

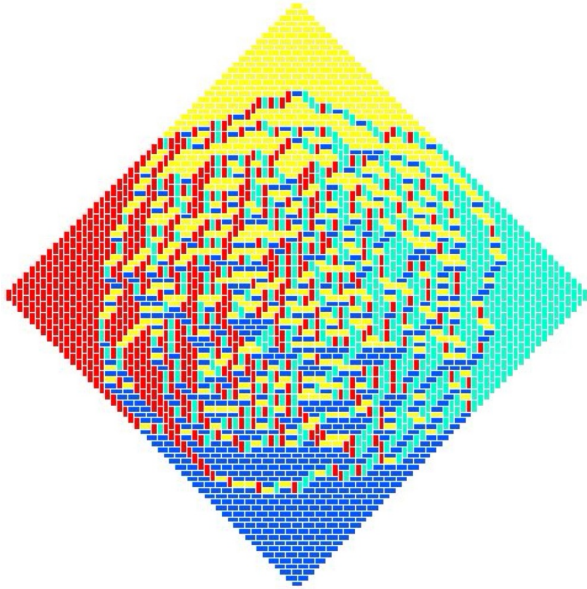
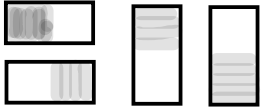
LIMIT SHAPES

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(+B. Debin, E. Guiller, M. Lapa)

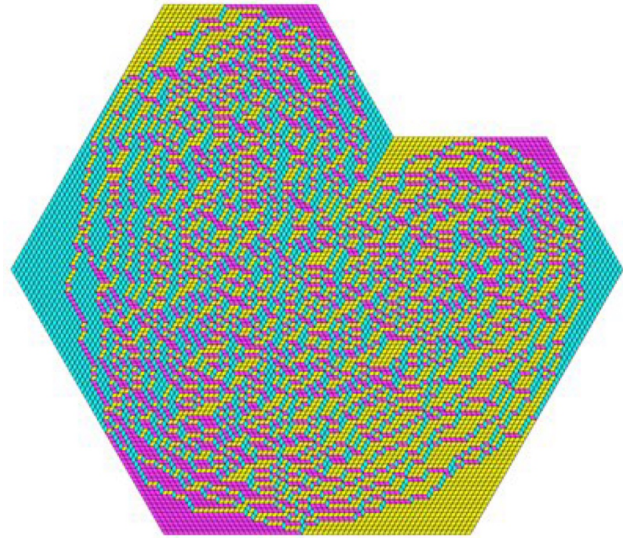
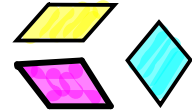
- Path models
- The tangent method
- Application 1: Limit shape of Rhombus Tilings
- Application 2: Limit shape of 6V model w/ DWBC
- Application 3: Limit shapes of 20V model
- The shear phenomenon: a simple example

LIMIT SHAPES

OBSERVATION: TILINGS

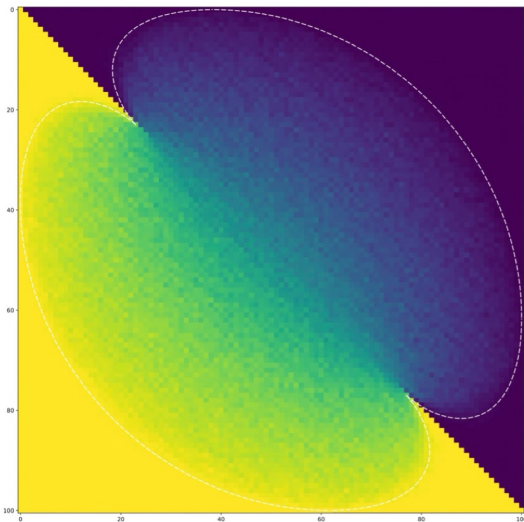
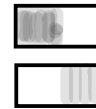
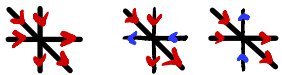


DOMINO TILING OF THE
AZTEC DIAMOND

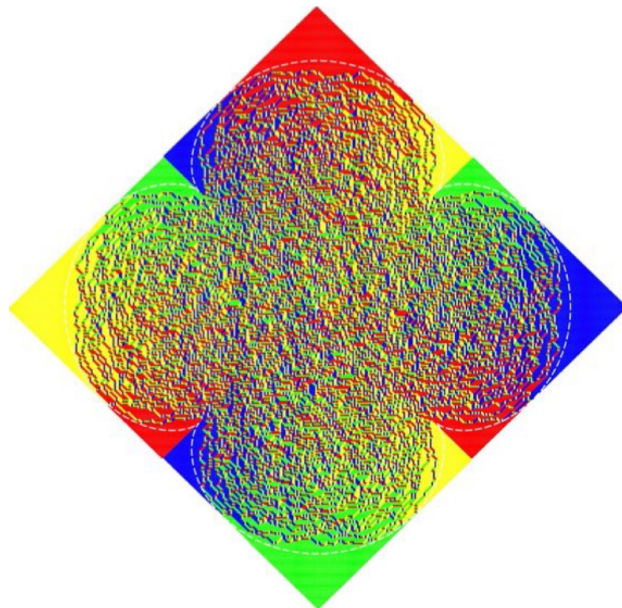


RHOMBUS TILING OF A CUT
HEXAGON

OBSERVATION : 20V and HASDT



20V w/ Domain Wall
Boundaries 2

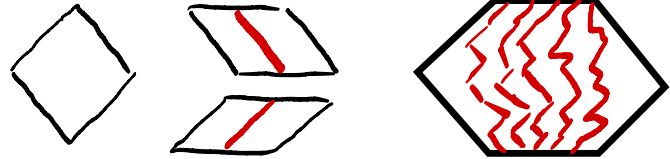


Holey Aztec Square Domino Tiling

ARCTIC CURVES : THE TANGENT METHOD

- STEP 1 : translate into path model

EX : Rhombus Tiling

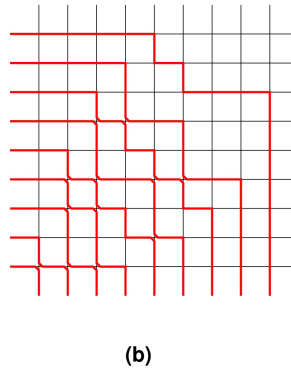
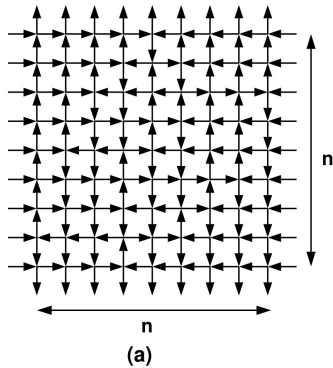
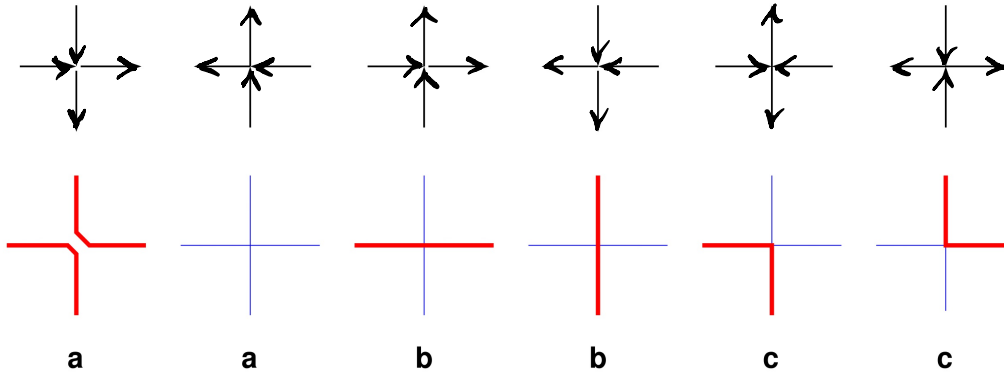


→ Non-Intersecting lattice paths

Works w/all Tiling models

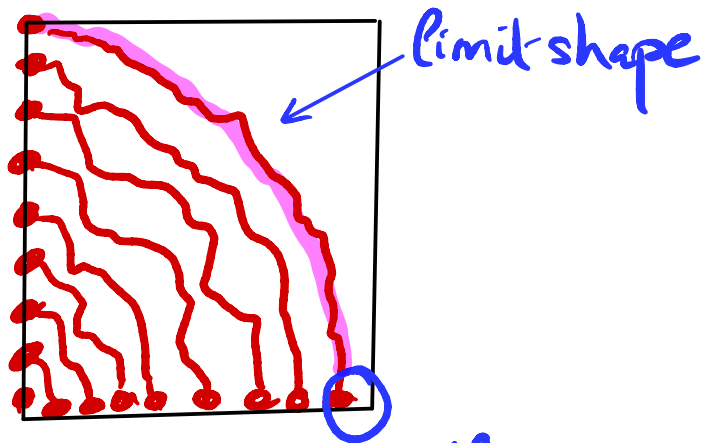
Works also with our vertex models

6V as a model of osculating paths



The Tangent Method

[Colomo-Sportiello '16]

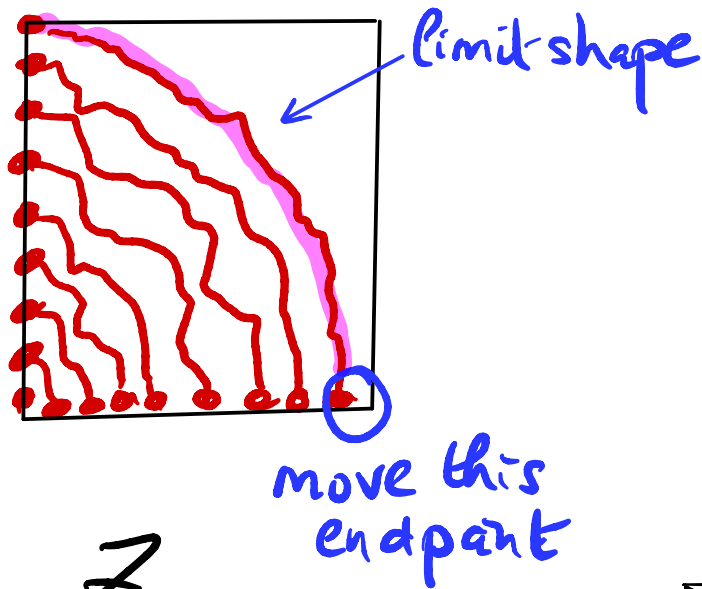


move this
endpoint

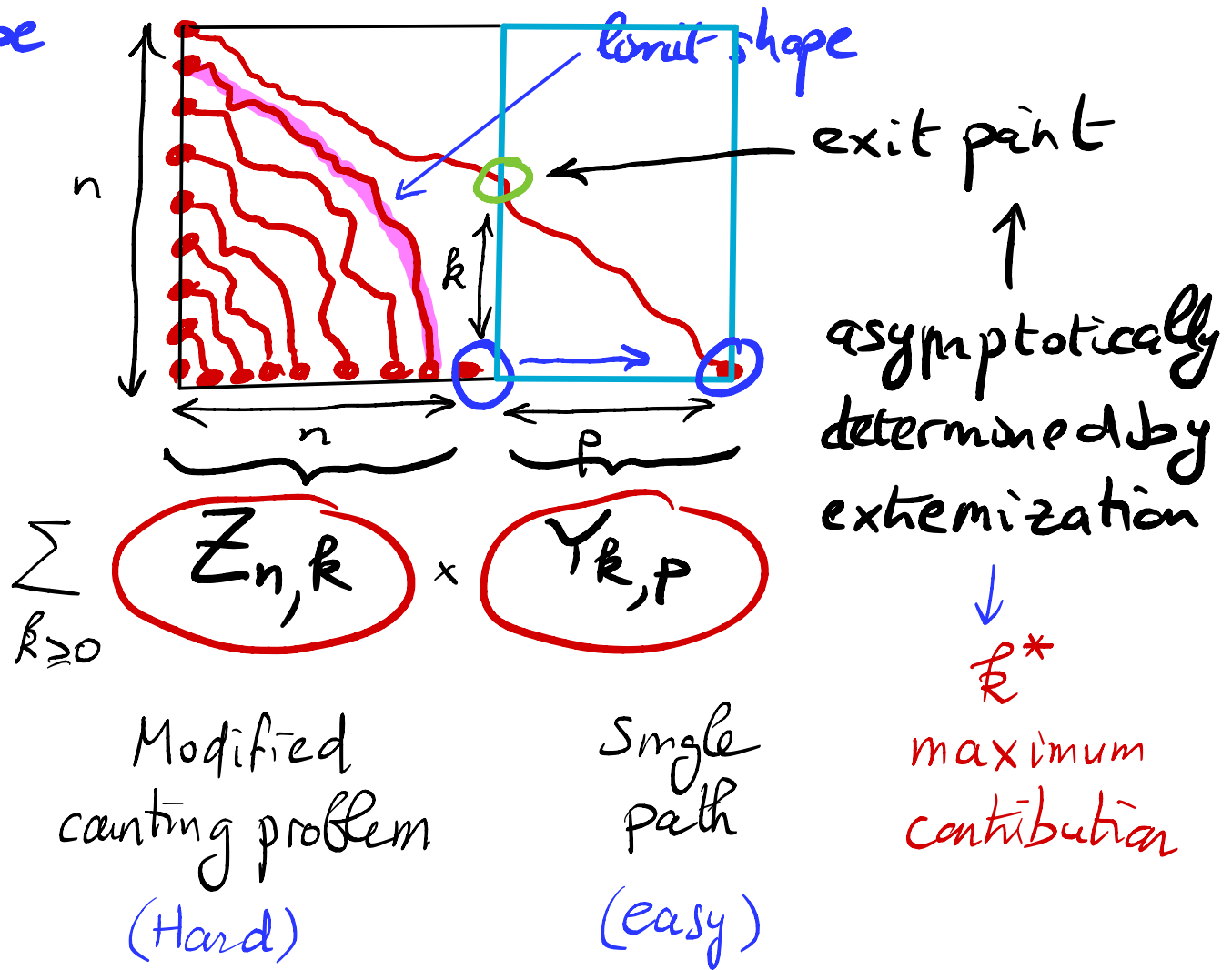
Z_n

The Tangent Method

[Colomo-Sportiello '16]

