

TILINGS, VERTEX MODELS AND OTHER REMARKABLE COMBINATORIAL OBJECTS

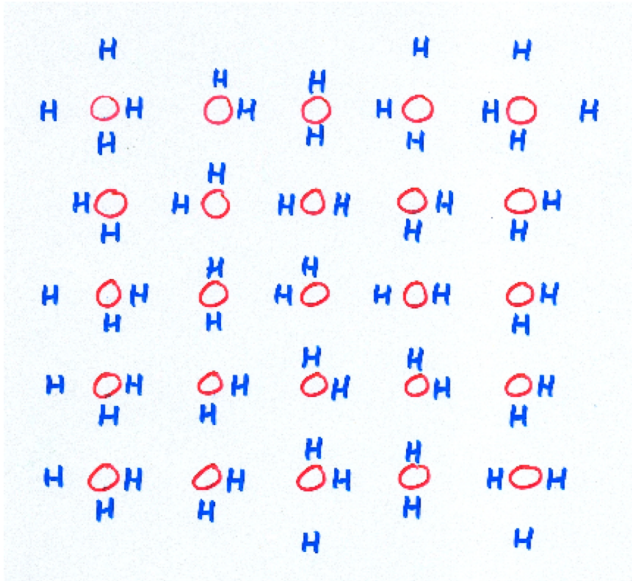
- The Six Vertex (6V) model and its relatives
- Integrability : the Yang-Baxter equation and the Izergin-Korepan determinant
- Descending Plane Partitions (DPP)
 - Proof of the 6V-DPP correspondence
- The Twenty Vertex (20V) model
- Holky Aztec square Domino Tilings
 - Proof of the 20V-HASDT correspondence
- other 20V correspondences and conjectures

6V
↕
IK
DPP

20V
HASDT

THE SIX VERTEX MODEL

Square Ice



Replace data by dipolar
momenta $\{\rightarrow, \leftarrow, \downarrow, \uparrow\}$

Ice Rule at each vertex

incoming arrows

= # outgoing arrows $\Rightarrow 6V$

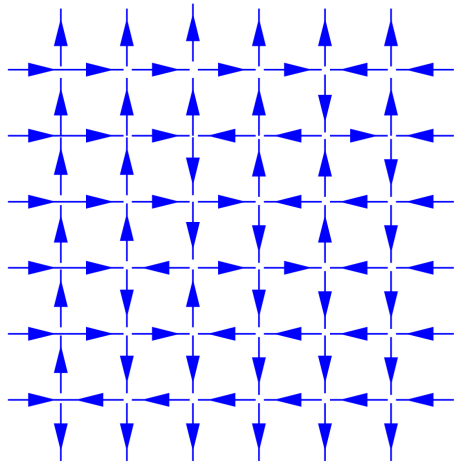
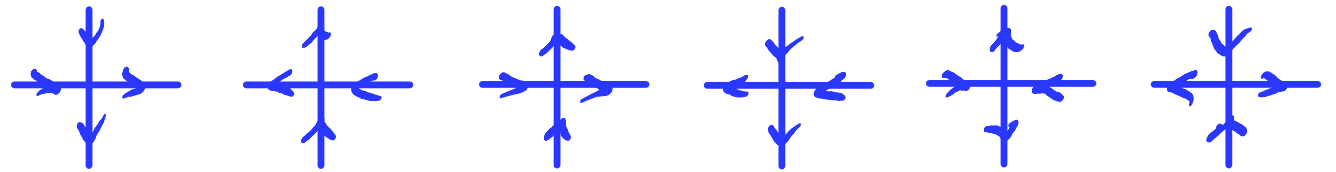
Solution: free energy per square = $(4/3)^{3/2}$
[Lieb 67]

6V

SIX VERTEX MODEL

- Model for 2D ice (H_2O)
- Arrows = dipolar momenta $O^- \xrightarrow{H^+}$
- Conservation law 2 entering arrows
2 outgoing

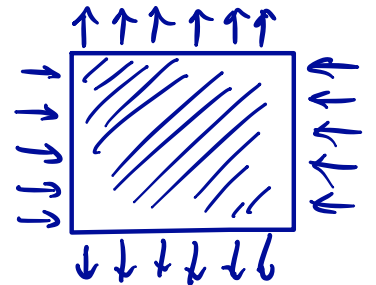
at each vertex



6 Vertex model

[Lieb 67]

+ Domain Wall Boundary Conditions
($n \times n$ square)

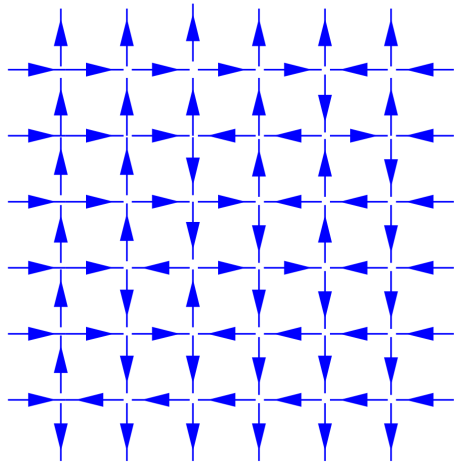


6V, ASM

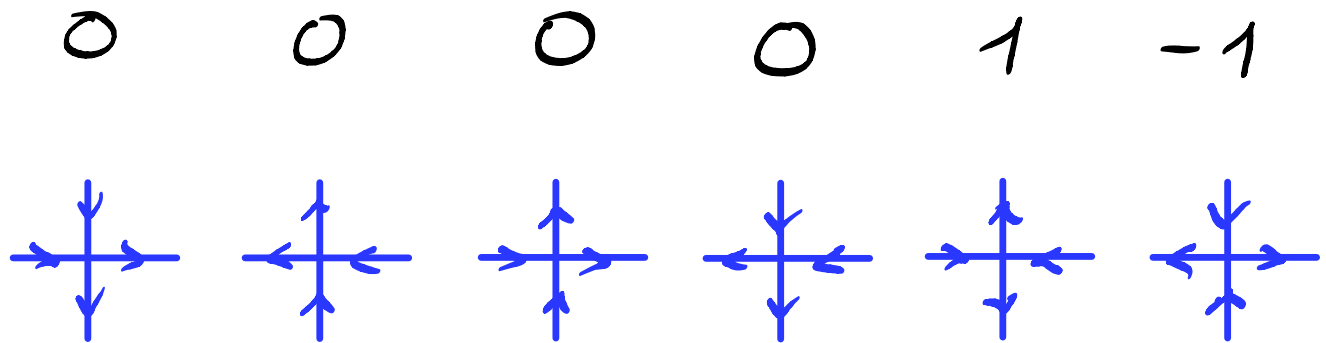
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- "Magic squares" with 0, 1, -1 entries
- generalizations of permutations

Alternating Sign Matrices
[Robbins-Rumsey 82]



6V



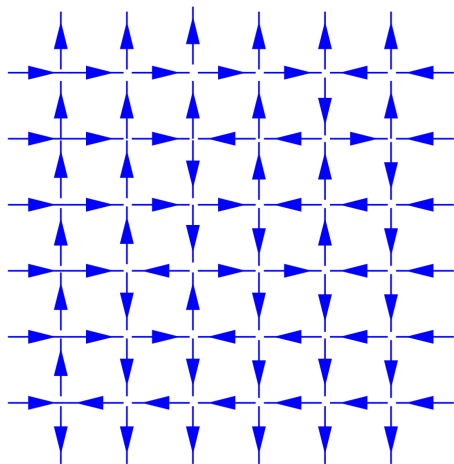
[Kuperberg '96]

