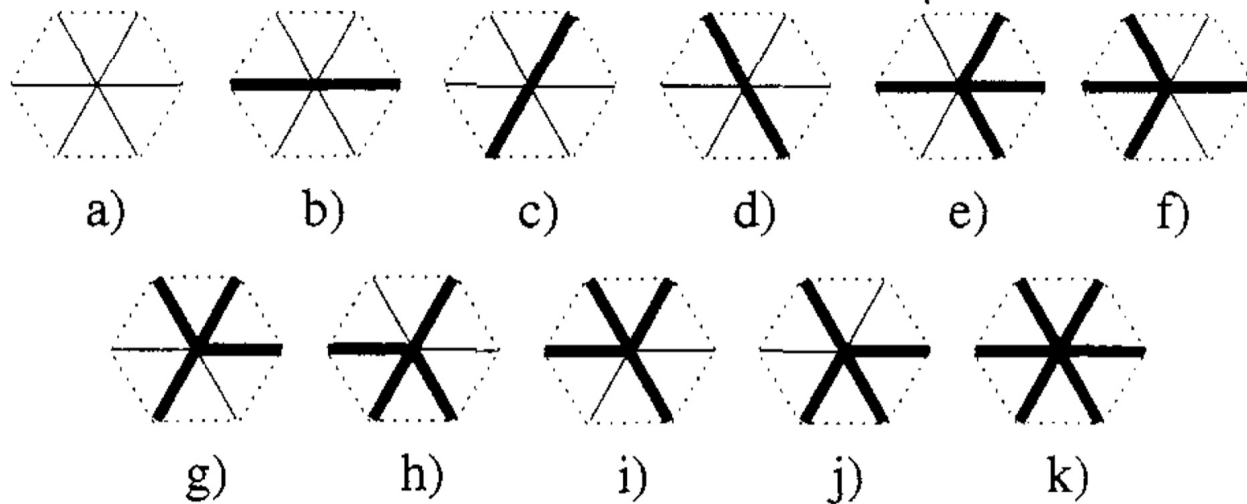
Entropy of Folding of the Triangular Lattice.

P. DI FRANCESCO and E. GUITTER

Service de Physique Théorique de Saclay() - F-91191 Gif-sur-Yvette Cedex, France*

(received 16 February 1994; accepted in final form 22 March 1994)

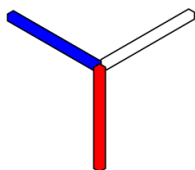
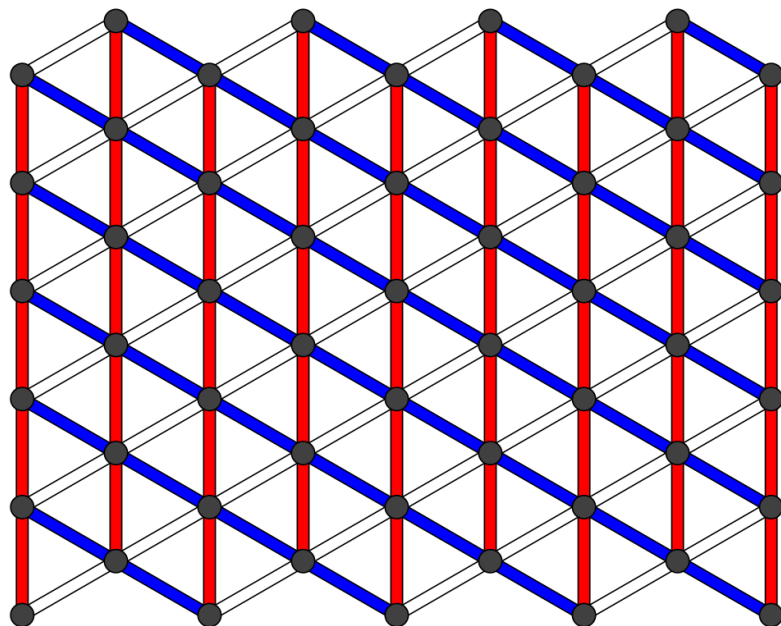
From 11-Vertex model



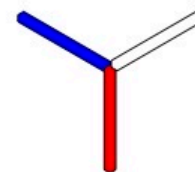
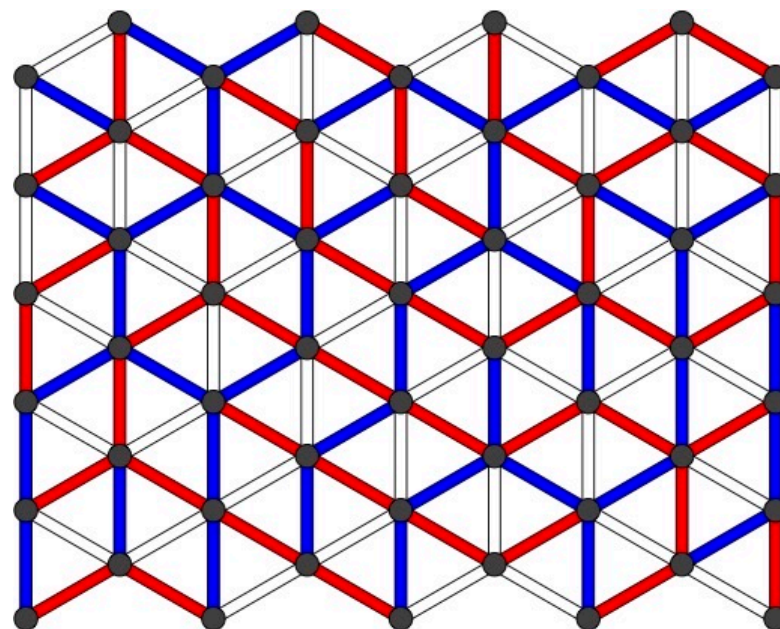
--- Q: compute the entropy of folding / triangle