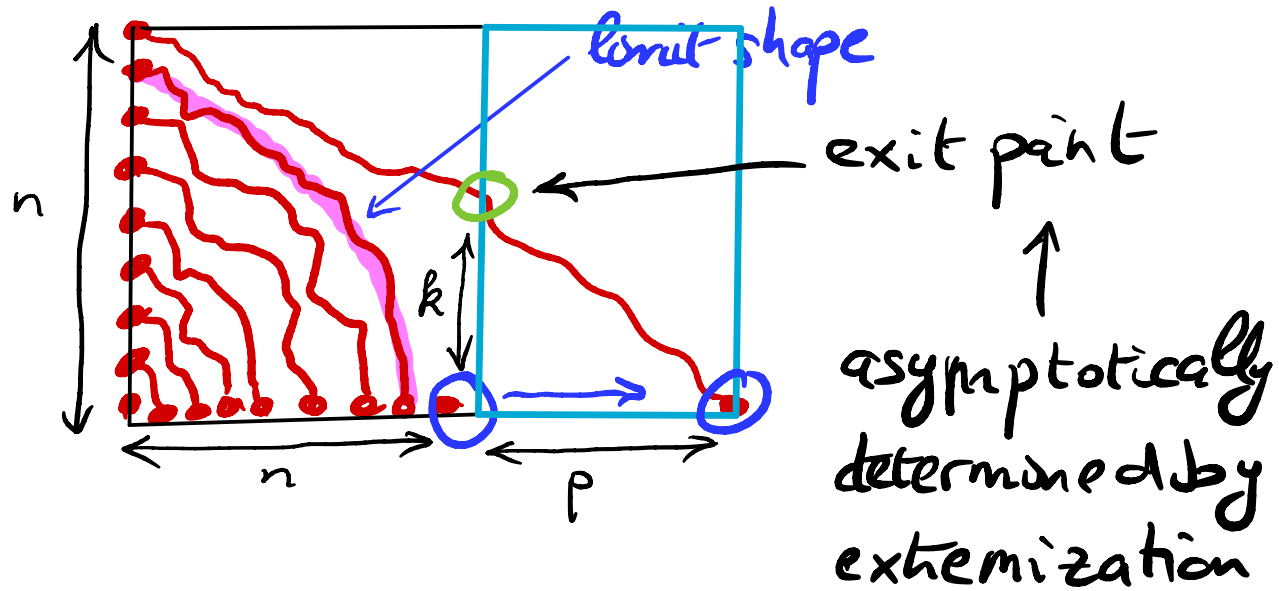
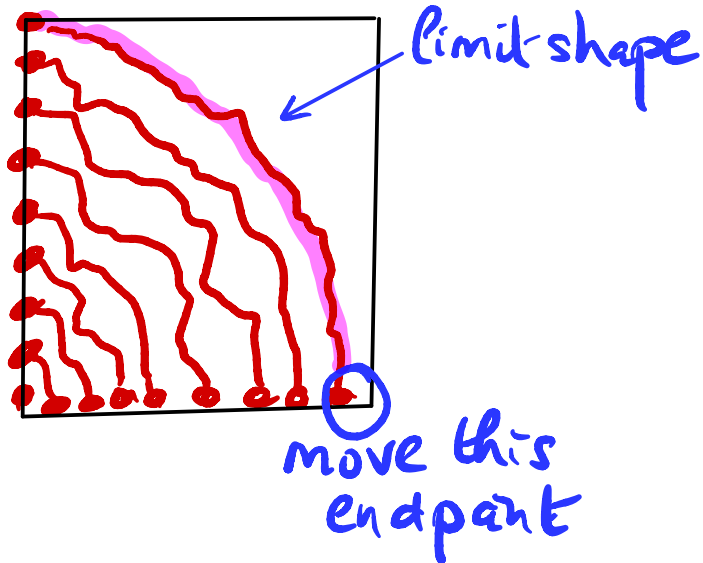


Z_n

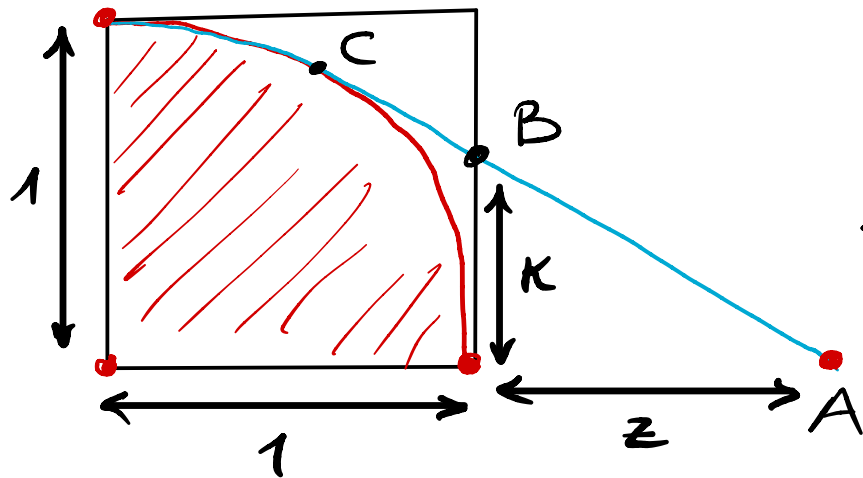


The Tangent Method

[Colomo-Sportiello '16]



Scaling Limit: $n \rightarrow \infty$ $\frac{p}{n} \rightarrow z$



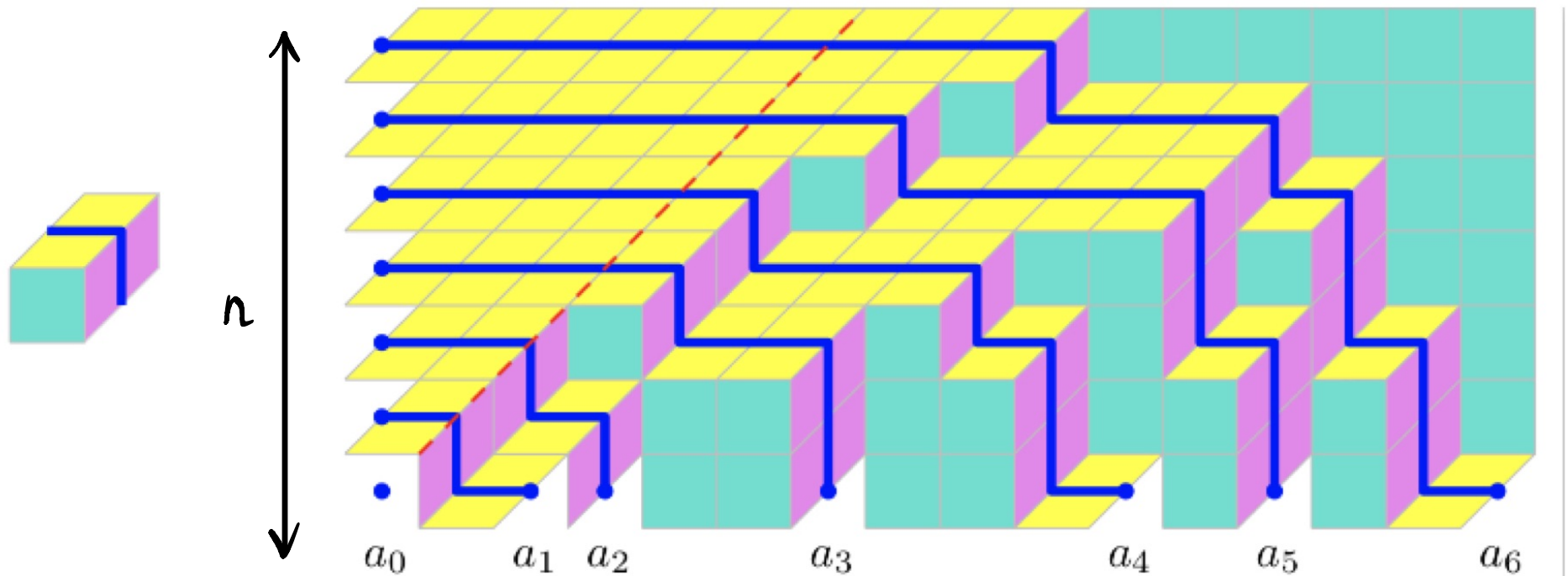
A fixed $\Rightarrow$ B ; line AB = tangent

The arctic curve is the envelope of the family of lines (AB).

APPLICATION I

(a free fermion case)

A FREE FERMION CASE : RHOMBUS TILINGS OF A RECTANGLE



→ a_i fixed but chosen arbitrarily

COUNTING:

Gessel-Viennot $\Rightarrow$

$$\text{THM} \quad Z_{a_0 a_1 \dots a_n} = \frac{\Delta(a_0, a_1, \dots, a_n)}{\Delta(0, 1, \dots, n)}$$

$$\Delta(x_1, \dots, x_m) = \prod_{i < j} (x_j - x_i)$$

Vandermonde

Proof: LU decomposition:

$$M_{\vec{j}} = \begin{array}{|c|} \hline i \\ \hline \end{array} \begin{array}{|c|} \hline \text{Step Function} \\ \hline \end{array} = \begin{pmatrix} a_{\vec{j}}^i + \bar{i} \\ \bar{i} \end{pmatrix} = (L \cdot U)_{\vec{j}}^i$$

$$L_{ij}^{-1} = \begin{cases} \frac{\prod_{k=0}^{i-1} a_i - a_k}{\prod_{\substack{k=0 \\ k \neq j}}^L a_i - a_k} & i=j \\ 0 & i \neq j \end{cases}$$

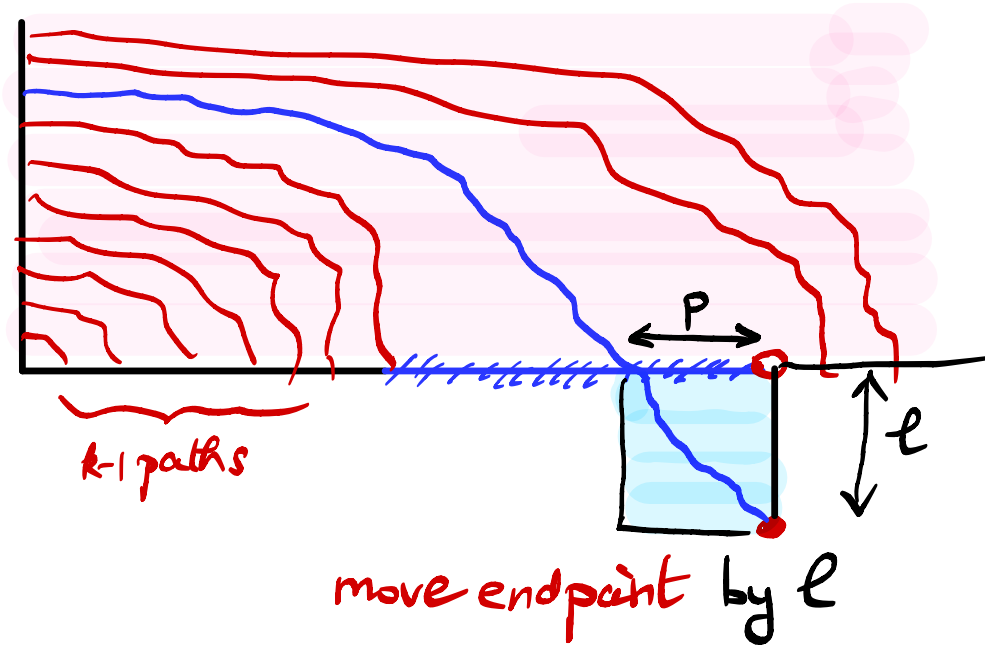
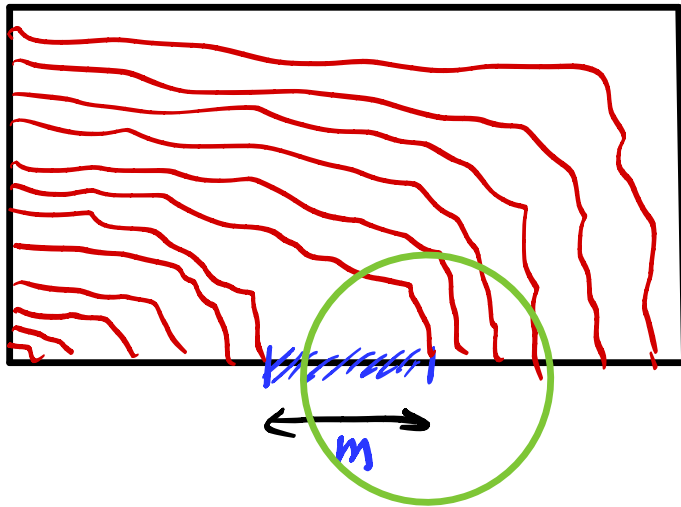
$$i \leq j$$

otherwise

$\Rightarrow U = L^{-1}M$ is upper

with
$$U_{i,i} = \frac{i-1}{i!} (a_i - a_k)$$

TANGENT METHOD : case of a gap



$$H_{N,P} = \frac{Z_{a_0, \dots, a_{k-1}, a_{k-P}, a_{k+1}, \dots, a_n}}{Z_{a_0, \dots, a_{k-1}, a_k, a_{k+1}, \dots, a_n}}$$