6V, ASM, TSSCPP

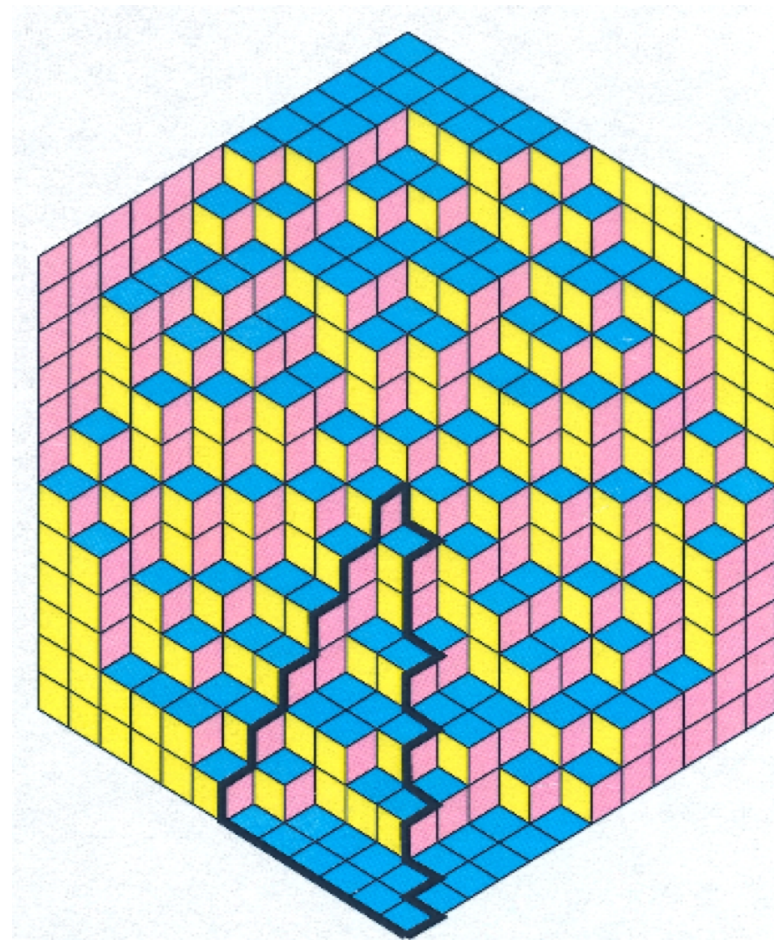
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

ASM

Rhombus Tilings
of a $(2n, 2n, 2n)$ hexagon
w/ all possible symmetries



6V



Totally Symmetric
Self-Complementary Plane
Partition

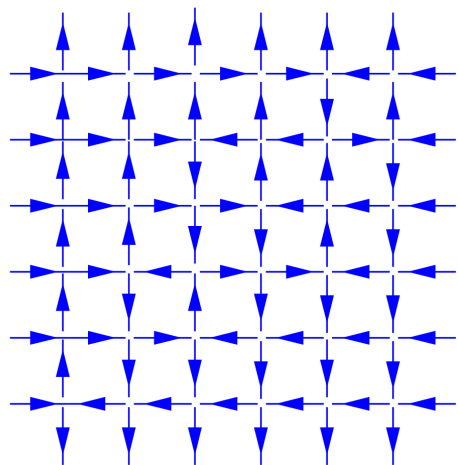
[Andrews 70's]

6V, ASM, TSSCPP, DPP

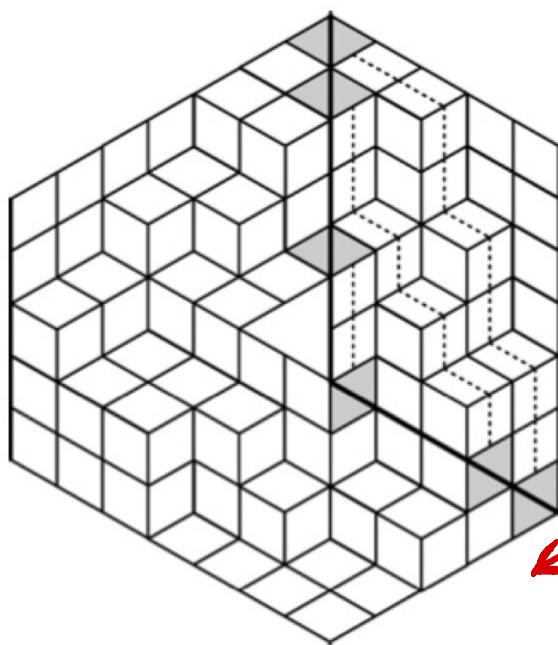
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

ASM

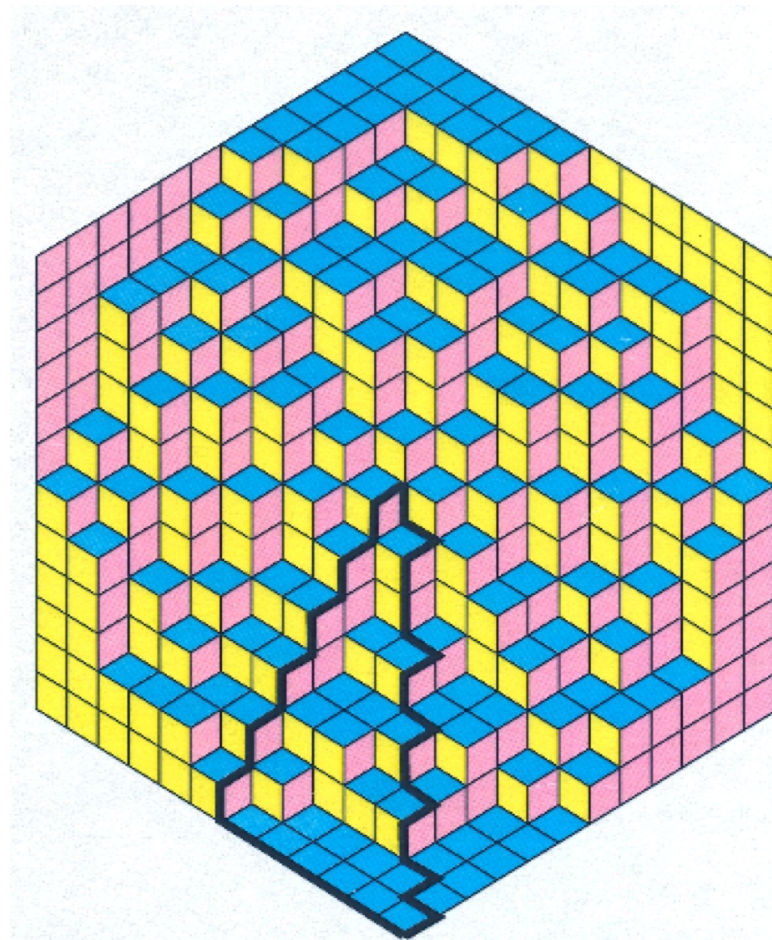
rhombus tilings of a
holey $(n, n+2, n+2, n+2)$
hexagon with $2\pi/3$
rotational symmetry



6V



$\frac{2\pi}{3}$



TSSCPP

Descending Plane Partition
[Andrews 70's] [Krattenthaler 00's]