$$q = 1.2087...$$

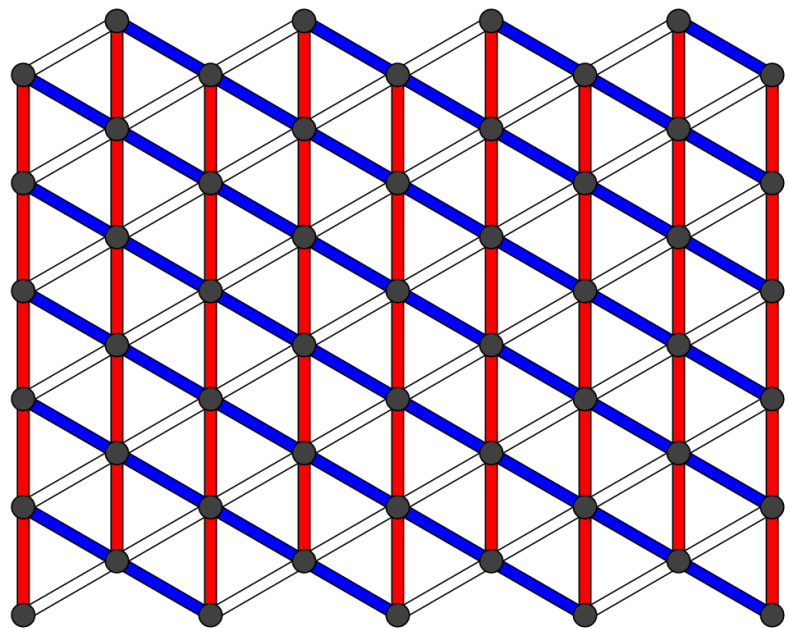
To the tricolor model



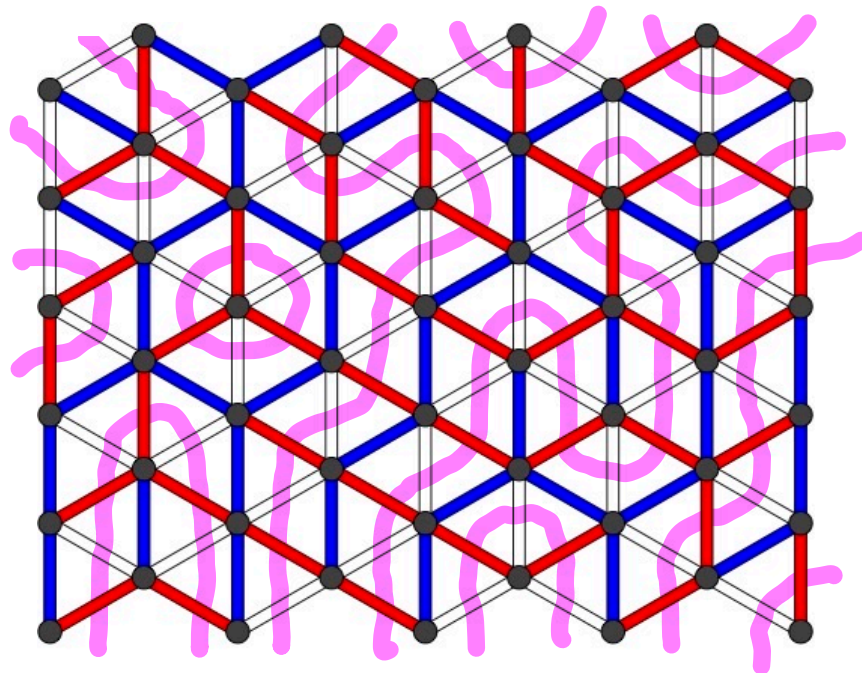
Flat sheet



Folded sheet



Flat sheet



Folded sheet

Fully-packed loop gas $O(n=2)$

[Baxter'70]

$$q = \prod_{n=1}^{\infty} \frac{(3n-1)}{\sqrt{3n(3n-2)}} = \frac{\sqrt{\Gamma(1/3)}}{\Gamma(2/3)} = \frac{\sqrt{3}}{2\pi} \Gamma(1/3)^{3/2}.$$



ELSEVIER

Nuclear Physics B 450 [FS] (1995) 463–494

NUCLEAR
PHYSICS B [FS]

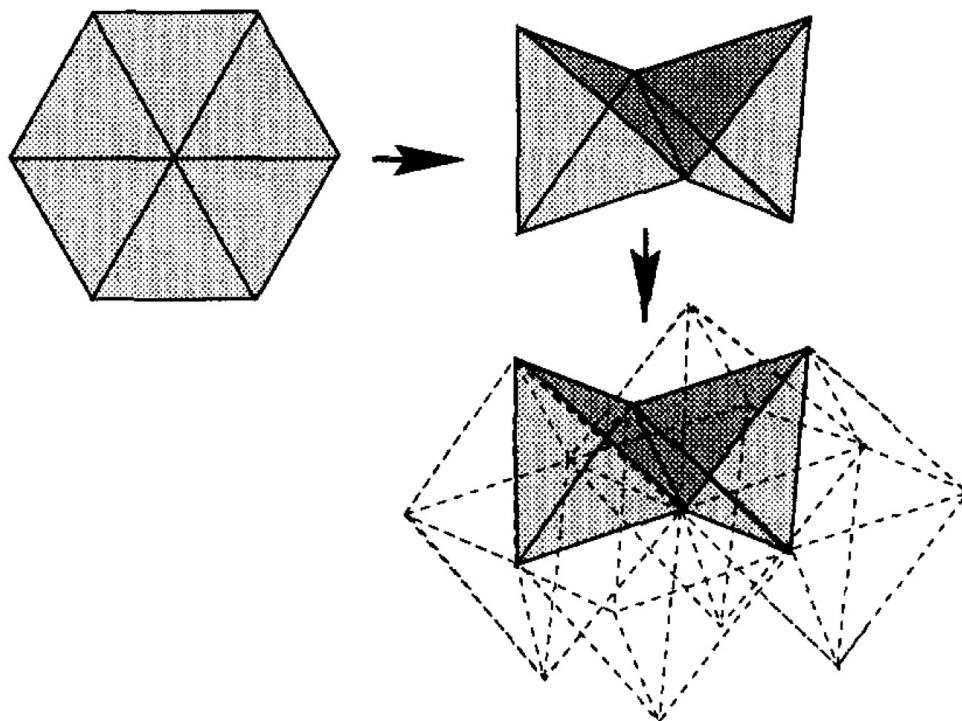
Three-dimensional folding of the triangular lattice

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Received 22 February 1995; revised 1 June 1995; accepted 2 June 1995



→ 96 vertex model!
No exact entropy but bounds

Folding transition of the triangular lattice

P. Di Francesco and E. Guitter

Service de Physique Théorique de Saclay, F-91191 Gif-sur-Yvette Cedex, France*

(Received 14 June 1994)

PHYSICAL REVIEW E

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J. Phys. A: Math. Gen. **30** (1997) L829–L838. Printed in the UK

LETTER TO THE EDITOR

Folding of the triangular lattice in the face-centred cubic lattice with quenched random spontaneous curvature

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Received 29 August 1997

Folding of the triangular lattice with quenched random bending rigidity

P. Di Francesco, E. Guitter, and S. Mori^{*}

Service de Physique Théorique, C.E.A. Saclay, F-91191 Gif-sur-Yvette Cedex, France

(Received 19 July 1996)



ELSEVIER



Nuclear Physics B 495 [FS] (1997) 583–607

Geometrical folding transitions of the triangular lattice in the face-centred cubic lattice

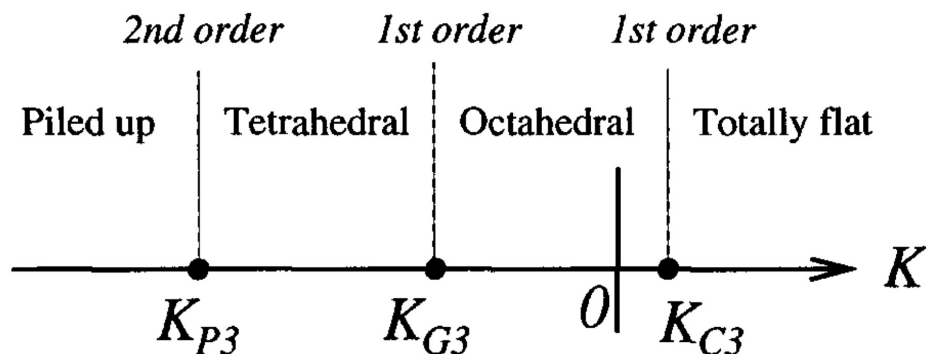
M. Bowick^{a,1}, O. Golinelli^{b,2}, E. Guitter^{b,3}, S. Mori^{c,4}

^a Physics Department, Syracuse University, Syracuse, NY 13244-1130, USA

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Received 20 November 1996; accepted 13 March 1997



What about 1D?

LE PROBLEME DES TIMBRES -POSTE

(Folding a strip of stamps)

Émile Lemoine



?