(1 - point function)

$$Y_{e,P} = \binom{P+l}{l}$$

(single path)

ASYMPTOTICS : $N \rightarrow \infty$

$$\begin{cases} p = \omega N \\ \ell = \lambda N \\ k = \kappa N \end{cases}$$

$$a_i \rightarrow N \alpha(\bar{c}_N) \quad \text{distribution of} \quad \underline{\text{end points}} \quad \alpha(u)$$

$$H_{N, \omega N} = e^{\sum_{i \neq k} \log \left(\frac{a_i - a_k + \omega N}{a_i - a_k} \right)} \sim e^{N \int_0^1 \log \left(\frac{\alpha(u) - \alpha(\kappa) + \omega}{\alpha(u) - \alpha(\kappa)} \right)}$$

$$Y_{\lambda N, \omega N} \sim e^{N [(\lambda + \omega) \log(\lambda + \omega) - \lambda \log \lambda - \omega \log \omega]}$$

Saddle point $\Rightarrow$ family of tangent lines

RESULT

THM

$$x(t) = e^{-\int_0^t \frac{du}{t - \alpha(u)}}$$

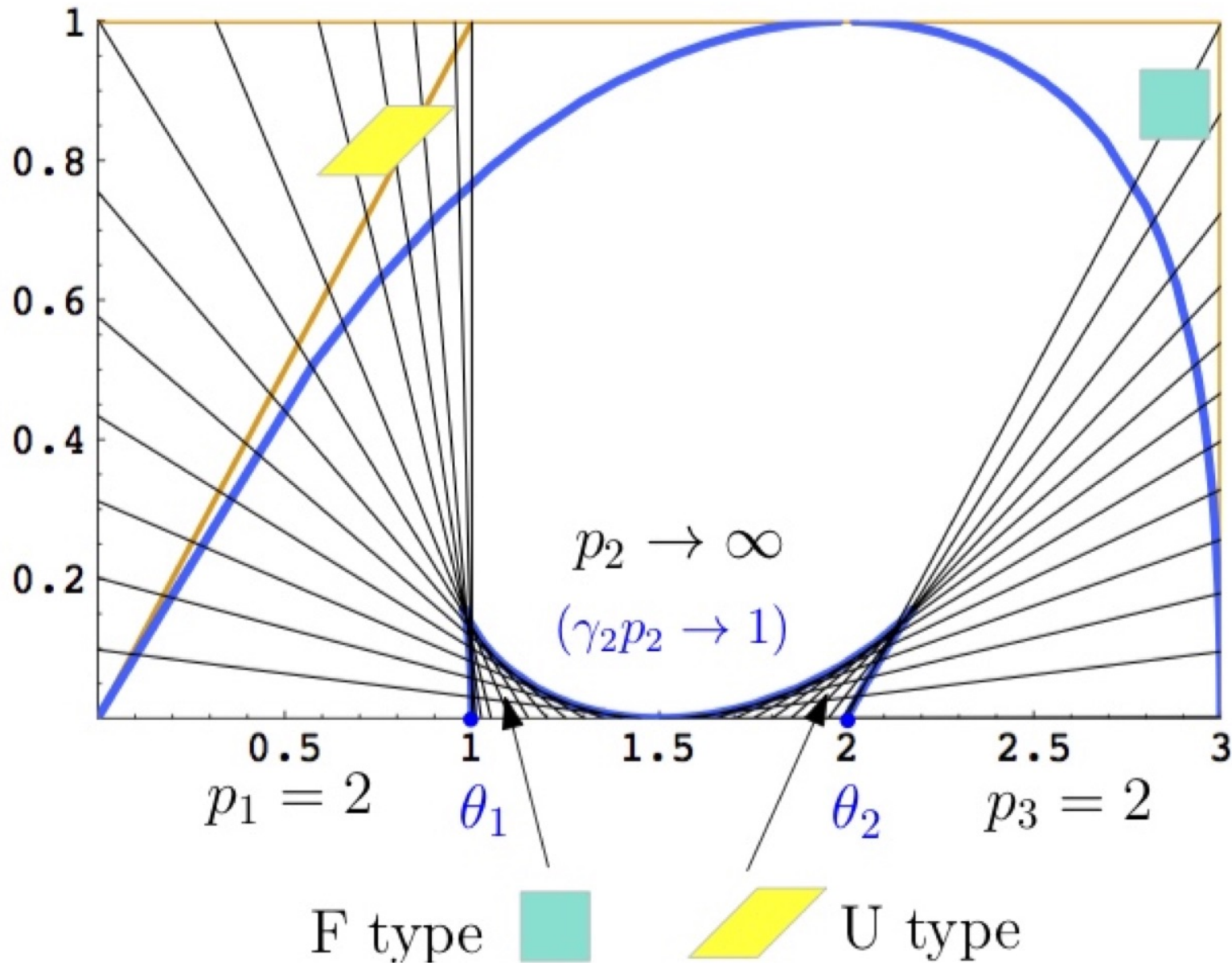
(moment-generating function
for $\alpha(u)$)

$$\begin{cases} X(t) = t - \frac{x(t)[1-x(t)]}{x'(t)} \\ Y(t) = \frac{[1-x(t)]^2}{x'(t)} \end{cases}$$

$t \in \mathbb{R}$

one-gap example:

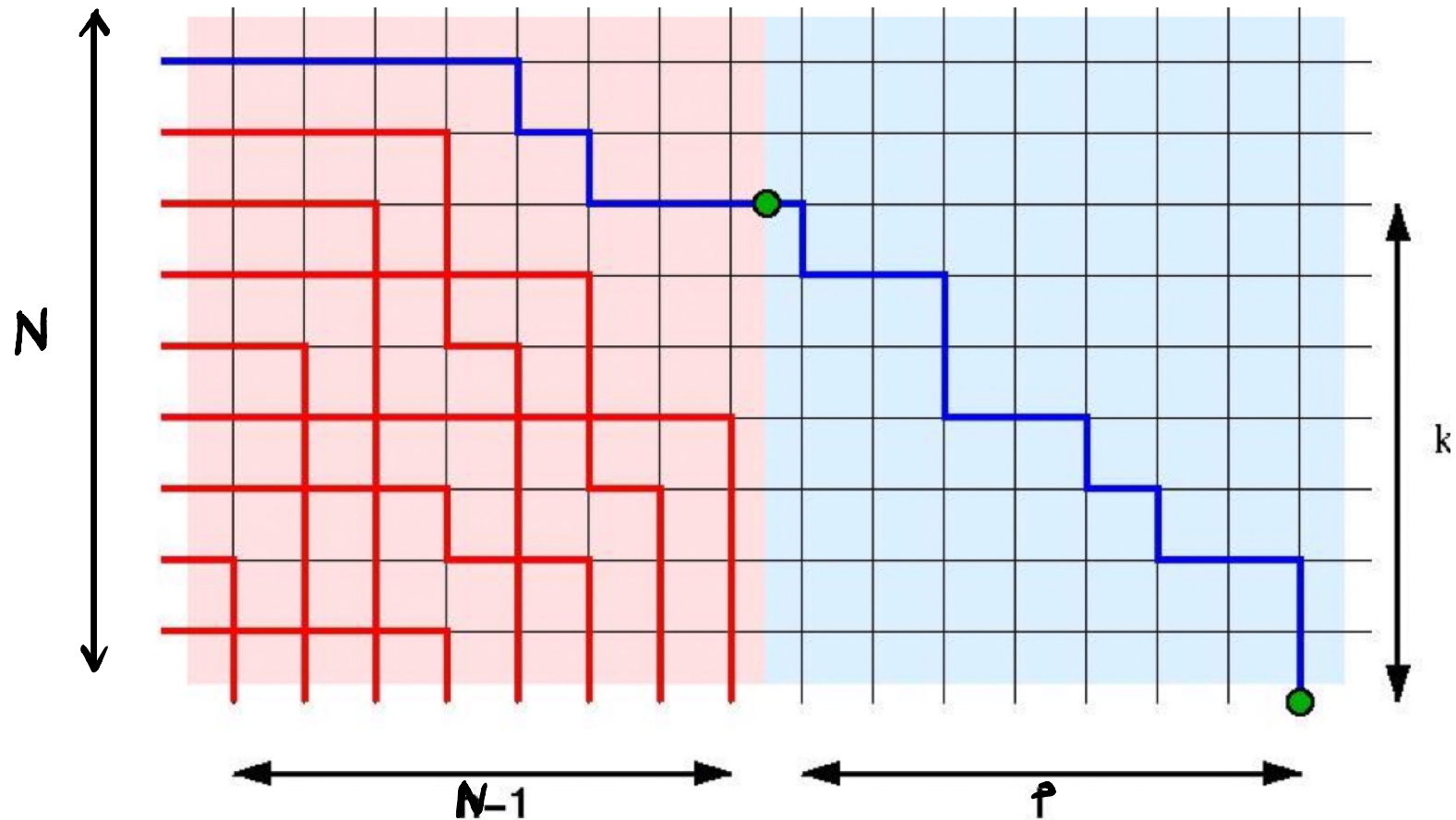
$$x(t) = \left(\frac{t-1}{t} \frac{3-t}{2-t} \right)^{\frac{1}{2}}, t \in \mathbb{R}.$$



APPLICATION II

6 V model with DWBC

Tangent Method for the 6V-DWBC model



$$H_{N,k} = \frac{Z_{N,k}}{Z_N}$$

1pt function

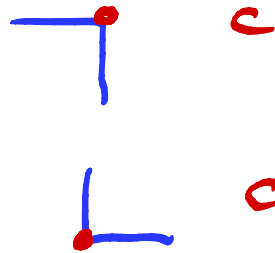
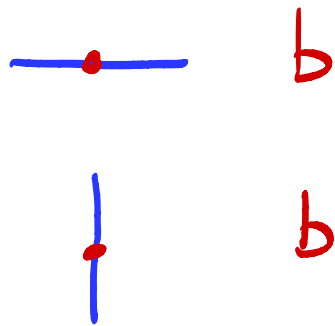


$$\Upsilon_{k,p}$$

1 path partition function

LARGE N ESTIMATES

- $Y_{k,p}$ = (relatively) easy, path with weights



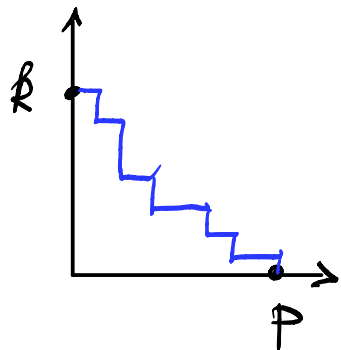
empty vertices

• a

Transfer matrix:

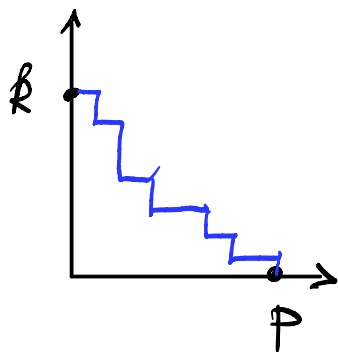
$$T = \begin{matrix} \text{---} \bullet & \text{---} \bullet \\ \bullet \text{---} & \bullet \text{---} \end{matrix} \begin{pmatrix} bu & cv \\ cu & bv \end{pmatrix}$$

$u/\text{step} \rightarrow$
 $v/\text{step} \downarrow$



k steps $\downarrow$
 p steps $\rightarrow$
tot = $k+p$ steps

$$Y_{k,p} = (1 \ 0) T^{k+p} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big|_{u^p v^{k+1}}$$



k steps $\downarrow$
 p steps $\rightarrow$
 tot = $k+p$ steps

$$Y_{k,p} = (1 \ 0) T^{k+p+1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big|_{u^p v^{k+1}}$$

$$= (1 \ 0) \sum_{m=0}^{\infty} T^m \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big|_{u^p v^{k+1}}$$

$$y(u,v) = (1 \ 0) (I - T)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sum_{k,p \geq 0} Y_{k,p} u^p v^{k+1}$$

$$= \frac{c v}{1 - bu - bv \left(1 + \frac{c^2 - b^2}{b} u\right)} = \frac{c v}{1 - bu} \sum_{k \geq 0} v^k \left(\frac{1 + \frac{c^2 - b^2}{b} u}{1 - bu} \right)^k b^k$$

$$Y_{k,p} = c b^k \frac{\left(1 + \frac{c^2 - b^2}{b} u\right)^k}{(1 - bu)^{k+1}} \Big|_{u^p} = c \sum_{\substack{r \geq 0 \\ 0 < s \leq k \\ r+s=p}} \binom{r+k}{k} \binom{k}{s} \left(\frac{c^2 - b^2}{b}\right)^{k+r} b^s$$

$$Y_{k,p} = c \sum_{\substack{r \geq 0 \\ 0 \leq s \leq k \\ r+s=p}} \binom{r+k}{k} \binom{k}{s} \left(\frac{c^2 - b^2}{b} \right)^{k+r} b^s$$

large $\begin{cases} k = K N \\ p = \lambda N \end{cases} \quad N \rightarrow \infty \quad \begin{cases} r \sim \xi N \\ s \sim \sigma N \\ \xi = \lambda - \sigma \end{cases} \quad \sum_s \rightarrow \int d\sigma$

$$Y_{K N, \lambda N} \sim \int_0^K d\sigma e^{-N S_{\text{path}}(K, \lambda, \sigma)}$$

$$S_{\text{path}}(K, \lambda, \sigma) = \sigma \log \left[\frac{\sigma b}{c^2 - b^2} \right] + (K - \sigma) \log(K - \sigma) + (\lambda - \sigma) \log(\lambda - \sigma) - (K + \lambda - \sigma) \log[b(K + \lambda - \sigma)]$$

PARTITION FUNCTION

Z_N = homogeneous limit of the Izergin-Korepin det.

$$(z_j \rightarrow e^{iu}, w_j \rightarrow e^{iv}, q = e^{\varepsilon\eta}, j=1, 2, \dots, N)$$

$$Z_{N(u-v)} = \alpha \det_{0 \leq i, j \leq N-1} \left(\left(\frac{d}{du} \right)^{i+j} w(u-v) \right) ; w(u) = \frac{\sin(2\eta)}{\sin(u+\eta)\sin(u-\eta)}$$

↪ "Wronskian type"

$$\begin{cases} c_{j+1} = \frac{d}{du} c_j \\ \bar{c}_{j+1} = \frac{d}{du} \bar{c}_j \end{cases}$$

Desnanot-Jacobi (Plücker) relations:

The diagram illustrates the Desnanot-Jacobi (Plücker) relation for determinants. It shows a sequence of square matrices with red diagonal lines. The first matrix is labeled with dimensions $N+1$ and $N-1$. The second matrix is labeled with dimensions $N-1$ and N . The third matrix is labeled with dimensions N and $N-1$. The fourth matrix is labeled with dimensions $N-1$ and N . The fifth matrix is labeled with dimensions N and $N-1$. The sixth matrix is labeled with dimensions $N-1$ and N . The relation is shown as a sequence of matrices connected by equals and minus signs.