6V, ASM, TSSCPP, DPP, NV

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

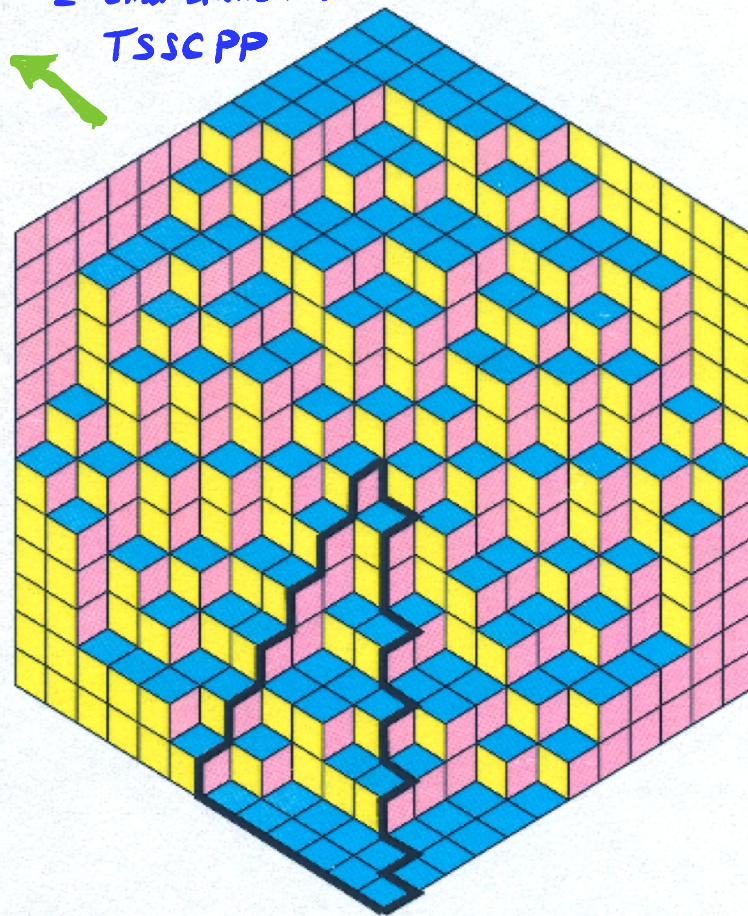
ASM

$V = \{ M \in M_{2n}(\mathbb{C}), M^2 = 0 \text{ and strictly upper triangular} \}$

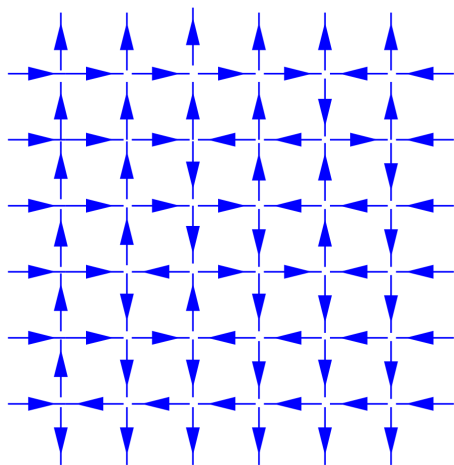
Nilpotent Variety
(degree)

[DF + Zinn Justin 05]

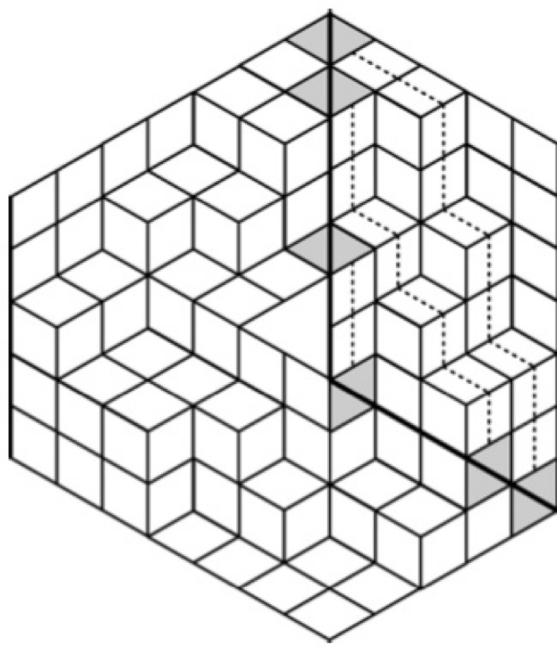
2-enumeration of
TSSCPP



TSSCPP



6V



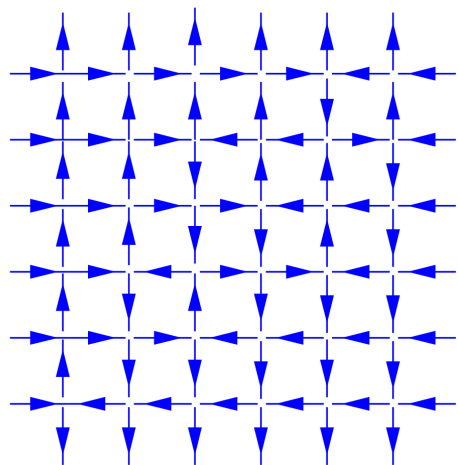
DPP

6V, ASM, TSSCPP, DPP

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

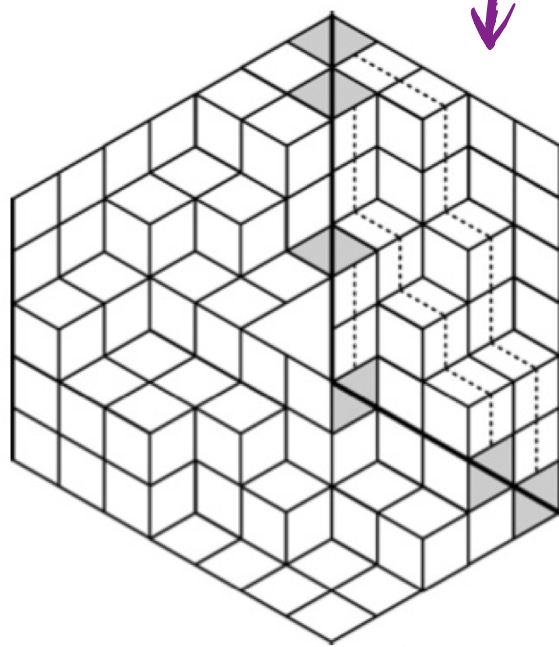
ASM

↕ bijection



6V

↔ ref.