Édouard Lucas



1891

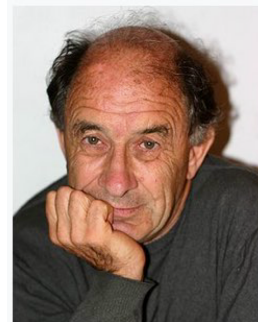
Henri Poincaré



1912

Vladimir Arnold

Владимир Арнольд



1991

Édouard Lucas



1891

THÉORIE DES NOMBRES

PAR

Edouard LUCAS.

« Le goût pour les sciences abstraites, en général, et surtout pour les mystères des nombres, est fort rare; on ne s'en étonne pas. Les charmes enchanteurs de cette science sublime ne se décèlent dans toute leur beauté qu'à ceux qui ont le courage de l'approfondir. »

[Lettre de C.-F. GAUSS à M^{lle} SOPHIE GERMAIN, du 30 avril 1807].

TOME PREMIER.

LE CALCUL DES NOMBRES ENTIERS. — LE CALCUL DES NOMBRES RATIONNELS.
LA DIVISIBILITÉ ARITHMÉTIQUE.



PARIS,

GAUTHIER-VILLARS ET FILS, IMPRIMEURS-LIBRAIRES
DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,
Quai des Grands-Augustins, 55.

1891

(Tous droits réservés.)

4746
23/1

ACTUAL FOLDING IN 1D

Exemple II. — La bande de timbres-poste. — De combien de manières peut-on replier, sur un seul, une bande de p timbres-poste?

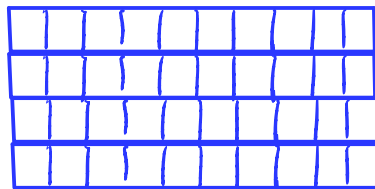
Exemple III. — La feuille de timbres-poste. — De combien de manières peut-on replier, sur un seul, une feuille rectangulaire de pq timbres-poste?

Nous ne connaissons aucune solution de ces deux problèmes difficiles proposés par M. EM. LEMOINE.

- 1D Folding



- 2D Folding



Henri Poincaré

SUR UN THÉORÈME DE GÉOMÉTRIE.

Par M. H. Poincaré (Paris).

Adunanza del 10 marzo 1912.

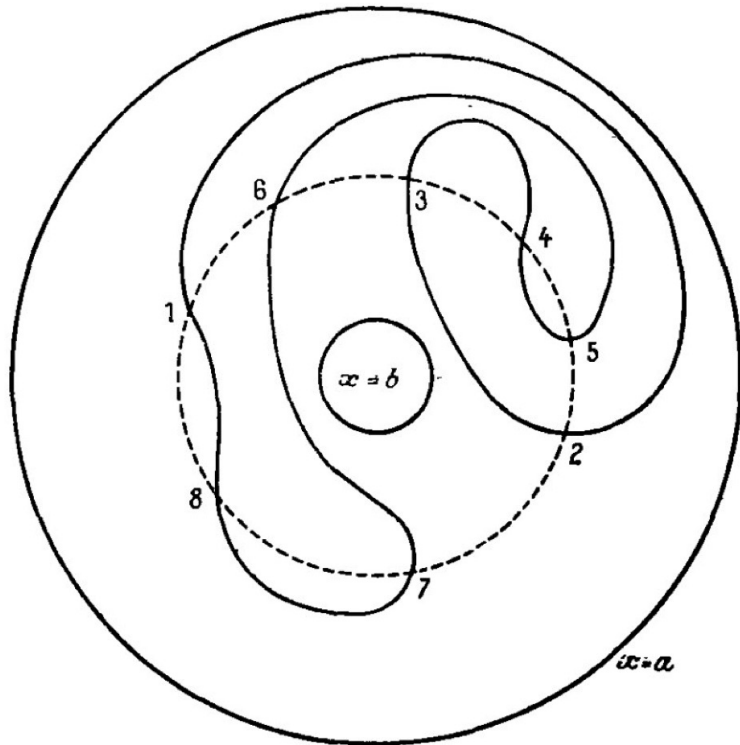
§ I.

INTRODUCTION.

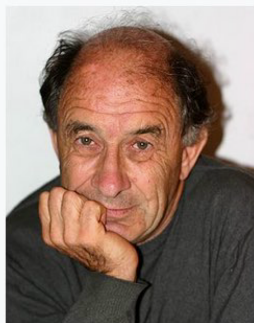
Je n'ai jamais présenté au public un travail aussi inachevé; je crois donc nécessaire

1912

GEODESICS ON CONVEX SURFACES



(Fig. 1).



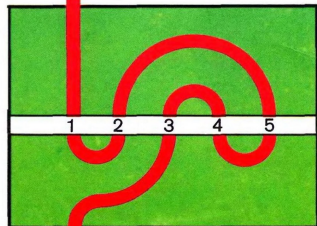
1991

Искусство

Научно-популярный
физико-математический журнал

ISSN 0130-2221

1 2 3 4 5 6 7 8 9 10 11 12



3 4 5 2 1

1991

По традиции на первых заседаниях научного семинара В. И. Арнольда для студентов Московского университета его участникам предлагается несколько новых задач — точнее, тем для исследования. Среди них иногда встречаются чрезвычайно красивые задачи, формулировки которых понятны даже школьнику. Вот одна из этих тем.

МЕАНДРЫ

Академик В. АРНОЛЬД

Шоссе, идущее с запада на восток, пересекает несколько раз реку, текущую с юго-запада также на восток. Занумеруем мосты в порядке их следования вдоль шоссе (с запада на восток). Проплывая под мостами вниз по реке, мы будем встречать их, вообще говоря, в другом порядке. Так, например, река на рисунке 1 проходит мосты в порядке 3, 4, 5, 2, 1. Таким образом, эта река определяет перестановку чисел от 1 до 5: (3 4 5 2 1). Ясно, что другая река могла бы протекать иначе и задавать другую перестановку. Но далеко не любая перестановка чисел (мостов) может быть реализована таким образом. (Попробуйте, например, придумать реку, проходящую мосты в порядке 2, 1, 3, 4, 5.) Мы будем называть перестановку меандром, если ее можно задать с помощью подходящей реки.

Основной вопрос для нас будет такой: сколько существует различных меандров (т. е. сколько перестановок номеров реализуется), если общее число мостов равно n ? Обозначим число различных меандров с n мостами через $a(n)$. Легко видеть, что $a(1) = 1$, $a(2) = 1$, $a(3) = 2$, $a(4) = 3$ (рис. 2).

Задача 1. Найдите следующие члены последовательности $a(n)$.

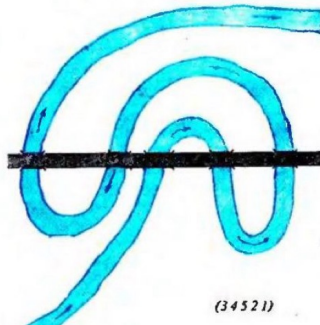
Ответ: 1, 1, 2, 3, 8, 14, 42, 81, 262, 538, 1828, 3926, 13820, ...

Общая формула для $a(n)$ неизвестна. Неизвестны даже асимптотики $a(n)$ и $a(n+1)/a(n)$ при $n \rightarrow \infty$.

Задача 2. Докажите, что река впервые пересекает шоссе под мостом, номер которого нечетен.