IZERGIN-KOREPIN Determinant

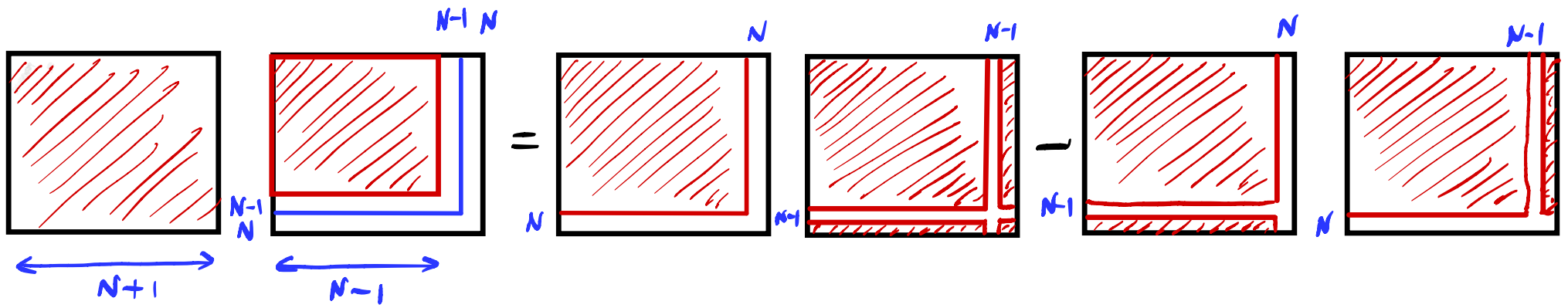
$$Z_{6V} \left[\begin{array}{c} w_1, w_2, \dots, w_N \\ \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \\ z_1 \\ \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \\ z_2 \\ \vdots \\ \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \\ z_n \end{array} \end{array} \right] = \frac{\prod_{i,j=1}^N a(z_i, w_j) b(z_i, w_j)}{\prod_{1 \leq i < j \leq N} (z_i - z_j)(w_i - w_j)} \times$$

$$\times \det \left(\frac{c(z_i, w_j)}{a(z_i, w_j) b(z_i, w_j)} \right)_{1 \leq i, j \leq N}$$

uniform limit: $q = e^{i\pi/6}$
 $z_i \rightarrow 1 \quad w_i \rightarrow 1$

$$Z_n = \prod_{i=0}^{n-1} \frac{(\beta + i)!}{(n+i)!} = \text{ASM}_n$$

[Kuperberg 96]



$$Z_{N+1}(u) Z_{N-1}(u) = Z_N(u) \frac{d^2}{du^2} Z_N(u) - \left(\frac{d}{du} Z_N(u) \right)^2$$

reinstatement $\alpha_N = \prod_{i,j=0}^{N-1} \frac{1}{i! j!}$

$$\frac{Z_{N+1}(u) Z_{N-1}(u)}{Z_N(u)^2} = \frac{1}{N^2} \frac{d^2}{du^2} \log(Z_N(u))$$

with $\begin{cases} Z_0(u) = 1 \\ Z_1(u) = w(u) \end{cases}$, this determines $Z_N(u)$ uniquely.

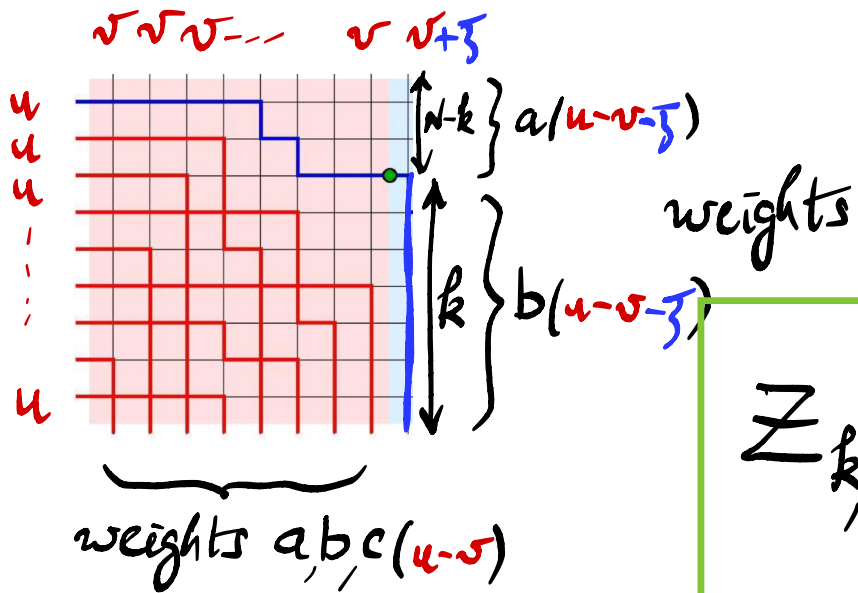
ONE-POINT FUNCTION

keep one inhomogeneity in the limit :

$$z_1, \dots, z_N \rightarrow e^{iu}$$

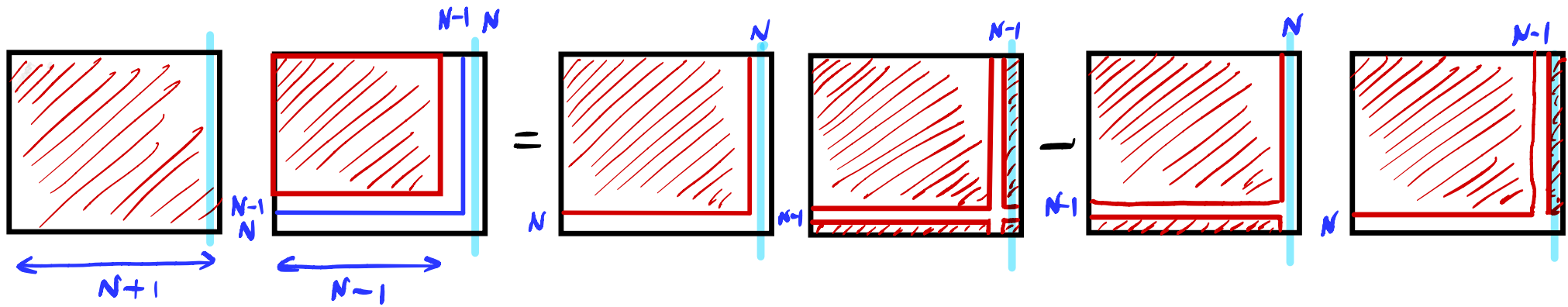
$$w_1, \dots, w_{N-1} \rightarrow e^{iv}$$

$$w_N \rightarrow e^{i(\zeta+v)}$$



$$Z_{k,N} = Z_N(u, \dots, u; v, \dots, v, v+\zeta) \Big|_{b(u-v-\zeta)^k a(u-v-\zeta)^{N-k}}$$

$$H_{k,N} = \frac{Z_{k,N}(u-v)}{Z_N(u-v)}$$



$$Z_{N+1}(u, \zeta) Z_{N-1}(u) = Z_N(u) \frac{d}{du} [Z_N(u, \zeta)] - \frac{d}{du} [Z_N(u)] Z_N(u, \zeta)$$

$$H_N(u, \zeta) := \frac{Z_N(u, \zeta)}{Z_N(u)}$$

$$\frac{Z_{N+1}(u) Z_{N-1}(u)}{Z_N(u)^2} \frac{H_{N+1}(u, \zeta)}{H_N(u, \zeta)} = - \frac{1}{N} \frac{d}{du} [\text{Log } H_N(u, \zeta)]$$

LARGE N ESTIMATES

$$\frac{Z_{N+1}(u) Z_{N-1}(u)}{Z_N(u)^2} = \frac{1}{N^2} \frac{d^2}{du^2} \log(Z_N(u)) \Rightarrow$$

$$Z_N(u) \sim e^{-N^2 f(u)}$$
$$e^{-2f(u)} + \frac{d^2}{du^2} f(u) = 0$$

LIOUVILLE eqn

$$\frac{Z_{N+1}(u) Z_{N-1}(u)}{Z_N(u)^2} \frac{H_{N+1}(u, \zeta)}{H_N(u, \zeta)} = -\frac{1}{N} \frac{d}{du} [\log H_N(u, \zeta)] \Rightarrow H_N(u, \zeta) \sim e^{-N \psi(u, \zeta)}$$

$$\frac{d}{du} \psi(u, \zeta) = e^{-2f(u) - \psi(u, \zeta)}$$

+ Boundary conditions, can be integrated explicitly

Solution: • $f(u, v) = f(u - v)$

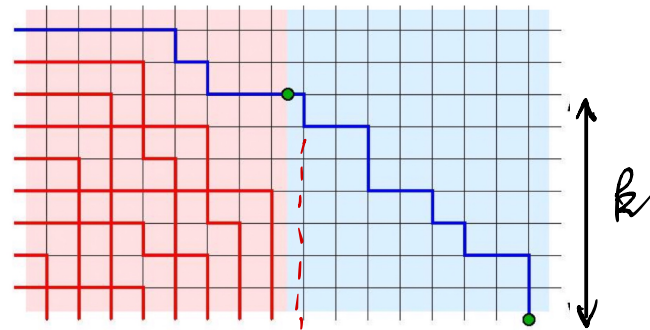
$$f(u) = -\text{Log} \left[\alpha \frac{\sinh(u + \eta) \sinh(u - \eta)}{\sinh(\alpha(u - \eta))} \right]$$

where $\alpha = \frac{\pi}{\pi - 2\eta}$, $q = e^{i\eta}$.

• $\psi(u, v; \zeta) = \psi(u - v; \zeta)$

$$\psi(u, \zeta) = \text{Log} \left[\frac{\alpha \sinh(\alpha(u - \zeta - \eta))}{\sinh(\alpha \zeta) \sinh(\alpha(u - \eta))} \right] + f(u)$$

Back to the tangent method:



$$\bullet W_{N,p} = \sum_k \frac{Z_{N,k}}{Z_N} \Upsilon_{k,p}$$

$\nwarrow$ $k \sim k^*$ dominant contribution?

$$\bullet \frac{Z_{N,k}}{Z_N} = \oint \frac{d\tau}{2i\pi\tau} \frac{1}{\tau^k} H_{N,k}(\tau) \quad \text{for } \tau = \tau(\zeta) = \frac{b(\zeta)}{a(\zeta)} \frac{a(0)}{b(0)}$$

$k = k_N$

$e^{-Nk \log(\tau(\zeta))} \rightsquigarrow e^{-N\psi(\zeta)}$

$$\bullet \Upsilon_{k,p} = \text{a refined version of } \binom{k+p}{p} \rightsquigarrow \text{Stirling asymptotics}$$

$k = k_N$
 $p = \lambda N$

$\sim \int_0^k d\sigma e^{-N S_{\text{path}}(k, \sigma)}$

Result (6V-DWBC)