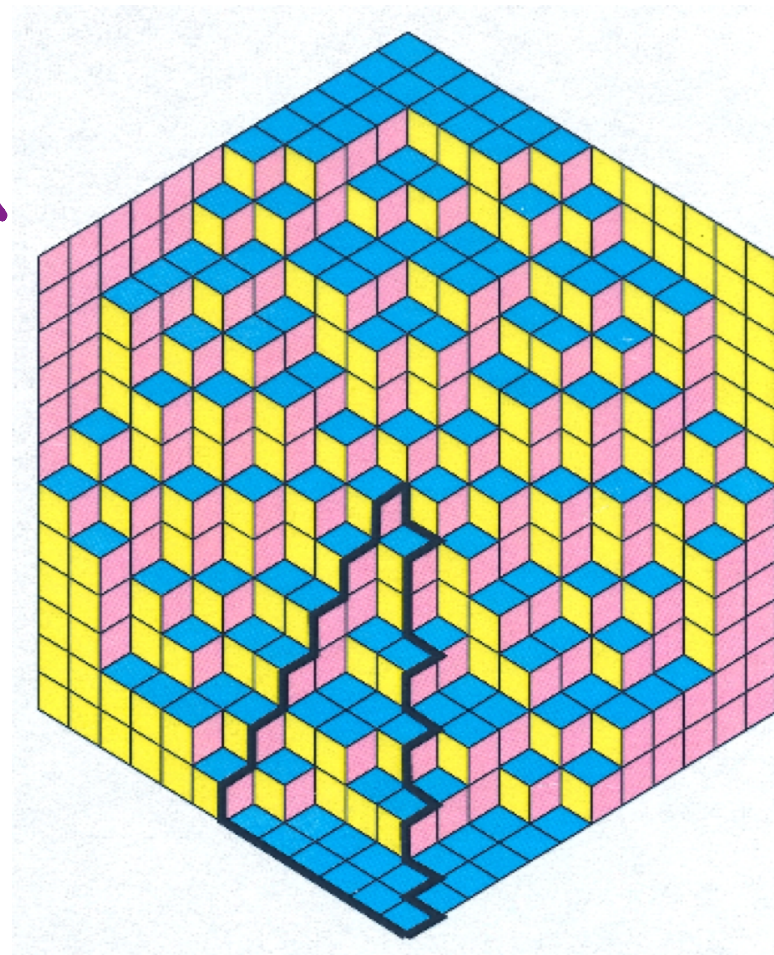


DPP

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

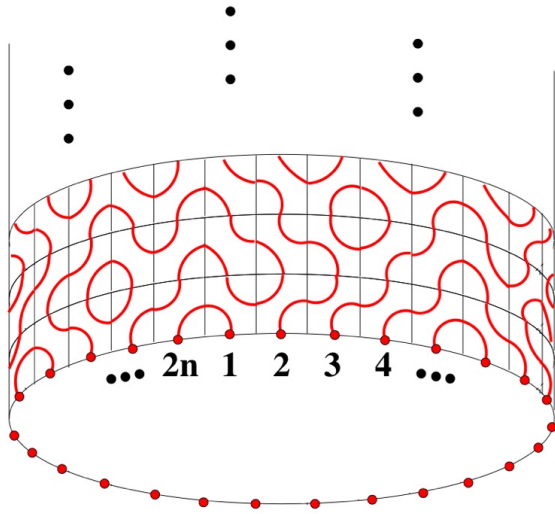
[Zeilberger 96]
[Kuperberg 96]

↓

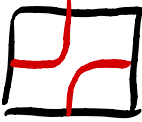
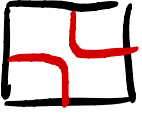


TSSCPP

LOOP MODEL (Physics strikes back)



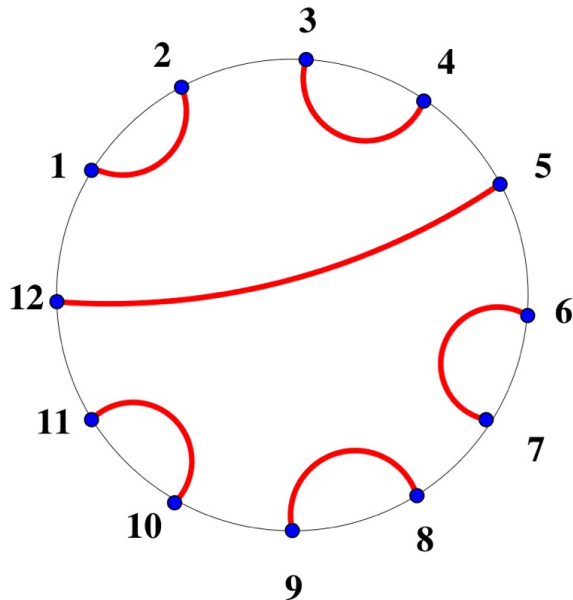
$O(n)$

each square face on the semi-infinite cylinder
is  or  with probabilities

p

$1-p$

$p \in (0, 1)$



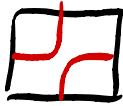
$\text{Proba}(\text{matching})$

$= ?$

RAZUMOV-STROGANOV CONJECTURE

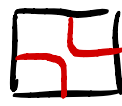
each square face on the semi-infinite cylinder

is



p

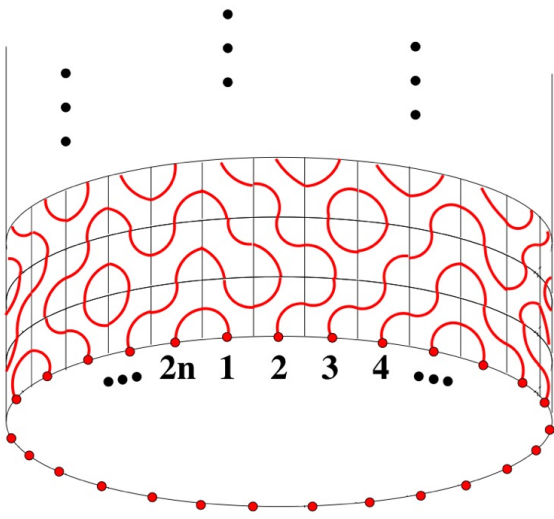
or



$1-p$

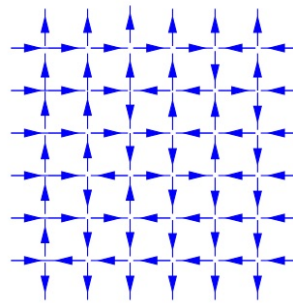
with probabilities

$p \in (0,1)$



$O(n)$

[Nienhuis
De Gier
Razumov
Stroganov 00]



b_{ij}

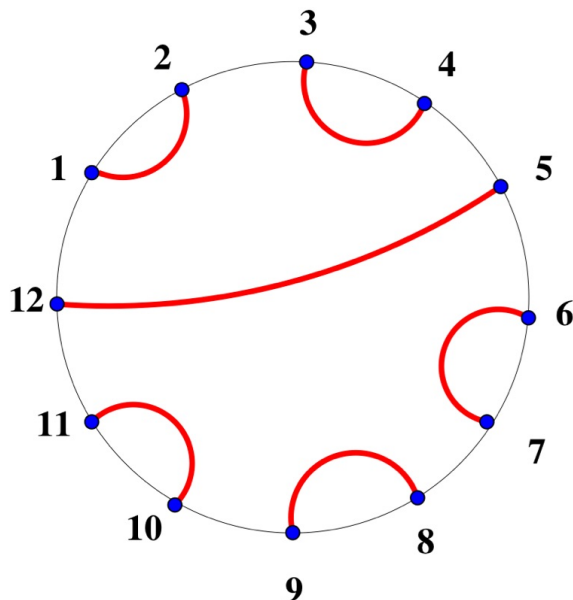
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6 Vertex

Alternating Sign Matrices
(ASM)

b_{ij}

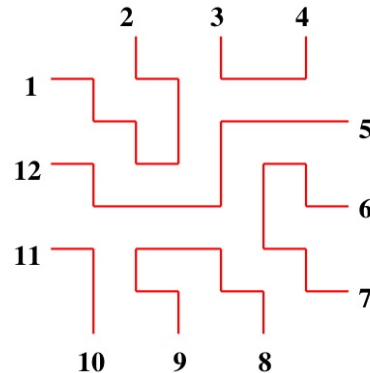
b_{ij}



$\text{Proba}(\text{matching})$

[PDF, Zinn-Justin 04]

[Cantini-Spottiglio 10]