Задача 3. Докажите, что номера 1-го, 3-го, 5-го... мостов (вдоль реки) нечетны, если нумеровать мосты вдоль шоссе, а номера 2-го, 4-го... мостов — четны.

Начнем классифицировать меандры с того, что зафиксируем первый вдоль реки мост. Обозначим через $a(n)$ число меандров с n мостами, для которых река впервые пересекает шоссе под i -м (вдоль шоссе) мостом. Согласно задаче 2,



(3 4 5 2 1)

Рис. 1.

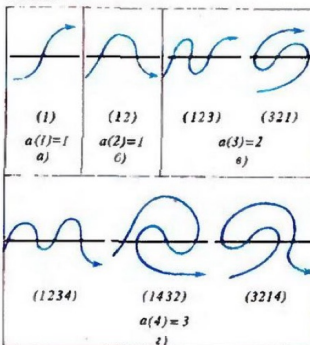
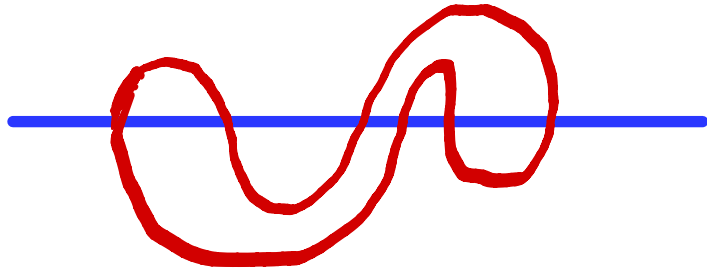


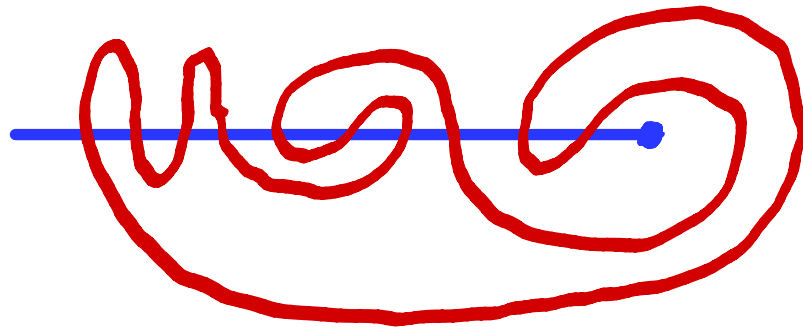
Рис. 2.

MEANDERS



closed meander

= meander # M_n



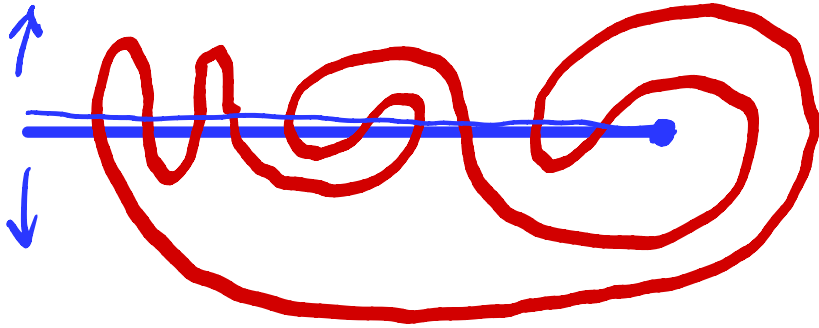
open meander

= semi-meander # $\tilde{M}_n$

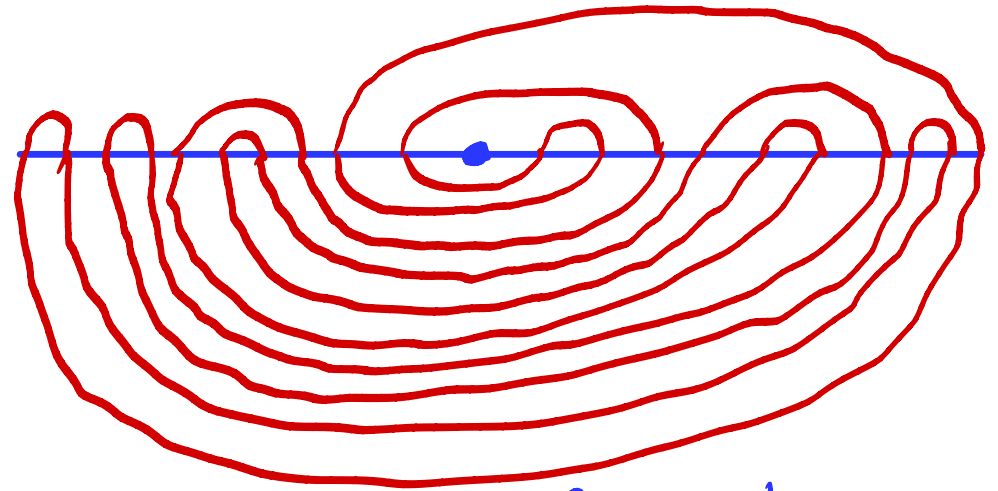
Questions:

- count these for fixed n
- asymptotics? universality classes?

REMARK: OPEN MEANDER = SEMI-MEANDER

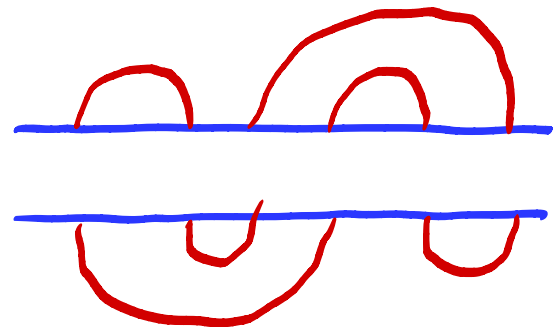
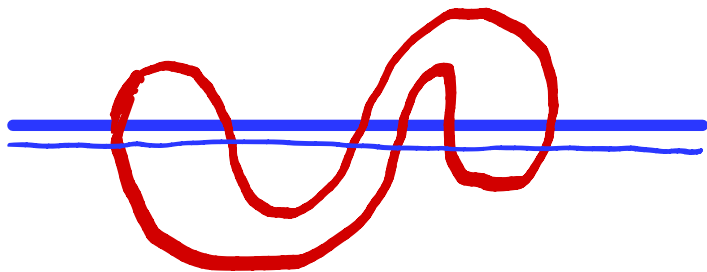


arch configuration



rainbow configuration

MEANDER = PAIR OF ARCH CONFIGURATIONS



a

b^t

COUNTING : Golinelli's algorithm For semi-meanders

```

PARAMETER (nmax = 14)
INTEGER A(-nmax+1:nmax)
INTEGER Sm(nmax)
INTEGER n
INTEGER j
DATA n, Sm/0, nmax*0/
A(0) = 1
A(1) = 0
2 n = n + 1
  Sm(n) = Sm(n) + 1
  j = -n + 1
1 IF ((n.EQ.nmax).OR.(j.EQ.n+1)) GOTO 3
  A(A(j)) = n+1
  A(n+1) = A(j)
  A(j) = -n
  A(-n) = j
  GOTO 2
3 A(A(-n+1)) = A(n)
  A(A(n)) = A(-n+1)
  j = A(n)+1
  n = n - 1
  IF (n .GT. 1) GOTO 1
PRINT '(i3, i15)', (n, Sm(n), n = 1, nmax)
END