- Family of tangent lines

$$y + A(\zeta)x - B(\zeta) = 0$$

- Arctic curve (NE portion)

$$x = \frac{B'(\zeta)}{A'(\zeta)} ; \quad y = B(\zeta) - A(\zeta) \frac{B'(\zeta)}{A'(\zeta)}$$

$$\zeta \in [u + \eta - \pi, 0]$$

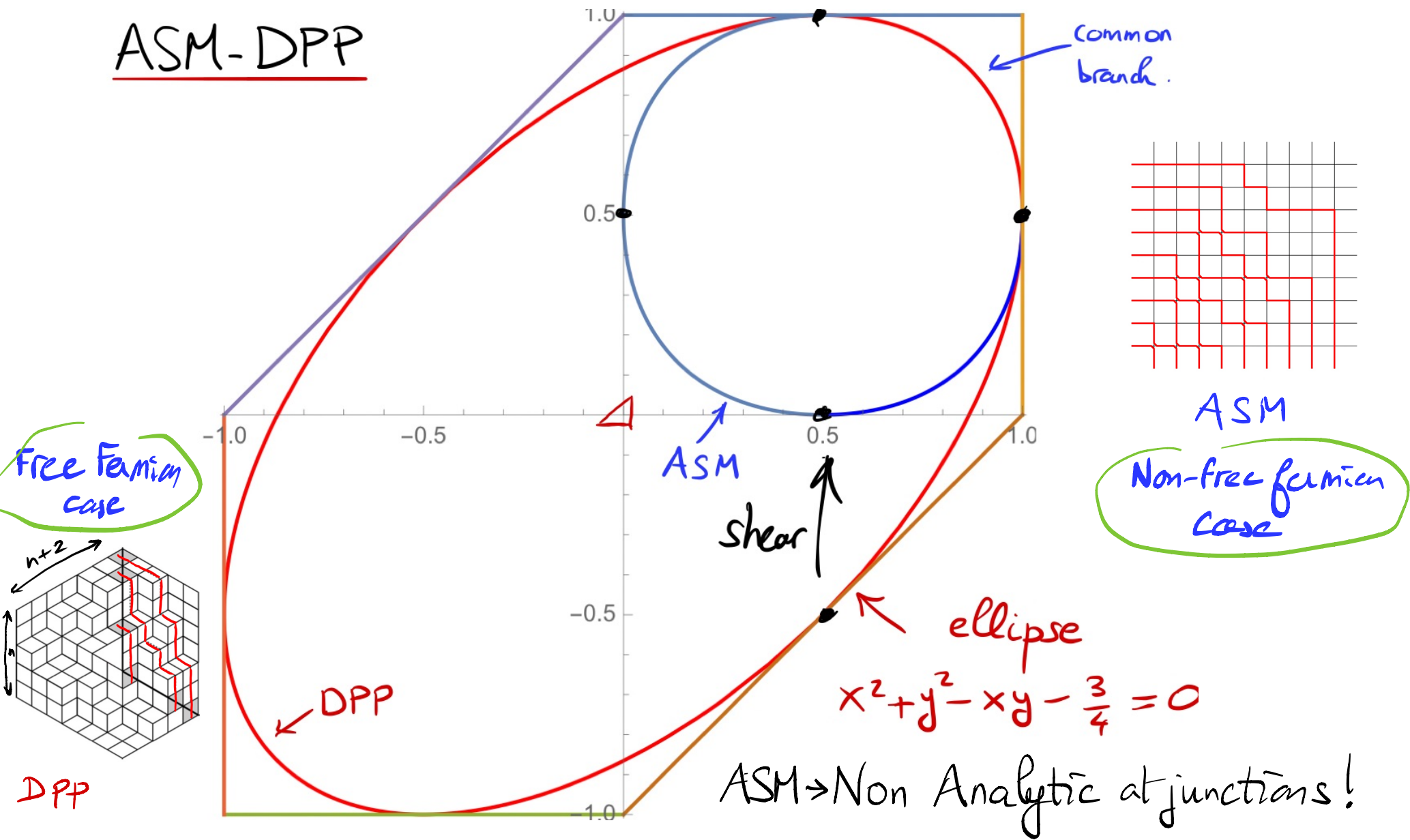
$$A[\xi] = \frac{\sin(u - \xi + \eta) \sin(u - \xi - \eta)}{\sin(\xi) \sin(\xi - 2\eta)}$$

$$B[\xi] = \{ \cot(u - \xi - \eta) + \cot(\xi) - \alpha \cot(\alpha\xi) - \alpha \cot(\alpha(u - \xi - \eta)) \} \\ \times \frac{\sin(u - \xi + \eta) \sin(u - \xi - \eta)}{\sin(2\eta)}$$

Uniform case: $(\eta = \frac{\pi}{6}, u-v = \frac{\pi}{2})$

the full curve is the union of 4 arcs of ellipse.

ASM-DPP

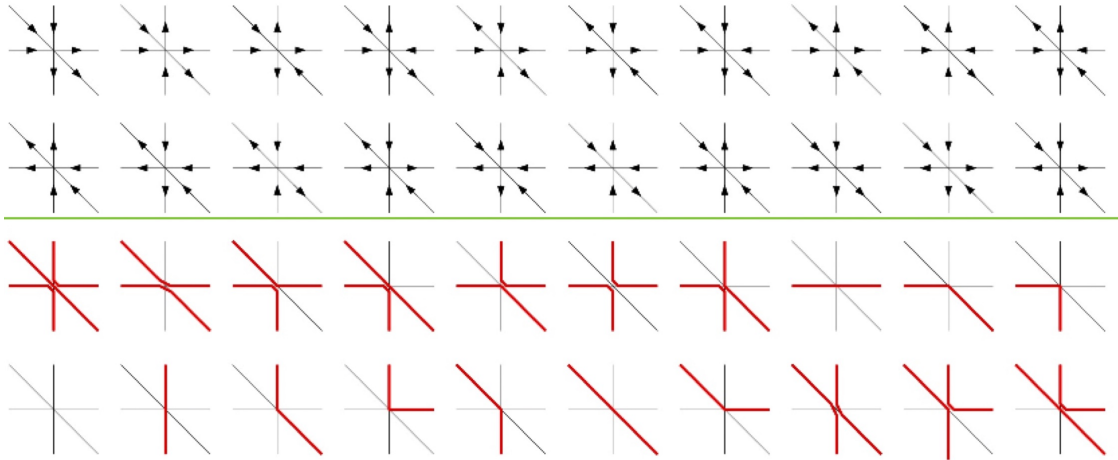
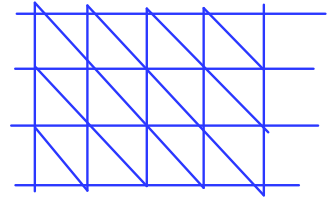


APPLICATION III

The 20 V model with DW/BC

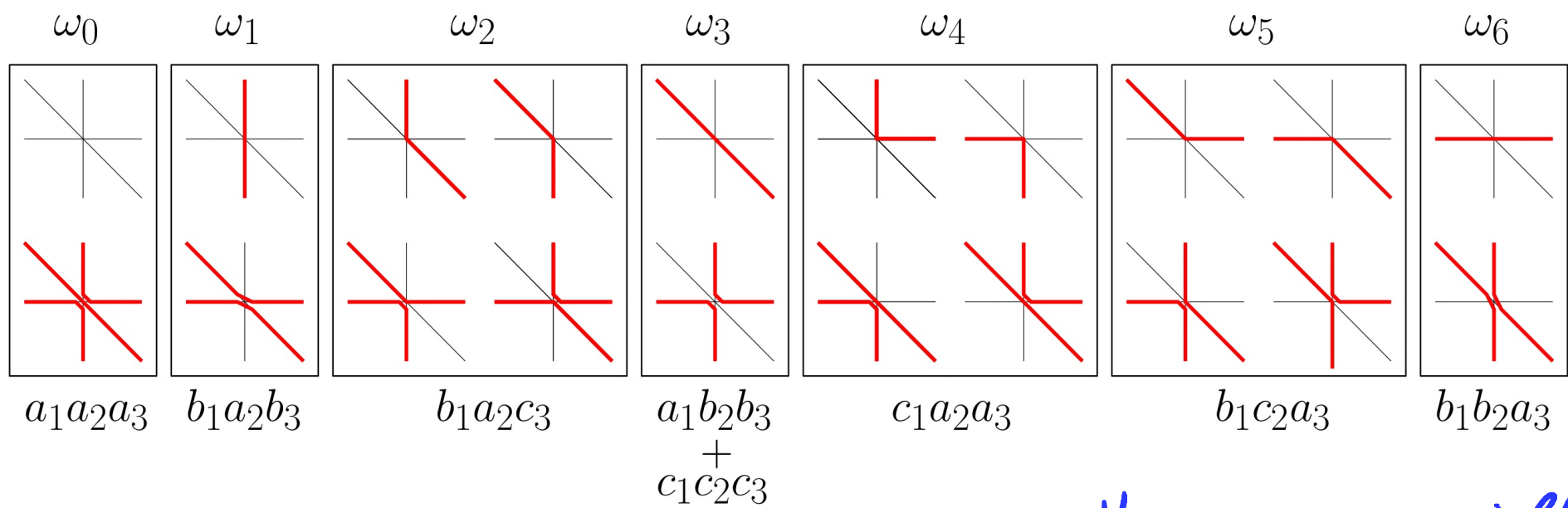
2D Vertex model

- Same idea but triangular lattice
- 20 local environments



Paths

osculating Schröder paths (h, v, d steps).



$$\omega_0 = \sin(\lambda + \eta) \sin\left(\frac{\lambda + 3\eta + \mu}{2}\right) \sin\left(\frac{\lambda + 3\eta - \mu}{2}\right)$$

$$\omega_1 = \sin(\lambda - \eta) \sin\left(\frac{\lambda - \eta + \mu}{2}\right) \sin\left(\frac{\lambda + 3\eta - \mu}{2}\right)$$

$$\omega_2 = \sin(2\eta) \sin(\lambda - \eta) \sin\left(\frac{\lambda + 3\eta - \mu}{2}\right)$$

$$\omega_3 = \sin(2\eta)^3 + \sin(\lambda + \eta) \sin\left(\frac{\lambda - \eta + \mu}{2}\right) \sin\left(\frac{\lambda - \eta - \mu}{2}\right)$$

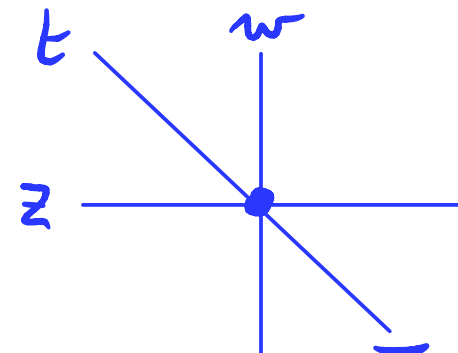
$$\omega_4 = \sin(2\eta) \sin\left(\frac{\lambda + 3\eta + \mu}{2}\right) \sin\left(\frac{\lambda + 3\eta - \mu}{2}\right)$$

$$\omega_5 = \sin(2\eta) \sin(\lambda - \eta) \sin\left(\frac{\lambda + 3\eta + \mu}{2}\right)$$

$$\omega_6 = \sin(\lambda - \eta) \sin\left(\frac{\lambda + 3\eta + \mu}{2}\right) \sin\left(\frac{\lambda - \eta - \mu}{2}\right),$$

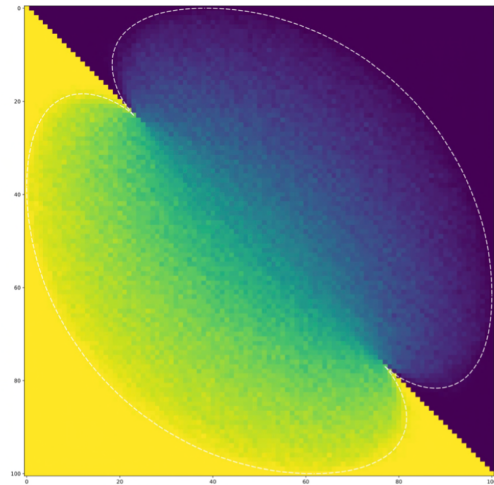
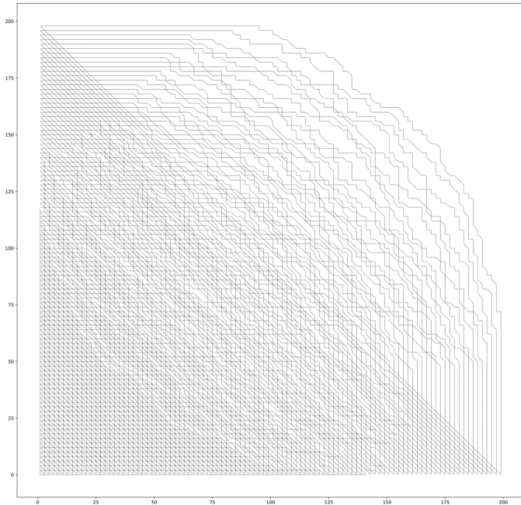
Homogeneous weights:

$$\begin{cases} q = e^{i\eta} \\ z = e^{i(\eta+\lambda)} \\ w = e^{-i(\eta+\lambda)} \\ t = e^{i\mu} \end{cases}$$



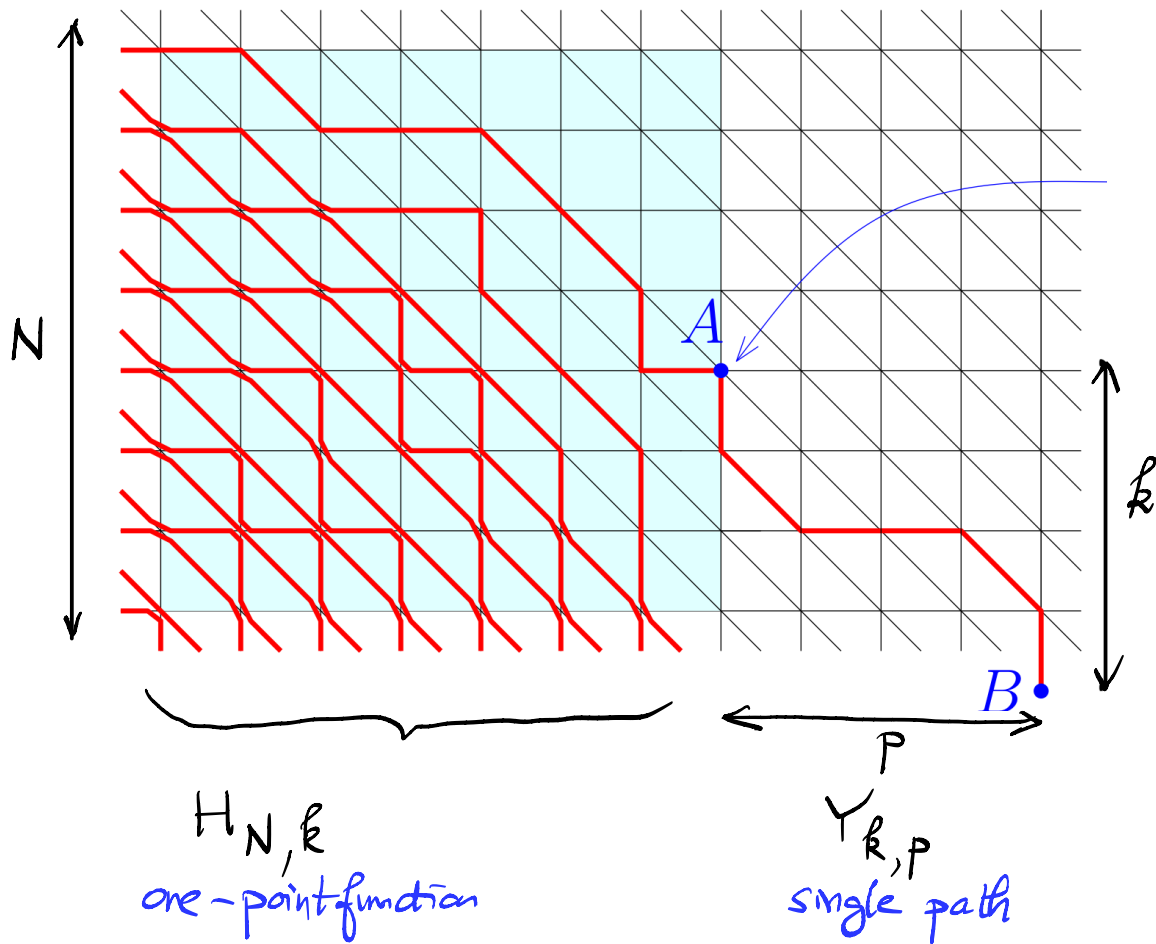
Uniform case $\eta = \frac{\pi}{8}$ $\lambda = \frac{5\pi}{8}$ $\mu = 0$

A FRANCO-BELGIAN COLLABORATION



(Brian Debin ~ 2019) + non-uniform cases
as well, by Glauber dyn.