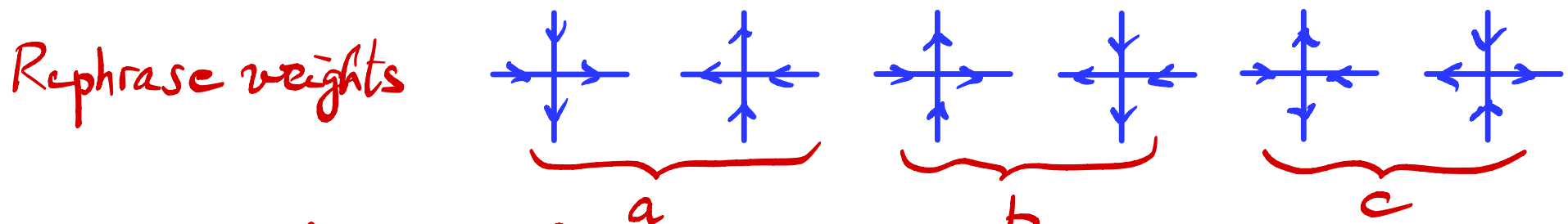


Number of Fully Packed Loops (FPL)
with same matching



INTEGRABILITY

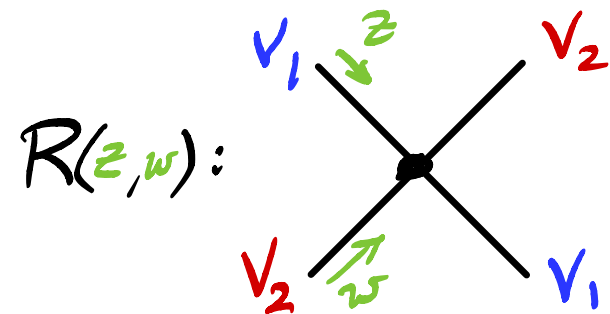
R-operator formulation



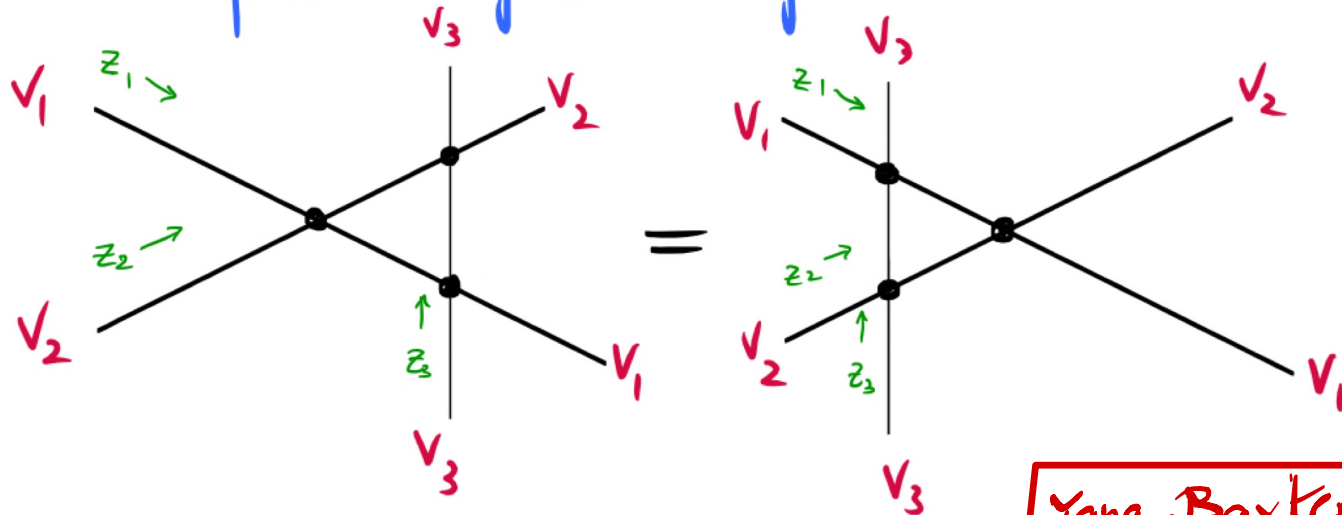
as entries of matrix of an operator $R: V_1 \otimes V_2 \rightarrow V_2 \otimes V_1$

$$V_1 \cong \mathbb{C}^2 = \text{Vect}(\rightarrow, \leftarrow)$$

$$V_2 \cong \mathbb{C}^2 = \text{Vect}(\uparrow, \downarrow)$$



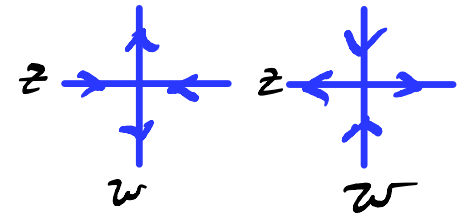
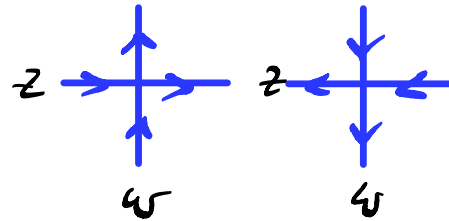
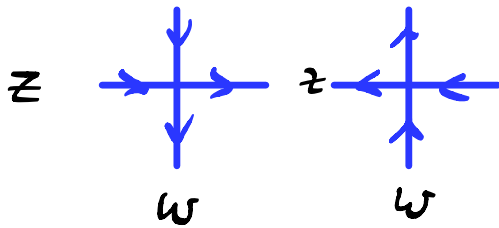
Fact: One can pick "integrable weights" such that:



Yang-Baxter equation

(Trigonometric) Solution to the Yang - Baxter equation

$$a(z, w) = qz - q^{-1}w ; \quad b(z, w) = q^{-1}z - q^1w ; \quad c(z, w) = (q^2 - q^{-2})\sqrt{zw}$$



(= Matrix Elements of R !)

IZERGIN-KOREPIN Determinant

$$Z_{6V} \left[\begin{array}{c} w_1, w_2, \dots, w_N \\ \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \\ z_1 \\ \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \\ z_2 \\ \vdots \\ \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \\ z_n \end{array} \end{array} \right] = \frac{\prod_{i,j=1}^N a(z_i, w_j) b(z_i, w_j)}{\prod_{1 \leq i < j \leq N} (z_i - z_j)(w_i - w_j)} \times$$

$$\times \det \left(\frac{c(z_i, w_j)}{a(z_i, w_j) b(z_i, w_j)} \right)_{1 \leq i, j \leq N}$$

uniform limit: $q = e^{i\pi/6}$
 $z_i \rightarrow 1 \quad w_i \rightarrow 1$

$$Z_n = \prod_{i=0}^{n-1} \frac{(\beta + i)!}{(n+i)!} = \text{ASM}_n$$

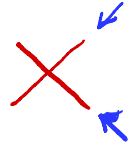
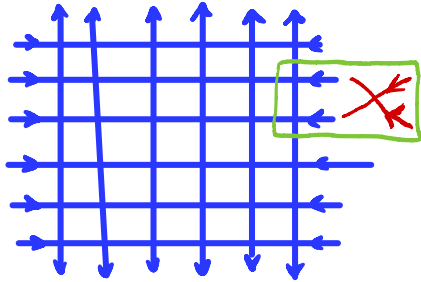
[Kuperberg 96]

PROOF

①

Symmetry

- Multiply by R matrix
- evaluate at entering states



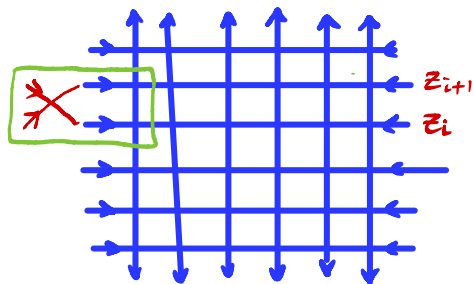
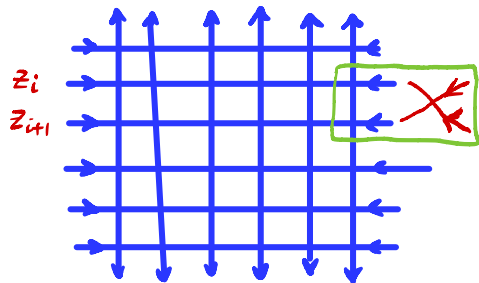
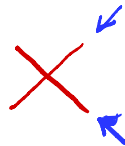
PROOF

①

Symmetry (from Integrability)

- Multiply by R matrix
- evaluate at entering states

- Use Yang-Baxter relation



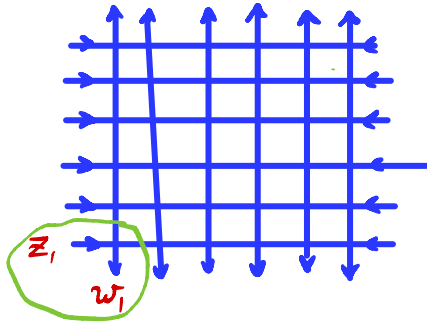
$$\Rightarrow Z(z_1, \dots, z_i, z_{i+1}, \dots, w_1, \dots, w_n)$$

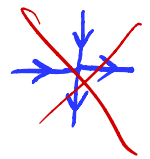
$$= Z(z_1, \dots, z_{i+1}, z_i, \dots, w_1, \dots, w_n)$$

+ same for w .