```

! maximal order
! arch representation
! semi-meander counter
! current depth (or order)
! next branch to visit
! n and Sm initialized to 0
! single-arch semi-meander

! a new node is visited

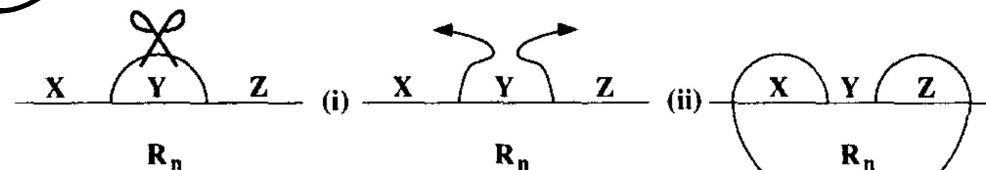
! leftmost (exterior) arch
! up or down?
! go down with process (I)

! going up

! next arch to break

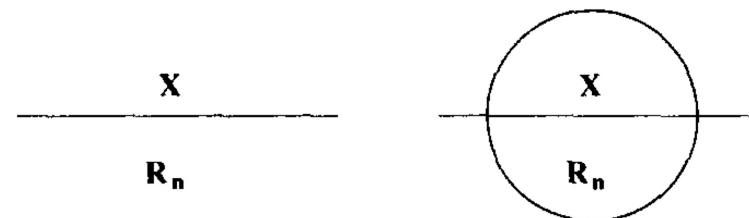
2 Moves :

(I)



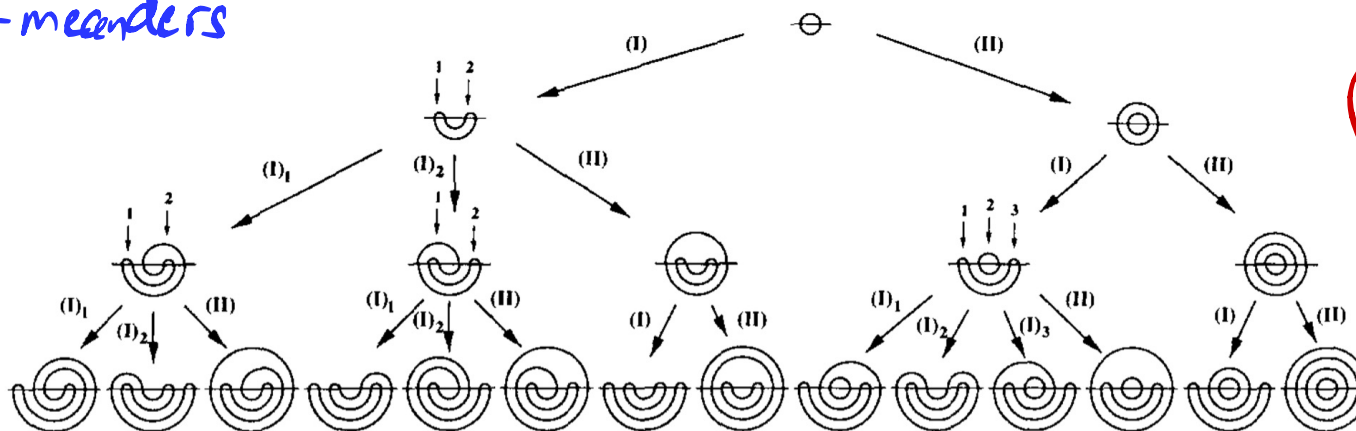
Preserves # C.C.

(II)



Changes # C.C. $\rightarrow$ # C.C. + 1

Tree of semi-meanders



use I only!

Semi-Meander numbers $\tilde{M}_n$ $\tilde{M}_n^{(k)}$

n	$\tilde{M}_n$	n	$\tilde{M}_n$	k	$\tilde{M}_{27}^{(k)}$	k	$\tilde{M}_{27}^{(k)}$
1	1	16	1053874	1	369192702554	16	2376167414
2	1	17	3328188	2	2266436498400	17	628492938
3	2	18	10274466	3	6454265995454	18	153966062
4	4	19	32786630	4	11409453277272	19	34735627
5	10	20	102511418	5	14161346139866	20	7159268
6	24	21	329903058	6	13266154255196	21	1333214
7	66	22	1042277722	7	9870806627980	22	220892
8	174	23	3377919260	8	6074897248976	23	31851
9	504	24	10765024432	9	3199508682588	24	3866
10	1406	25	35095839848	10	1483533803900	25	374
11	4210	26	112670468128	11	619231827340	26	26
12	12198	27	369192702554	12	236416286832	27	1
13	37378	28	1192724674590	13	83407238044		
14	111278	29	3925446804750	14	27346198448		
15	346846			15	8352021621		

ALTERNATIVE DESCRIPTIONS

I - ALGEBRA

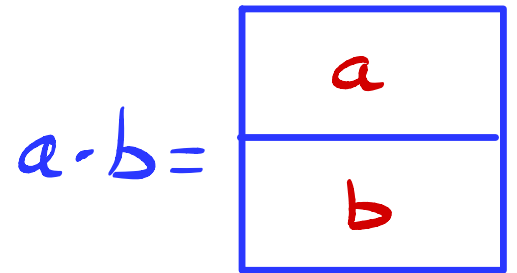
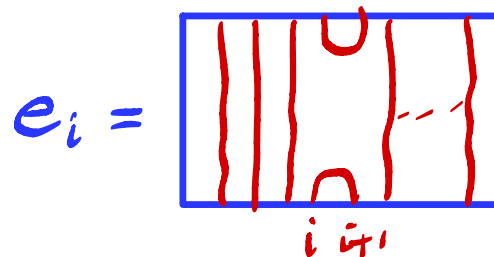
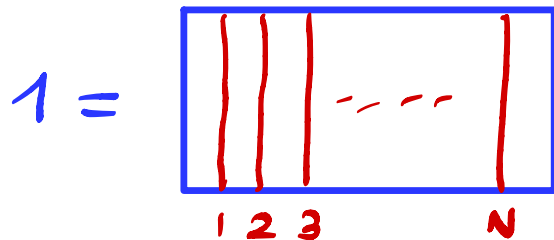
• Temperley-Lieb Algebra $TL_N(\beta) = \langle e_1, \dots, e_{N-1} \mid \mathcal{R} \rangle_{\text{over } \mathbb{C}}$

$$\mathcal{R} : \quad (1) \quad e_i^2 = \beta e_i \quad i=1, \dots, N-1$$

$$(2) \quad e_i e_{i \pm 1} e_i = e_i$$

(Integrable lattice models = Potts Model; Knot theory = Jones; etc--)

• Pictorial representation



(Concatenation)

• Relations R:

$$e_i^2 = \begin{array}{|c|} \hline \text{Diagram: 2 vertical lines, a loop on the right line crossing itself.} \\ \hline \end{array} = \beta \begin{array}{|c|} \hline \text{Diagram: 2 vertical lines, a loop on the right line.} \\ \hline \end{array} = \beta e_i$$

$i \quad i+1$ $i \quad i+1$

$$e_i e_{i+1} e_i = \begin{array}{|c|} \hline \text{Diagram: 3 vertical lines, a loop on the middle line crossing the other two.} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Diagram: 2 vertical lines, a loop on the right line.} \\ \hline \end{array} = e_i$$