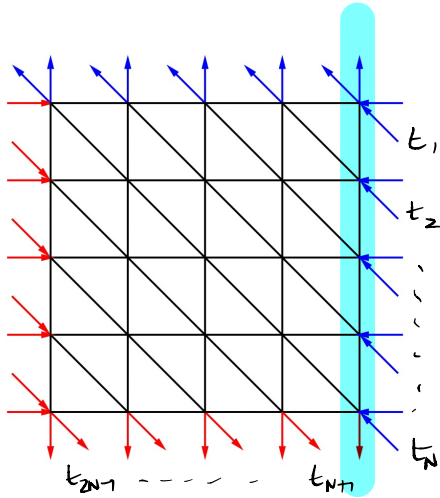
TANGENT METHOD



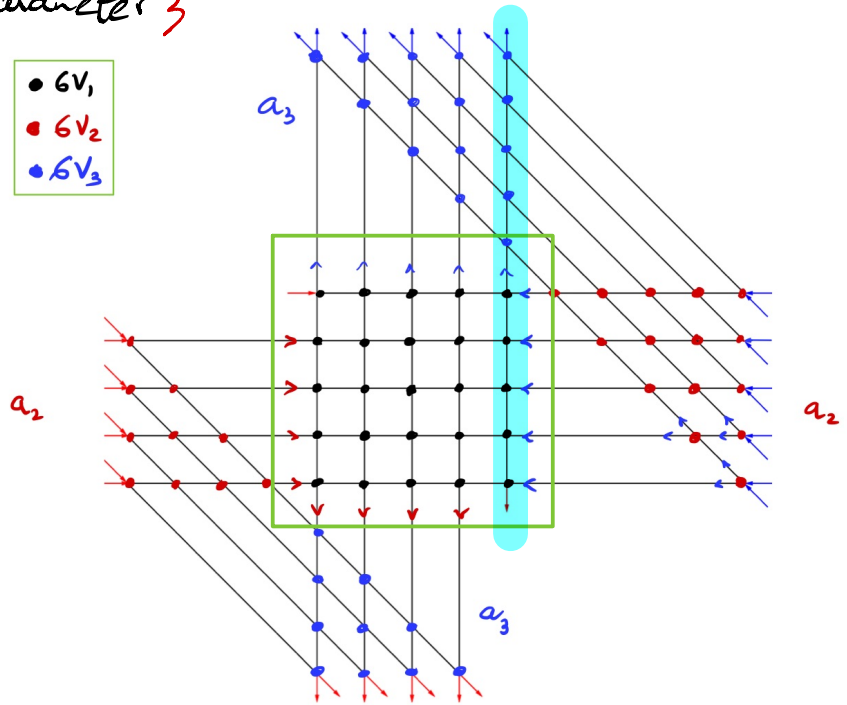
One-Point Function

(Domain-Wall Boundary conditions)

← different spectral parameter λ



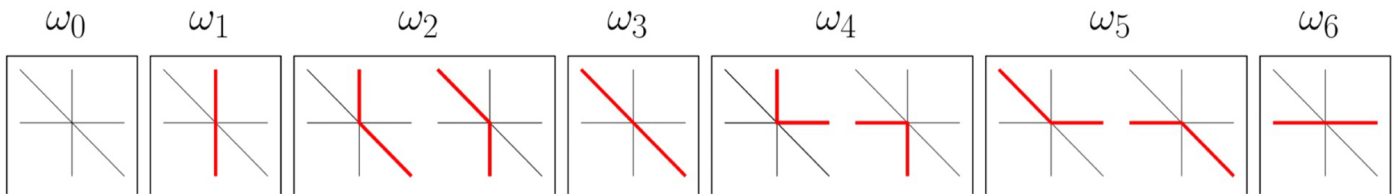
- $6V_1$
- $6V_2$
- $6V_3$



$$H^{20V}(\lambda) \propto H^{6V}(\lambda)$$

integrability!

Single Path Partition function



use transfer matrix technique :

$$T = \begin{matrix} & \begin{matrix} \text{---} \\ \text{---} \\ \text{---} \end{matrix} \\ \begin{matrix} \text{---} \\ \text{---} \\ \text{---} \end{matrix} & \begin{bmatrix} \omega_1 v & \omega_2 uv & \omega_4 u \\ \omega_2 v & \omega_3 uv & \omega_5 u \\ \omega_4 v & \omega_5 uv & \omega_6 u \end{bmatrix} \end{matrix}$$

generating function :

$$Y(u, v) = (001)(I - T)^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Asymptotics are entirely determined by the determinant of $(I - T)$:

$$\Delta(u, v) = \det(I - T) = 1 - \alpha_1 u - \alpha_2 v - \alpha_3 uv - \alpha_4 u^2 v - \alpha_5 uv^2 - \alpha_6 u^2 v^2$$

$$\alpha_1 = \frac{\omega_1}{\omega_0}, \quad \alpha_2 = \frac{\omega_6}{\omega_0}, \quad \alpha_3 = \frac{\omega_0 \omega_3 + \omega_4^2 - \omega_1 \omega_6}{\omega_0^2}$$

$$\alpha_4 = \frac{\omega_2^2 - \omega_1 \omega_3}{\omega_0^2}, \quad \alpha_5 = \frac{\omega_5^2 - \omega_6 \omega_3}{\omega_0^2}, \quad \alpha_6 = \frac{2\omega_2 \omega_4 \omega_5 + \omega_1 \omega_6 \omega_3 - \omega_3 \omega_4^2 - \omega_1 \omega_5^2 - \omega_6 \omega_2^2}{\omega_0^3}.$$

function of the 7 weights!

(only dependence on $\omega_0 \dots \omega_6$ is from $\Upsilon_{k,p}$)

RESULT: ARCTIC CURVES

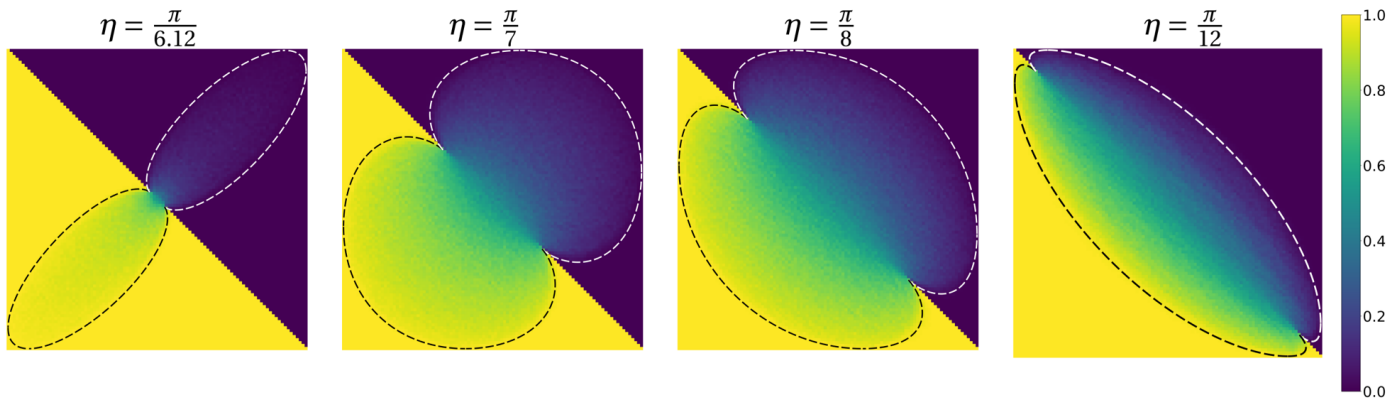
Theorem 5.1. *The arctic curve for the 20V model with DWBC2 at arbitrary admissible values of the parameters η , λ and μ is made generically of three portions, denoted “normal”, “shear” and “final” together with their images under 180° rotation. The three branches have respectively parametric equations:*

$$\begin{aligned} \text{Normal:} \quad x_n(\xi) &= 1 + \frac{\partial_\xi R_n(\xi)}{\partial_\xi S_n(\xi)}, \quad y_n(\xi) = R_n(\xi) - S_n(\xi) \frac{\partial_\xi R_n(\xi)}{\partial_\xi S_n(\xi)}, \quad \xi \in [0, \pi - \lambda - \eta] \\ \text{Shear:} \quad x_s(\xi) &= 1 + \frac{\partial_\xi R_s(\xi)}{\partial_\xi S_s(\xi)}, \quad y_s(\xi) = R_s(\xi) - S_s(\xi) \frac{\partial_\xi R_s(\xi)}{\partial_\xi S_s(\xi)}, \quad \xi \in \left[-\frac{\lambda - \eta + \mu}{2}, 0\right] \\ \text{Final:} \quad x_f(\xi) &= R_f(\xi) - S_f(\xi) \frac{\partial_\xi R_f(\xi)}{\partial_\xi S_f(\xi)}, \quad y_f(\xi) = 1 + \frac{\partial_\xi R_f(\xi)}{\partial_\xi S_f(\xi)}, \quad \xi \in \left[-\frac{\lambda - \eta - \mu}{2}, 0\right] \end{aligned}$$

where

$$\begin{aligned} R_n(\xi) &= R_s(\xi) = (\cot(\xi + \lambda - \eta) - \cot(\xi) + \alpha \cot(\alpha \xi) - \alpha \cot(\alpha(\xi + \lambda - \eta))) \\ &\quad \times \frac{\sin(\xi + \lambda + \eta) \sin(\xi + \lambda - \eta) \sin\left(\xi + \frac{\lambda - \eta + \mu}{2}\right) \sin\left(\xi + \frac{\lambda + 3\eta + \mu}{2}\right)}{\sin(2\eta) \left(\sin(\xi + \lambda + \eta) \sin(\xi + \lambda - \eta) + \sin\left(\xi + \frac{\lambda - \eta + \mu}{2}\right) \sin\left(\xi + \frac{\lambda + 3\eta + \mu}{2}\right) \right)}, \\ R_f(\xi) &= (\cot(\xi + \lambda - \eta) - \cot(\xi) + \alpha \cot(\alpha \xi) - \alpha \cot(\alpha(\xi + \lambda - \eta))) \\ &\quad \times \frac{\sin(\xi + \lambda + \eta) \sin(\xi + \lambda - \eta) \sin\left(\xi + \frac{\lambda - \eta - \mu}{2}\right) \sin\left(\xi + \frac{\lambda + 3\eta - \mu}{2}\right)}{\sin(2\eta) \left(\sin(\xi + \lambda + \eta) \sin(\xi + \lambda - \eta) + \sin\left(\xi + \frac{\lambda - \eta - \mu}{2}\right) \sin\left(\xi + \frac{\lambda + 3\eta - \mu}{2}\right) \right)}, \\ S_n(\xi) &= \frac{\sin(\xi + \lambda + \eta) \sin(\xi + \lambda - \eta) \left(\sin(\xi) \sin(\xi + 2\eta) + \sin\left(\xi + \frac{\lambda - \eta + \mu}{2}\right) \sin\left(\xi + \frac{\lambda + 3\eta + \mu}{2}\right) \right)}{\sin(\xi) \sin(\xi + 2\eta) \left(\sin(\xi + \lambda + \eta) \sin(\xi + \lambda - \eta) + \sin\left(\xi + \frac{\lambda - \eta + \mu}{2}\right) \sin\left(\xi + \frac{\lambda + 3\eta + \mu}{2}\right) \right)}, \\ S_s(\xi) &= \frac{\sin(\xi + \lambda + \eta) \sin(\xi + \lambda - \eta) \sin\left(2\xi + \frac{\lambda - \eta + \mu}{2}\right) \sin\left(\frac{\lambda + 3\eta + \mu}{2}\right)}{\sin(2\eta - \xi) \sin(\xi) \left(\sin(\xi + \lambda + \eta) \sin(\xi + \lambda - \eta) + \sin\left(\xi + \frac{\lambda - \eta + \mu}{2}\right) \sin\left(\xi + \frac{\lambda + 3\eta + \mu}{2}\right) \right)}, \\ S_f(\xi) &= \frac{\sin(\xi + \lambda + \eta) \sin(\xi + \lambda - \eta) \sin\left(2\xi + \frac{\lambda - \eta - \mu}{2}\right) \sin\left(\frac{\lambda + 3\eta - \mu}{2}\right)}{\sin(2\eta - \xi) \sin(\xi) \left(\sin(\xi + \lambda + \eta) \sin(\xi + \lambda - \eta) + \sin\left(\xi + \frac{\lambda - \eta - \mu}{2}\right) \sin\left(\xi + \frac{\lambda + 3\eta - \mu}{2}\right) \right)}, \end{aligned}$$

and $\alpha = \pi / (\pi - 2\eta)$.