PROOF

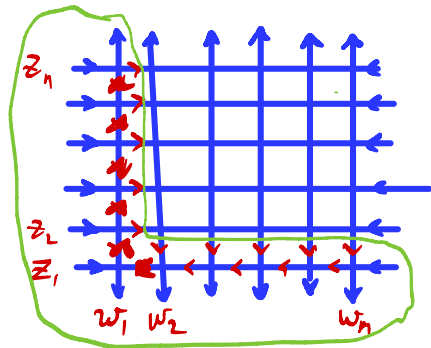
② Recursion relation



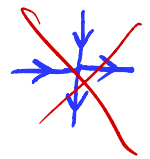
$$w_1 = q^2 z_1 \Rightarrow a(z_1, w_1) = 0$$
$$\Rightarrow \begin{array}{c} \uparrow \\ \rightarrow \quad \leftarrow \\ \downarrow \end{array} = c(z_1, w_1)$$


PROOF

② Recursion relation



$$w_1 = q^2 z_1 \Rightarrow a(z, w_1) = 0$$

$$\Rightarrow \begin{array}{c} \uparrow \\ \leftarrow \rightarrow \\ \downarrow \end{array} = c(z, w_1)$$


$$\Rightarrow Z_N(z_1, z_2, \dots, z_n; w_1, \dots, w_n) \Big|_{w_1 = q^2 z_1}$$

$$= \prod_{i=1}^n a(z_i, w_i) c(z_i, w_i) \prod_{i=2}^n b(z_i, w_i)$$

$$\times Z_{N-1}(z_2, \dots, z_n; w_2, \dots, w_n)$$

① and ② determine polynomial Z_N $\square$

HOMOGENEOUS LIMIT

$$\lim_{\substack{z_i \rightarrow r \\ w_i \rightarrow r^{-1}}} Z_N(z_1, \dots, z_N; w_1, \dots, w_N) = Z_N(x, y) \quad x = \left(\frac{c}{a}\right)^2 \quad y = \left(\frac{b}{a}\right)^2$$

$$= b^{n^2} \left(\frac{c}{b}\right)^n \sum_{\text{configs } bV} x^{\frac{Nc-N}{2}} y^{\frac{Nb}{2}}$$

$\downarrow$ $\downarrow$
 ASM $\#(-1) = Nc_2$; $\text{Inv}(A) - \#(-1) = Nb_1$

= refined g.f. for ASM.

From IK determinant:

$$Z_N(x, y) = (ab)^{N^2} \det_{0 \leq i, j \leq N-1} \left(\frac{\left(\frac{d}{du}\right)^i \left(\frac{d}{dv}\right)^j}{i! j!} \frac{c(u'v)}{a(u'v)b(u'v)} \right) \Bigg|_{\substack{u=r^{-1} \\ v=r^{-1}}}$$

↑
Taylor expⁿ and the limit.

Note:

$$\frac{c(u^{-1}, v)}{a(u^{-1}, v) b(u^{-1}, v)} = \frac{1}{uv - q^{-2}} - \frac{1}{uv - q^2}$$

$$\downarrow \qquad \qquad \downarrow$$

$$(A_-)_{ij} \qquad (A_+)_{ij} = \frac{\left(\frac{d}{du}\right)^i}{i!} \frac{\left(\frac{d}{dv}\right)^j}{j!} \frac{1}{uv - q^2} \Big|_{\substack{u=r^{-1} \\ v=r^{-1}}}$$

Thm ① $\sum_{i,j \geq 0} (A_+)_{ij} z^i w^j = \frac{1}{(r^{-1} + z)(r^{-1} + w) - q^2}$ ← obvious!

② $\det(A_- - A_+) \propto \det(I - A_-^{-1} A_+) \propto \det((1-v)I + vG)_{ij}$
 $\propto i, j \leq N-1$

where

$$v = \frac{r^2 - q^{-2}}{q^2 - q^{-2}}; \quad \sum_{i,j \geq 0} G_{ij} z^i w^j = \frac{1}{1 - z x - w - (y-x)zw}$$

$\Rightarrow$

$$Z_N(x, y) \propto \det_{0 \leq i, j \leq N-1} \left(\frac{1-v}{1-zw} + \frac{v}{1-zx-w-(y-x)zw} \right)$$

$\uparrow$
Finite truncation

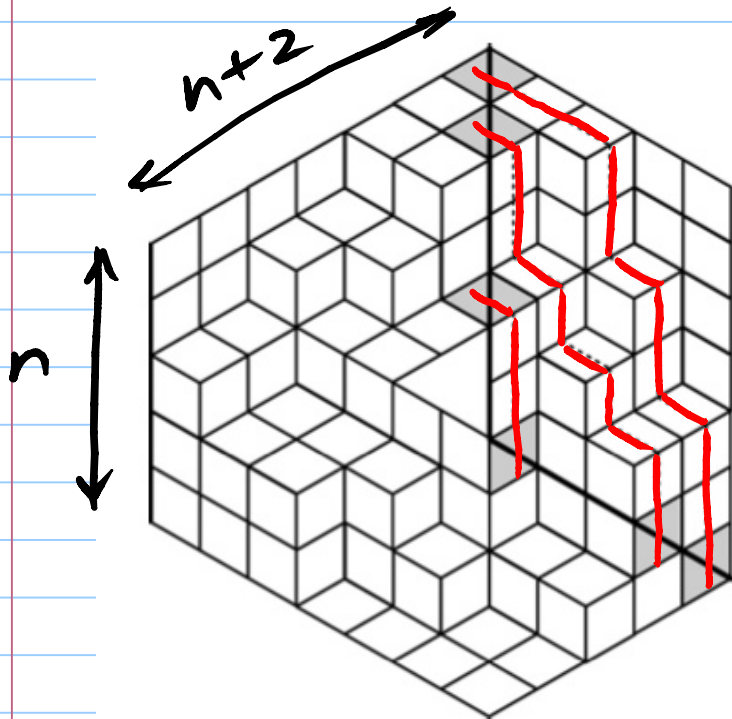
$\uparrow$
infinite matrix

D P P

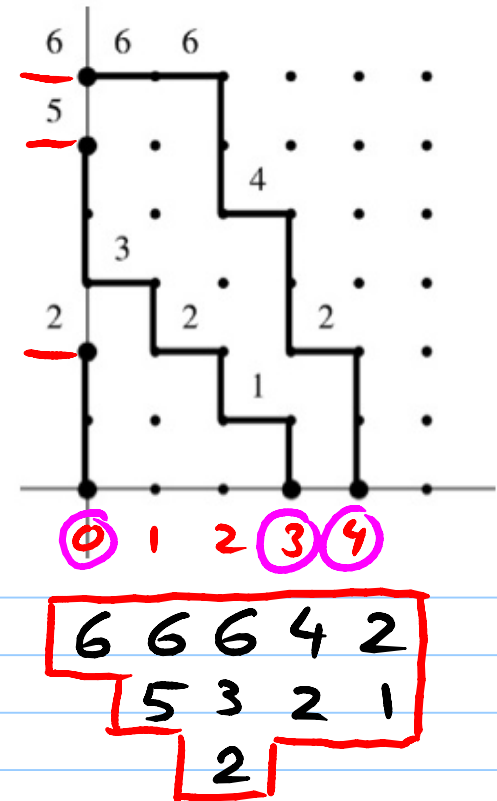
(Descending Plane Partitions)

DPP as Lattice Paths

(Krattenthaler '06)



$\begin{matrix} \textcircled{4} & 6 \\ \textcircled{3} & 5 \\ 2 & 4 \\ 1 & 3 \\ \textcircled{0} & 2 \\ -1 & 1 \\ -2 & 0 \end{matrix}$