$i \quad i+1 \quad i+2$ $i \quad i+1$

• Reduced elements = pull all strings, remove all loops

• Left ideals of $TL_{2N}(\beta)$

$$\begin{array}{|c|} \hline \text{Diagram: 2 vertical lines, a loop on the right line.} \\ \hline \end{array} \equiv \begin{array}{|c|} \hline \text{Diagram: 2 vertical lines, a loop on the right line.} \\ \hline \end{array}$$

$e_1 e_3 e_5 e_7 e_9 e_{11}$
 $e_8 e_{10} e_{12}$
 $e_2 e_4 e_{11}$

each configuration $\rightarrow$ linear basis of I_{2N}

$$I_{2N} = e_1 e_3 \dots e_{2N-1} \cdot TL_{2N}(\beta)$$

$$\dim(I_{2N}) = C_N = \text{Catalan \#}$$

- inner product = provided by the Trace

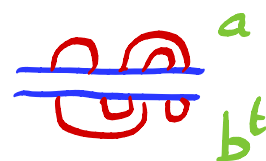
$$\text{Tr}(a) = \text{[diagram of a box labeled } a \text{ with a red loop around it]} = \beta^{\# \text{ loops}}$$

$$\begin{aligned} \langle a, b \rangle &= \frac{1}{\beta^N} \text{Tr}(a b^t) = \frac{1}{\beta^N} \text{[diagram of a box divided into two horizontal sections, the top section labeled } a \text{ and the bottom section labeled } b^t \text{, with a red loop around it]} \\ &= \beta^{\# \text{ loops}} \left(\text{[diagram of a box divided into two horizontal sections, the top section labeled } a \text{ and the bottom section labeled } b^t \text{]} \right) \leftarrow \beta^N \text{ trivial weight} \end{aligned}$$

counts # loops in meanders !!!

$$M_N(\beta) = \sum_{a,b} \langle a, b \rangle$$

Meander determinant



$$\mathcal{G}_n(\beta) \begin{pmatrix} \text{oo} & \text{oo} \\ \text{oo} & \text{oo} \end{pmatrix} \rightarrow \begin{pmatrix} \beta^2 & \beta \\ \beta & \beta^2 \end{pmatrix} \rightarrow \beta^2(\beta^2 - 1) = U_1^2 U_2$$

Thm

$$D_n(q) = \det(\mathcal{G}_n(q)) = \prod_{i=1}^n U_i(q)^{a_{n,i}}$$

$$a_{n,i} = \binom{2n}{n-i} - 2\binom{2n}{n-i-1} + \binom{2n}{n-i-2}$$

(U_i = Chebyshev of 2nd kind)

[DF + Guiller 96]

Proof = Gram-Schmidt!

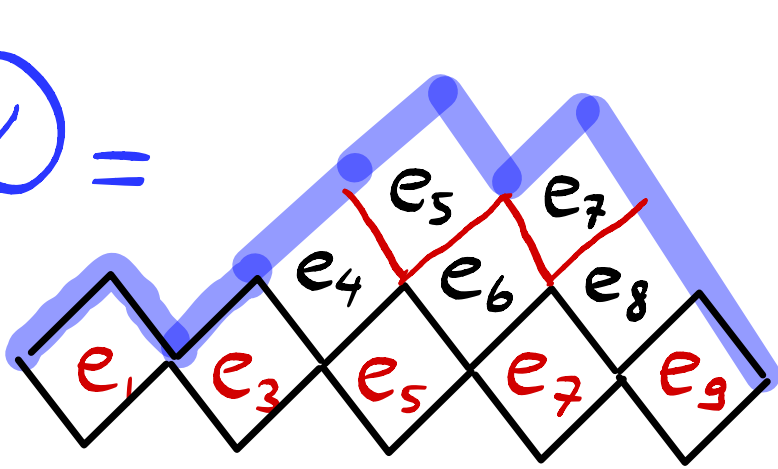
+ similar results for
semi-meanders
for ideals

$$I_{n,\ell} = e_1 \dots e_{2\ell} T L_n(\beta)$$

Idea of the proof: Dyck Paths

Arch configurations and

$\mathcal{D} =$



reduced element
 $\leftrightarrow e(\mathcal{D}) = e_1 e_3 e_5 e_7 e_9 e_4 e_6 e_8 e_5 e_7$
 $\in \mathbb{I}_{2N}$

- Change of basis that makes $\{e(\mathcal{D})\}$ orthonormal
- By local changes

$$e_i \xrightarrow{\text{height } h} \frac{U_h}{\sqrt{U_{h-1} U_{h+1}}} \left(e_i - \frac{U_{h-1}(\beta)}{U_h(\beta)} \right) \leftarrow \text{Chebyshev}$$

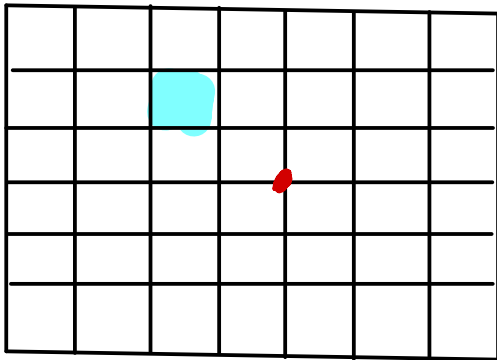
$\uparrow$ norm change $\Rightarrow$ determinant

□

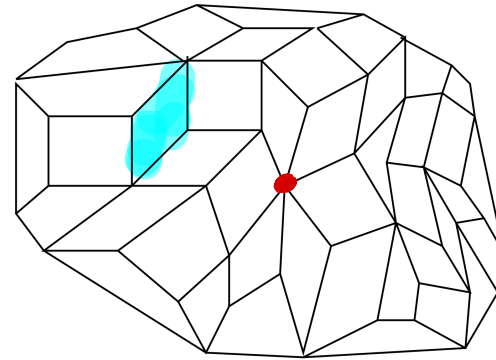
ALTERNATIVE DESCRIPTIONS

II - (Random) GEOMETRY

- A meander is a random map.
- ↕
- Discrete 2D Quantum gravity (Euclidian)



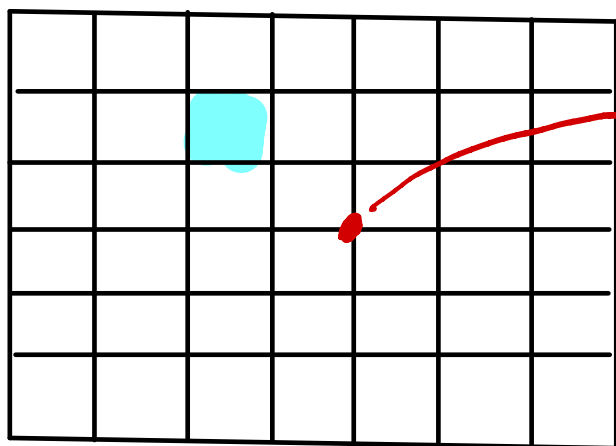
flat space (lattice)



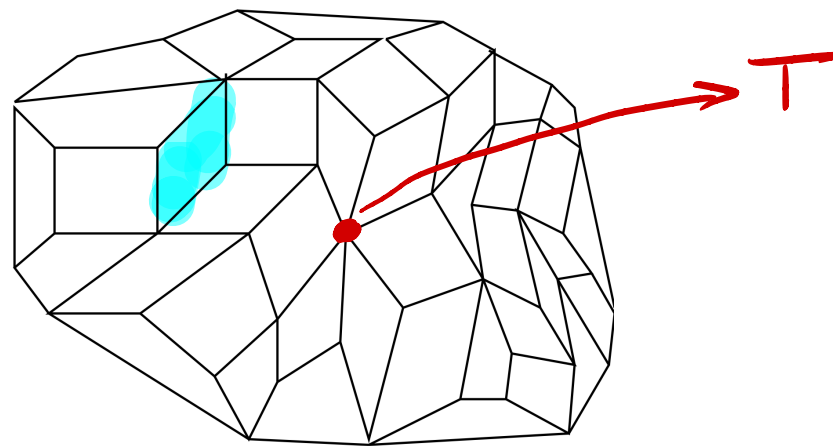
- area
- curvature

fluctuating space (map)

MATTER



lattice model



maps + matter

Continuum (large size, large area), critical limit

Conformal Field Theory

- central charge
- exponents

(Virasoro alg reps)

KPZ
→

Liouville Field Theory

- string susceptibility
- exponents

(Gravitational dressing)

[Knizhnik, Polyakov, Zamolodchikov '88]

KPZ formula = relates flat space scaling to random surface scaling.