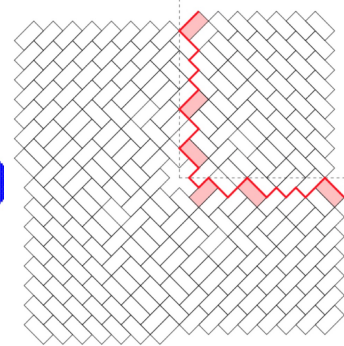
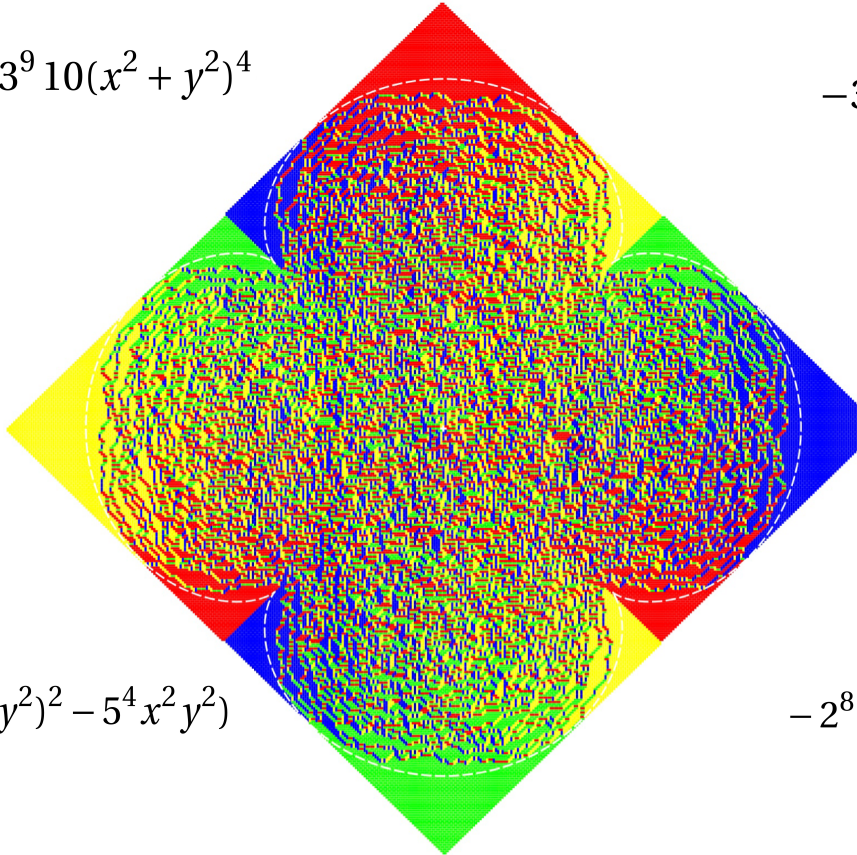
uniform

$$0 < \eta < \frac{\pi}{6}$$

Holey Aztec square domino tilings (uniform weights)

$$3^{11}(x^2 + y^2)^5 + 3^9 10(x^2 + y^2)^4$$

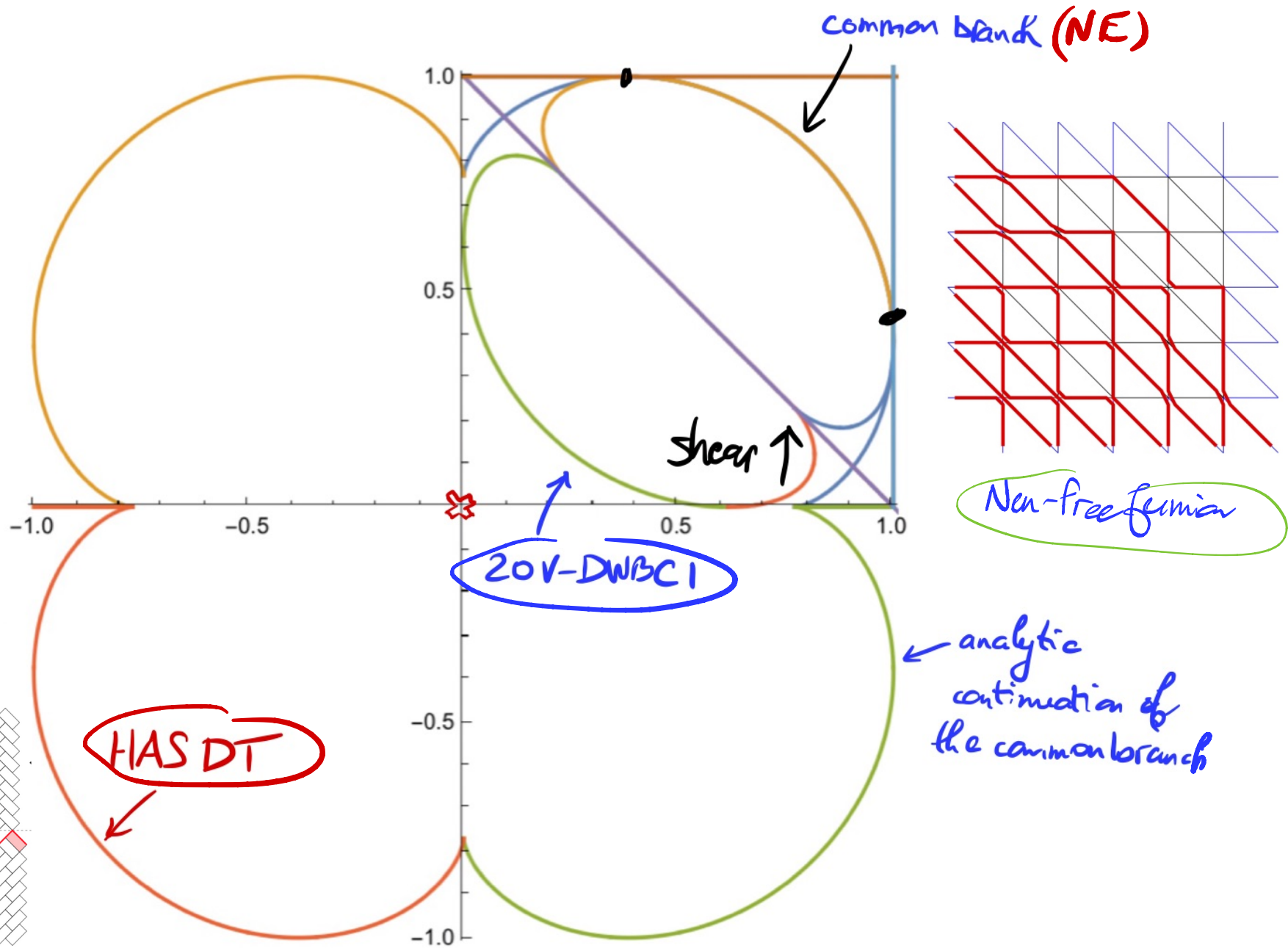
$$-3^6 5(x^2 + y^2)^3$$



$$+ 6^2 20(73(x^2 + y^2)^2 - 5^4 x^2 y^2)$$

$$- 2^8 15(x^2 + y^2) - 2^{12} = 0.$$

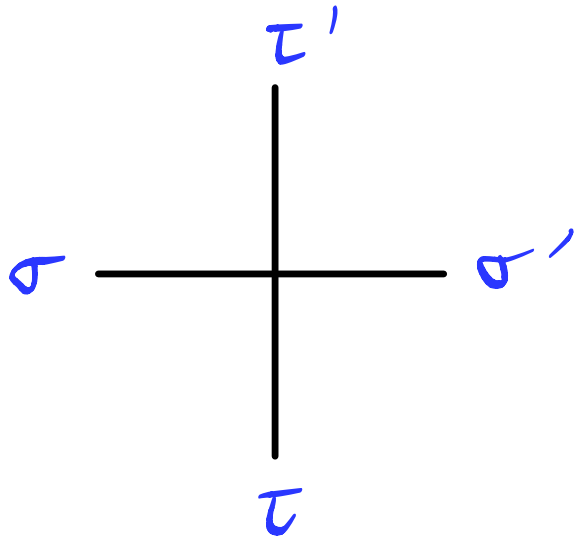
20V DWBC1 $\leftrightarrow$ HASDT correspondence of Arctic Curves



INTERACTING PATHS :

A SIMPLE MODEL

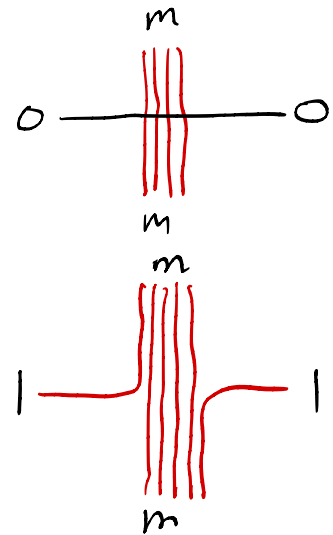
A NILP Model with multiple vertical edges



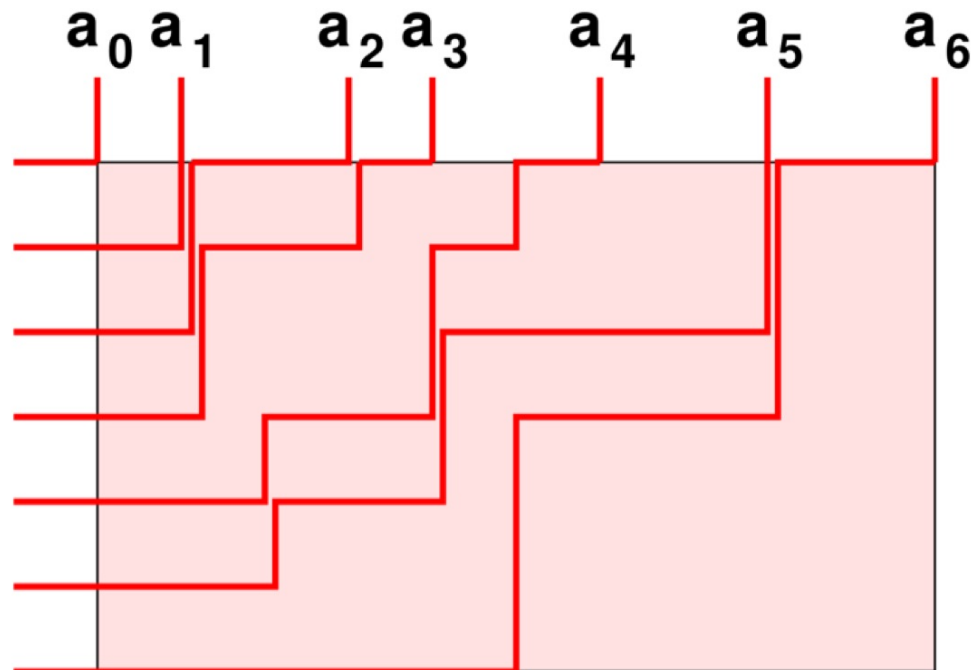
$$\text{weight} = \delta_{\sigma+\sigma', \tau+\tau'}$$

$$\sigma, \sigma' \in \{0, 1\} \quad \tau, \tau' \in \mathbb{Z}_+$$

(limit of colored vertex models
[Gorbati-Wheeler])



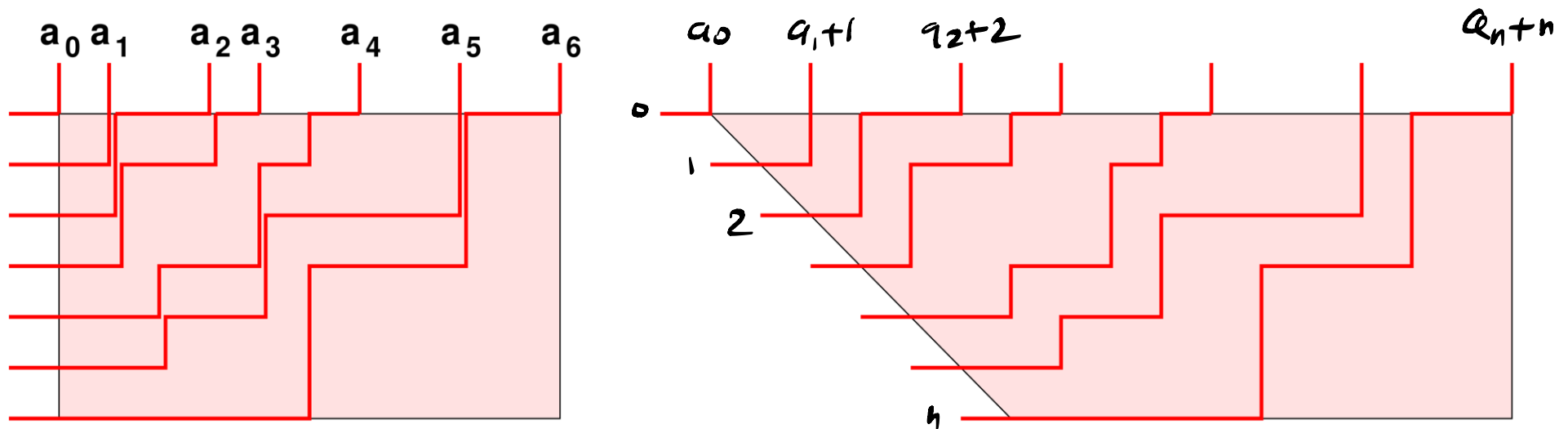
+ special
boundary
conditions



BIJECTION TO NILP WITH EXITS

at $a_i + i \quad i \in [0, n]$.