Rhombus Tilings of a Hexagon $(n, n+2, n, n+2, n, n+2)$ with $\triangle \text{hole}$ $\leftrightarrow$ Lattice Paths

$2 \times 2 \times 2$

COUNTING: GESSEL-VIENNOT

(Thm) (LGV) Partition fctn of (edge-weighted)

NON-INTERSECTING LATTICE PATHS w/ starting pts $s_1, \dots, s_n$ and endpoints $e_1, \dots, e_n$, such that $s_i \rightarrow e_j$ and $s_j \rightarrow e_i$ must intersect:

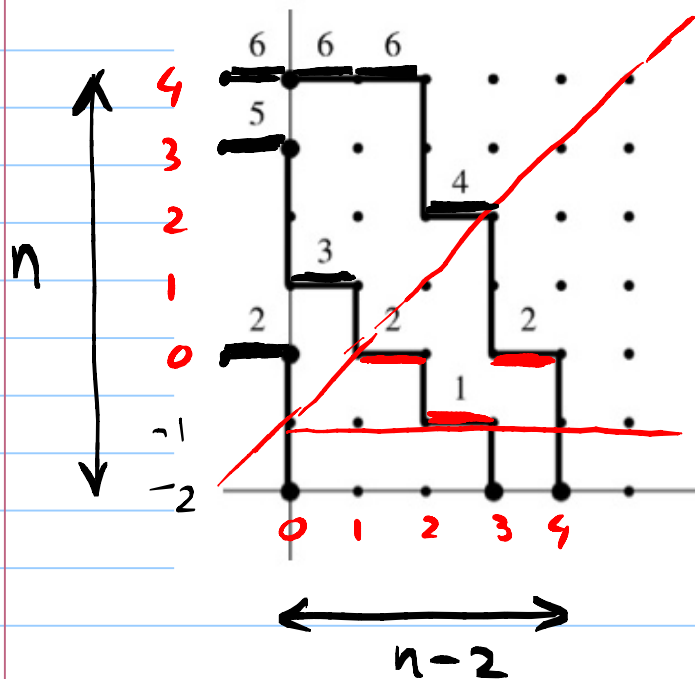
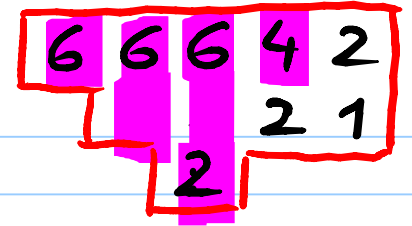
$$Z_{s_1, \dots, s_n; e_1, \dots, e_n} = \det_{1 \leq i, j \leq n} (Z_{s_i, e_j}) = \det(\underline{\underline{Z}})$$

(Cor) For a core ($s_i = e_i, \forall i$ form a segment s)

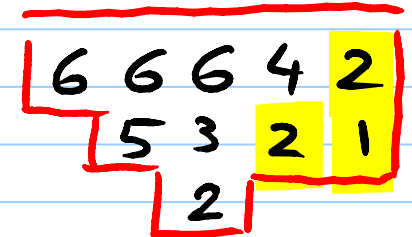
$$Z = \sum_{k \geq 0} \sum_{i_1 < \dots < i_k} \det \left(\underbrace{Z_{s_{i_1}, \dots, s_{i_k}; s_{i_1}, \dots, s_{i_k}}}_{\text{diagonal minors of } \underline{\underline{Z}}} \right) = \det(I + \underline{\underline{Z}})$$

[Cauchy-Binet
0.0]