(Based on the coupling of metric (Liouville) field to conformal (matter) field.)

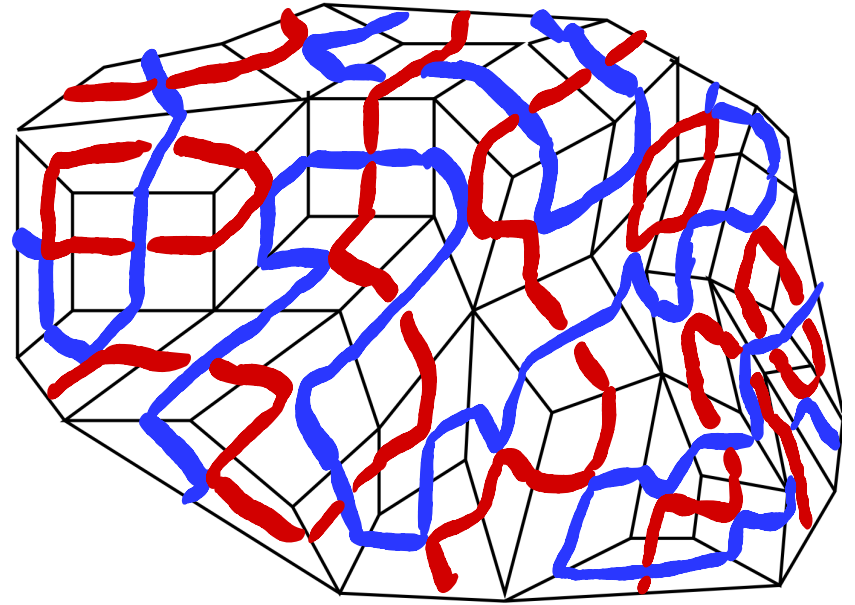
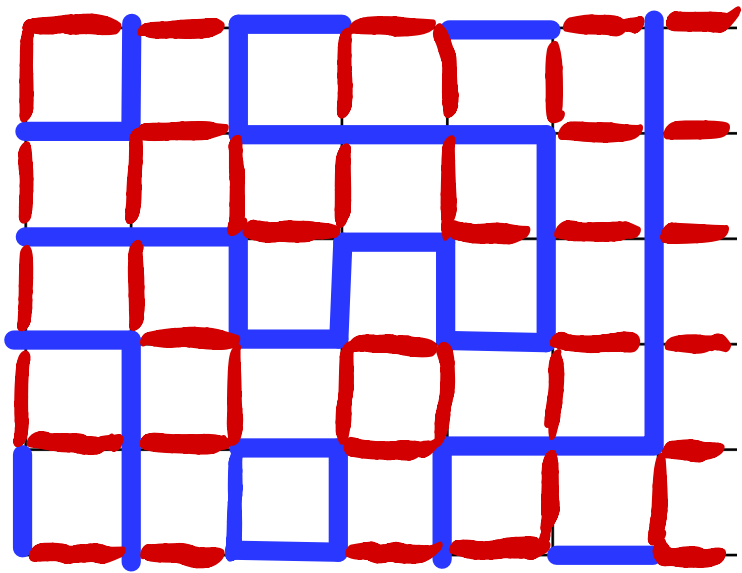
$$Z_{\text{random maps}}^{N \text{ triangles}} \underset{N \rightarrow \infty}{\sim} \frac{e^{\mu N}}{N^{\gamma-3}} ; \gamma = \frac{c-1 - \sqrt{(1-c)(25-c)}}{12}$$

c = central charge (flat space property)

γ = string susceptibility exponent (random space property)

Task = identify c !

Matter = Rivers and Roads



Fully-Packed loop
model FPL_2

Tangent meanders

Area = $\#$ (crossings &
tangency points)

$$C = -4$$

Prediction from Field Theory:

meander
number : $M_{2n} \sim_{n \rightarrow \infty} K \frac{A^n}{n^\alpha}$

$\left\{ \begin{array}{l} \sqrt{A} = \text{entropy / bridge} \\ \alpha = \text{configuration} \\ \text{exponent} \end{array} \right.$

$$\alpha = \frac{29 + \sqrt{145}}{12}$$

$$(= 2 - \gamma)$$

[PDF, O. Golinelli, E. Guitten]

semi-meander
number

$$\tilde{M}_n \sim \tilde{\kappa} \frac{A^n}{n^{\tilde{\alpha}}}$$

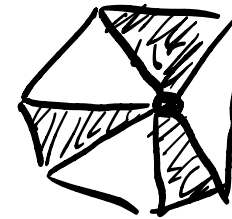
$$\tilde{\alpha} = 1 + \frac{\sqrt{11} (\sqrt{29} + \sqrt{5})}{24}$$

- numerical evidence: now up to 44 bridges, 3 decimals.

SOME RELATED COMBINATORIAL PROBLEM

Hamiltonian cycles on Random Eulerian Triangulations

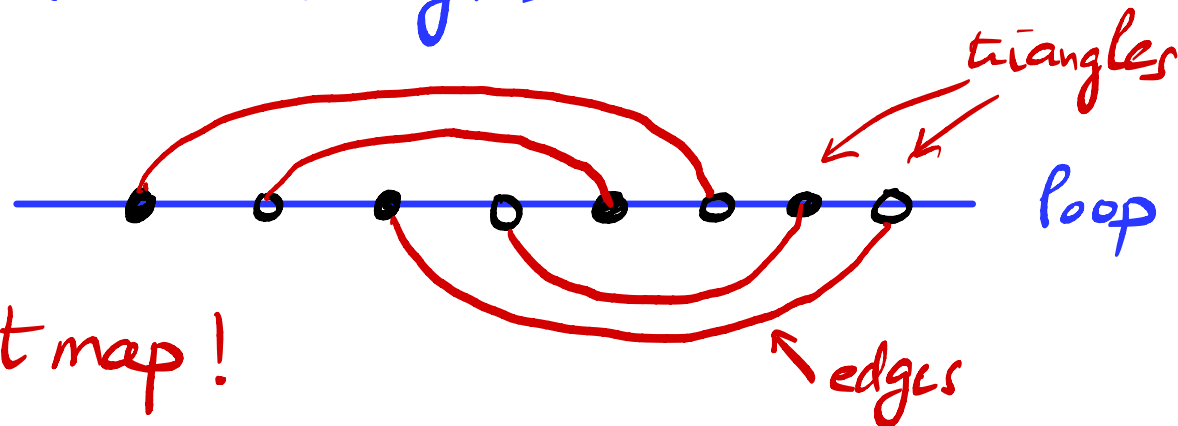
- Eulerian = face-bicolorable
- Hamiltonian cycle = single loop that visits all triangles