"Sliding": path i translated by $\vec{i} \rightarrow$

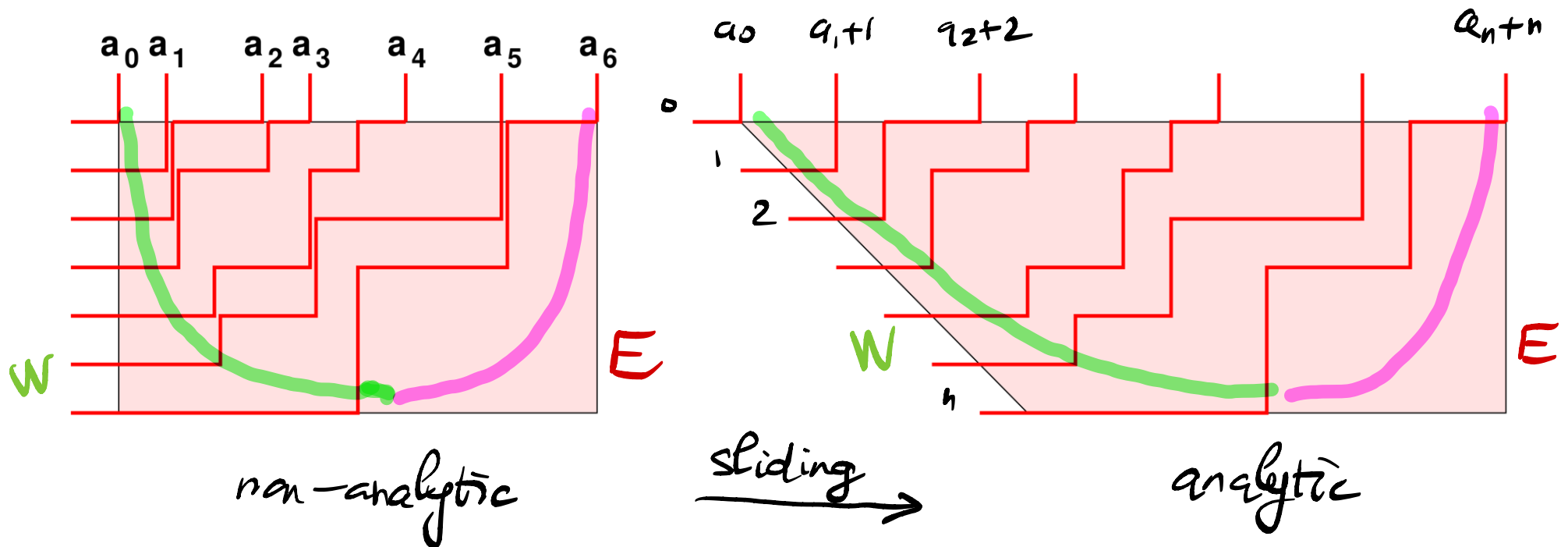


$$Z = Z^{\text{NILP}}(\{a_i + i\}) = \frac{\Delta(\{a_i + i\})}{\Delta(\{i\})} \quad [\text{DF+Gwiler 18}]$$

Ex $a_i = i$ (DNBC) $\Rightarrow Z = 2^{n/(n+1)/2}$

↖ Vandermonde determinant

ARCTIC CURVE



E branch is translated

W branch is sheared

$\Rightarrow$ non-analyticity at junction!

CONCLUSION

- Integrability

exact enumeration + refinements (eg 1 point function!)

- Tangent Method

very robust = applies both to free and non-free fermion cases
still not rigorous (Aggarwal)

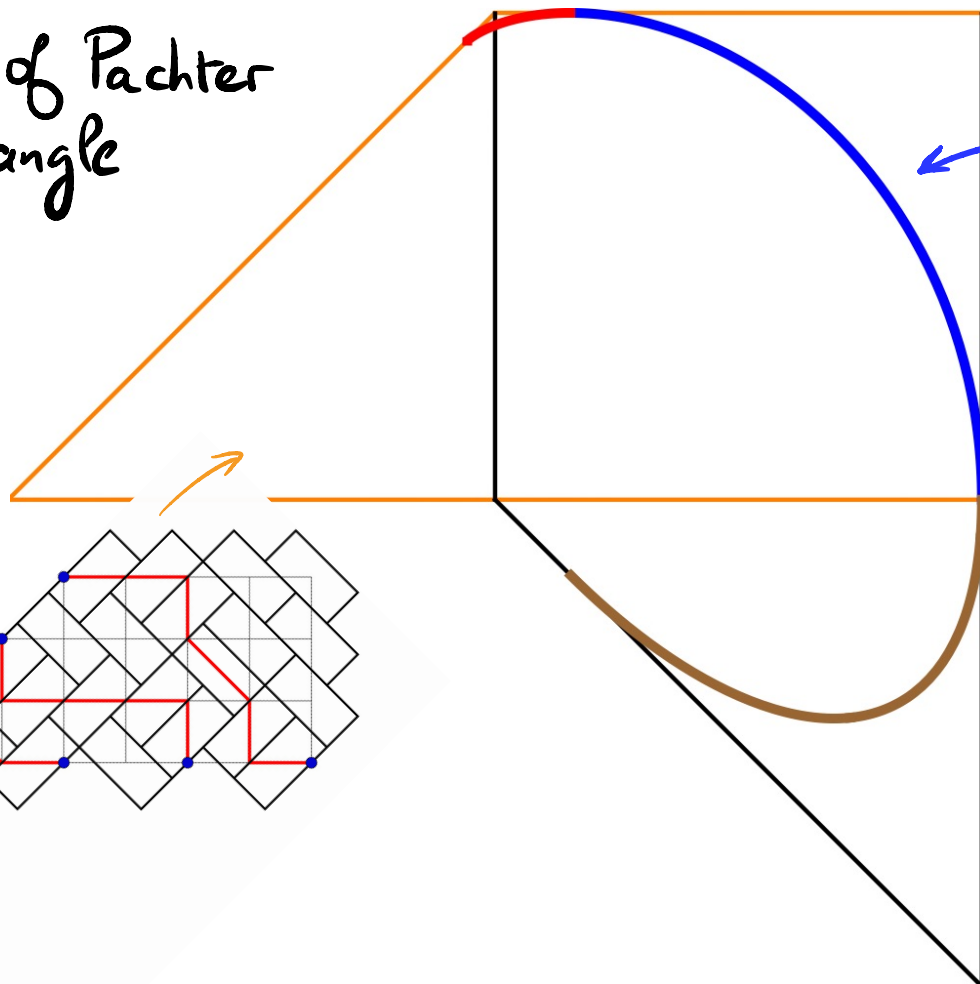
- Limit shapes v/s non-bijections

matching of limit shapes

shear phenomenon

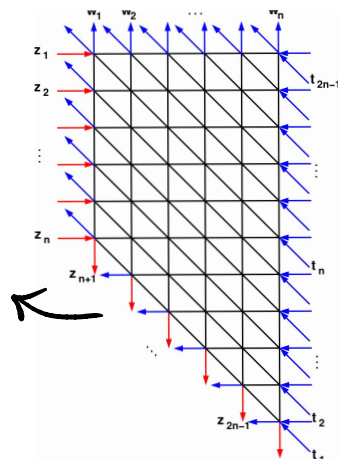
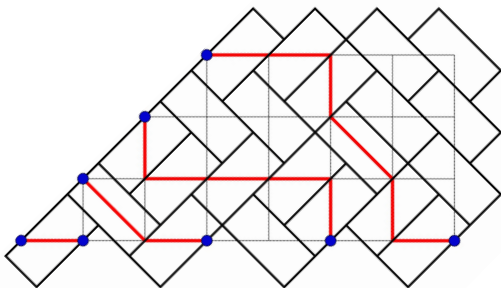
more examples? (20V DWBC3, DT of Pachter Triangle)

DT of Pachter
Triangle



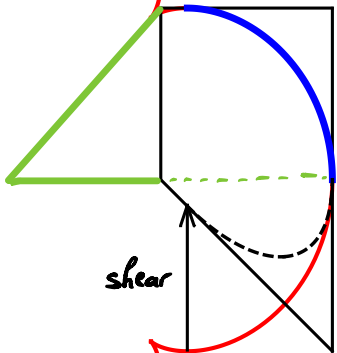
common
part of
Arctic curve

20V
DWBC3
(uniform)

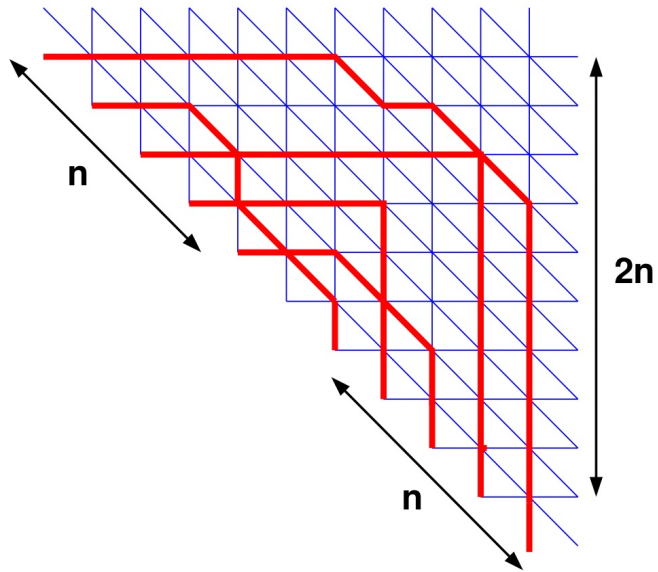


Shear
again!

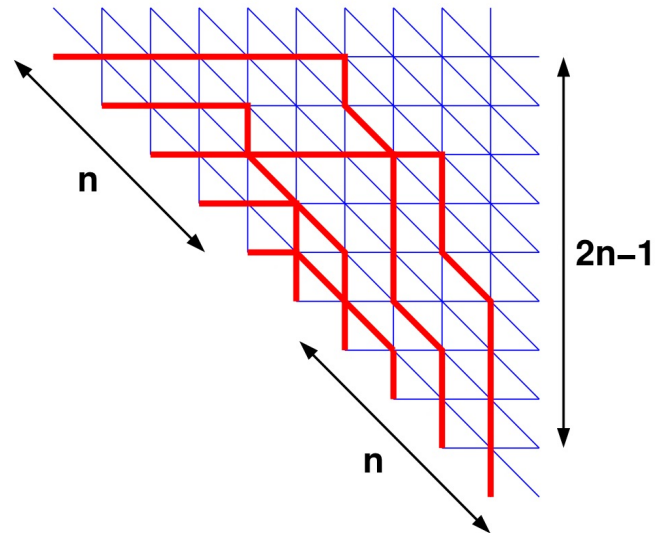
↖ analytic
continuation



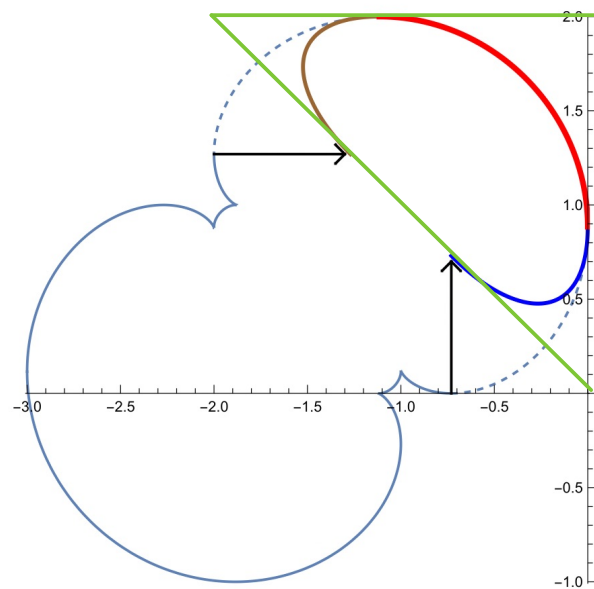
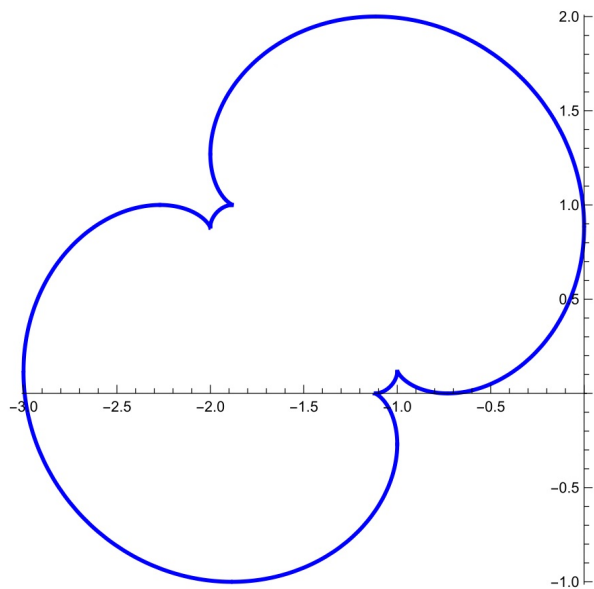
20V on a Triangle



(a)



(b)



uniform case = degree 10 algebraic curve