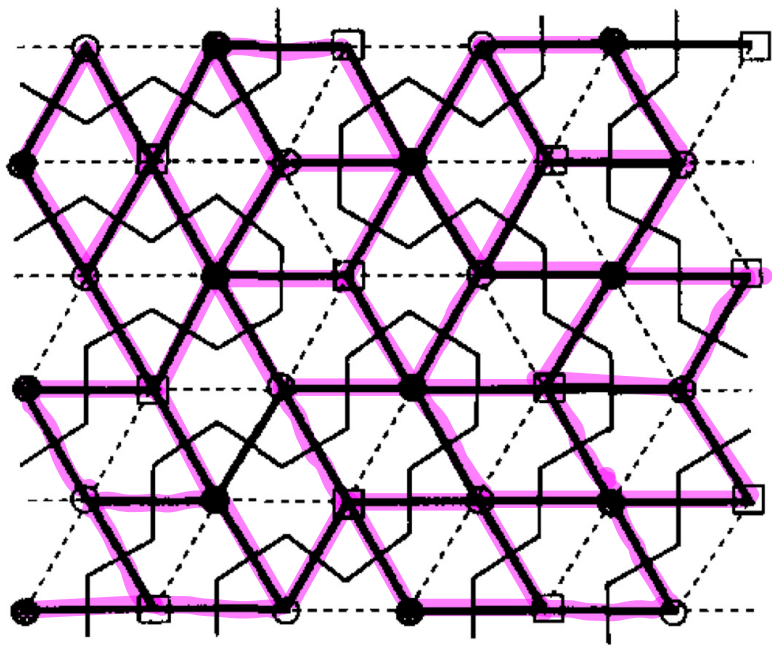
($\Rightarrow$) even-valent

Dually:

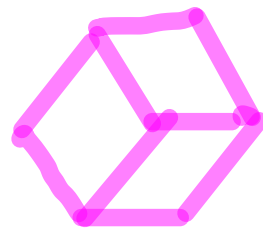
random trivalent map!



FULLY-PACKED LOOPS ON TRIANGULAR LATTICE



Fully-Packed
Loops



Dual
Rhombus Tiling

⇒ extra height (1 more scalar field!)

Conformal theory description :

$$C_{FPL} = C_{\text{Dense}} + 1$$

[cf Blöte Nienhuis]

dense loop
model

height
degree of freedom

$$C = -2 + 1 = -1$$

PREDICTION FROM FIELD THEORY

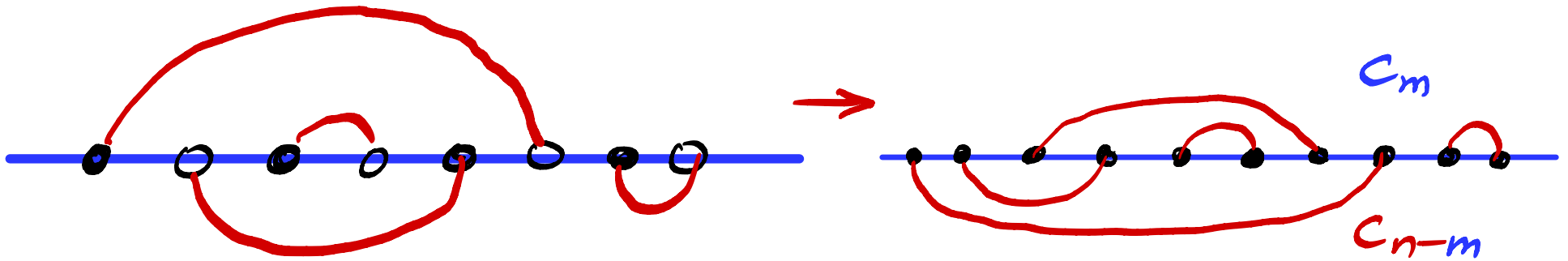
Hcycle on
Eulerian Triang.
w/ $2N$ triangles

$$H_{2N} \sim 1 \frac{B^N}{N^\delta}$$

$$\delta = \frac{13 + \sqrt{13}}{12}$$

$$(KPZ(c=-1))$$
$$\delta = 2 - \gamma$$

SOME FUN COMBINATORICS



NOTE: if we relax vertex bicolouration we kill the extra degree of freedom $\Rightarrow C = C_{\text{Dense}} = -2$

$$\text{KPZ}(c=-2) \Rightarrow \gamma = -1$$

$$\mathbb{Z}_{2n} = \sum_{m=0}^n \binom{2n}{2m} C_m C_{n-m} = C_n C_{n+1} \sim \frac{1}{\pi} \frac{4^{2n}}{n^3} \quad \leftarrow \delta = 2 - \gamma$$

Catalan numbers

Explicit confirmation of KPZ!

MEANDERS: THE END



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Nuclear Physics B 570 [FS] (2000) 699–712



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Meanders: exact asymptotics

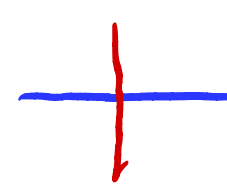
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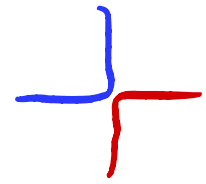
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2-loop FPL ($n_1 = n_2 = 0$) on
a random quadrangulation

2 types of
vertices



crossing



tangent

Dense

Dense

extra



$$c = (-2) + (-2) + 1 = -3$$

dual height $\in \mathbb{R}^3 = \langle A, B, C \rangle$

• Eulerian?
(vertex-bicolored)

if yes

• But not: $c = -4$ (no extra d.f.) but still 2 types of vertices

Field description:

$$\mathbb{R}^2 = \langle A, B \rangle$$

dual height $\in \mathbb{R}^2$ and tangent vertex irrelevant!

$$\begin{array}{c|c} h+A & h+A-B \\ \hline h & h-B \end{array}$$

$$\begin{array}{c|c} h+A & h \\ \hline h & \sqrt{h-B} \end{array}$$

Conclusion: $c = -4 \xrightarrow{\text{KPZ}} \gamma = \frac{5 - \sqrt{145}}{12}$.

$$M_n \sim e^{\mu n} / n^\alpha ; \quad \alpha = \frac{29 + \sqrt{145}}{12}$$

meander exponent



Nuclear Physics B 580 [FS] (2000) 757–795



Exact meander asymptotics: a numerical check

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• numerical evidence

(third digit $\alpha = 3.420$ [7])

(also I. Jensen, T. Guttman)

• more exponents!

associated w/ multisources/sinks

CONCLUSION

- 1D real Folding is complicated

exact enumeration

Templey-Lieb algebra = loop reps for $6V/O(n)$

model $\rightarrow$ integrable R-matrix = α  + β 

we only scratched the surface!

- This problem is a 2D problem in hiding
use of 2D quantum gravity $\equiv$ random surfaces
- Simpler surfaces: "pillowcases" $\rightarrow$ local properties of
meanders, related to Masur-Veech volumes
[Kontsevich, Zorich, Zograf, Delerrière, Gonjard]

- What about Ex3 of Emile Lemoine?

2D Beyond Phantom folding? To be continued!

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