horizontal steps — = non-special parts

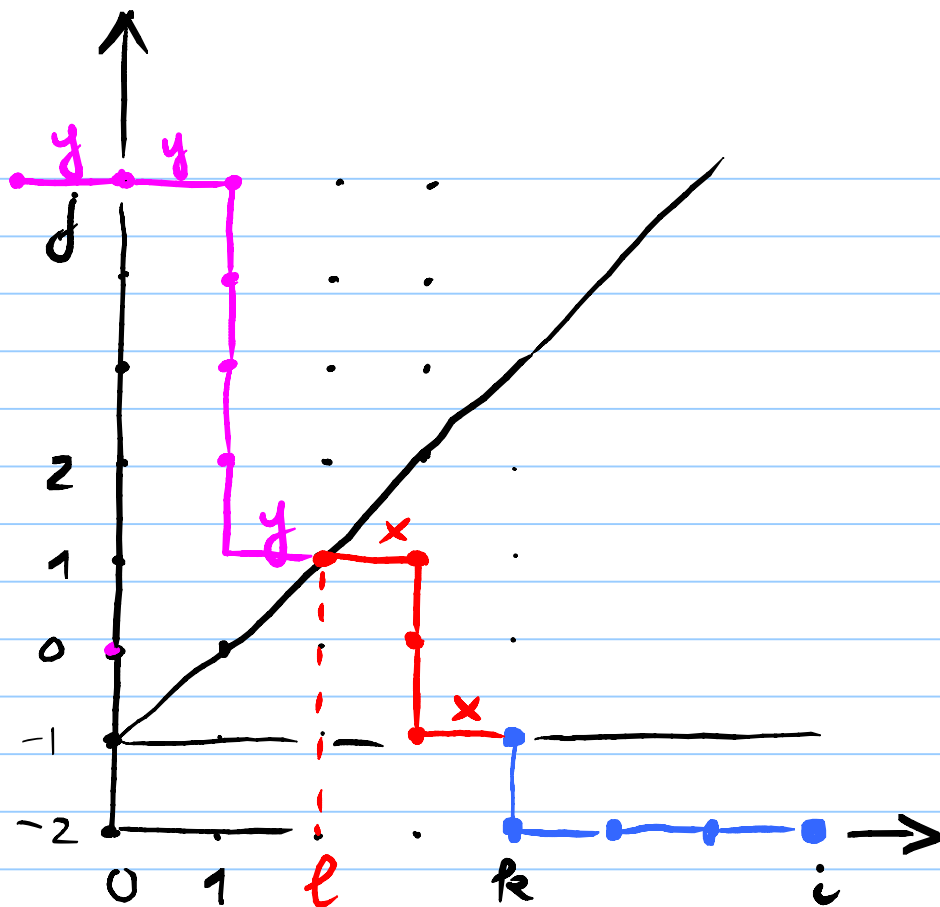


horizontal steps —
= special parts



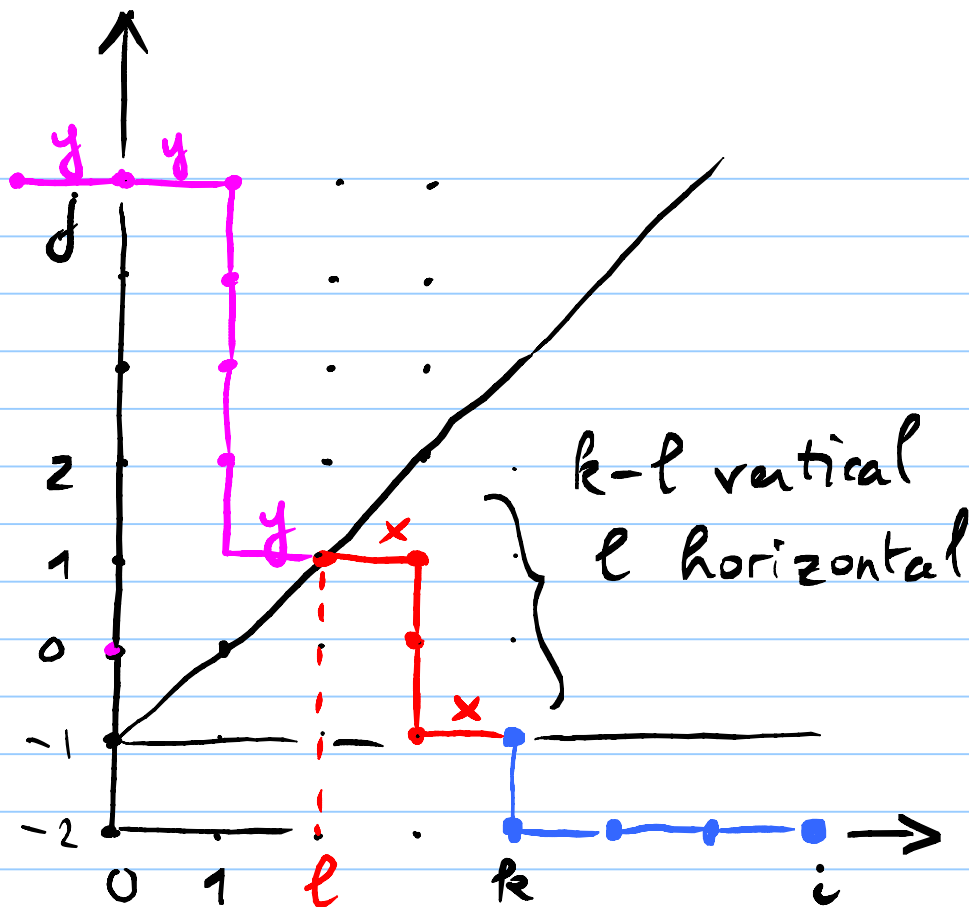
horizontal steps here do not count

$$Z_{DPP(n)} = \sum_{DPP(n)} x^{\#(\text{special parts})} y^{\#(\text{non-special parts})}$$



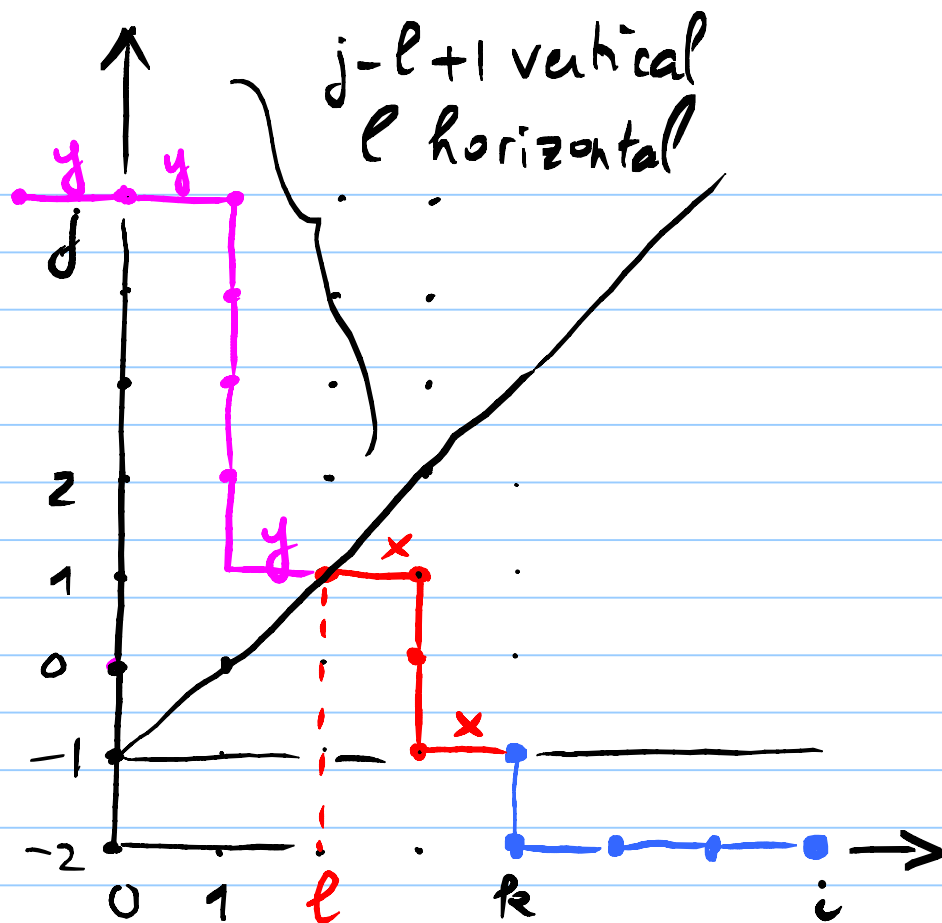
$$Z_{\text{DPP}}^{(n)}(\overset{\text{per special part}}{\underset{\uparrow}{x}}, \overset{\text{per non-special part}}{\underset{\uparrow}{y}}) = \det(I + M)$$

$$M_{i,j} = \sum_{k=0}^i$$



$$Z_{\text{DPP}}^{(n)}(\underset{\substack{\uparrow \\ \text{per special} \\ \text{part}}}{x}, \underset{\substack{\uparrow \\ \text{per non-special} \\ \text{part}}}{y}) = \det(I+M)$$

$$M_{i,j} = \sum_{k=0}^i \sum_{l \geq 0} \binom{k}{l} x^{k-l}$$



$$Z_{\text{DPP}}^{(n)}(\underset{\substack{\uparrow \\ \text{per special} \\ \text{part}}}{x}, \underset{\substack{\uparrow \\ \text{per non-special} \\ \text{part}}}{y}) = \det(I + M^{(n-1)})$$

$$M_{i,j} = \sum_{k=0}^i \sum_{l \geq 0} \binom{k}{l} x^{k-l} \binom{j+1}{l} y^{l+1}$$

Thm ①

[DF + Behrend + PZinn-Justin '11,12]

$$Z_{\text{DPP}(N)} = \det_{0 \leq i, j \leq N-1} \left(\frac{1}{1-zw} + \frac{1}{1-z} \frac{yz}{1-xz-w-(y-x)zw} \middle| \begin{matrix} i \\ j \end{matrix} \right)$$

Thm ②

$$Z_{\text{DPP}(N)}(x, y) = Z_{\text{GV}}(x, y)$$

+ refinements

PROOF

$$Z_{\text{DPP}(N)} = \det_{0 \leq i, j \leq N-1} \left(\frac{1}{1-zw} + \frac{1}{1-z} \frac{yz}{1-xz-w-(y-x)zw} \middle| z^i w^j \right)$$

$$Z_N(x, y) = \det_{0 \leq i, j \leq N-1} \left(\frac{1-v}{1-zw} + \frac{v}{1-zx-w-(y-x)zw} \middle| z^i w^j \right)$$

Both truncations of ∞ matrices = relate them!

Lemma

$$\det \left(\begin{array}{c|c} \begin{array}{c} 1 \\ \text{1's} \\ \text{diagonal} \end{array} & \begin{array}{c} \text{0} \\ \text{0's} \end{array} \\ \hline \text{n rows} \end{array} \times \begin{array}{c} \text{M} \\ \text{matrix} \end{array} \times \begin{array}{c|c} \begin{array}{c} \text{1's} \\ \text{diagonal} \end{array} & \begin{array}{c} \text{0's} \end{array} \\ \hline \text{n columns} \end{array} \right) = \det \left(\begin{array}{c} \text{M} \\ \text{n} \times \text{n} \text{ truncation} \end{array} \right)$$

PROOF

$$Z_{\text{DPP}(N)} = \det_{0 \leq i, j \leq N-1} \left(\frac{1}{1-zw} + \frac{1}{1-z} \frac{yz}{1-xz-w-(y-x)zw} \middle| z^i w^j \right)$$

$$Z_N(x, y) = \det_{0 \leq i, j \leq N-1} \left(\frac{1-v}{1-zw} + \frac{v}{1-zx-w-(y-x)zw} \middle| z^i w^j \right)$$

Both truncations of ∞ matrices = relate them!

(Thm) $(1-z)(1-(1-v)w) f_{\text{DPP}}(z, w) - (1-(y-xv-1)z)(1-w) f_{\text{ASM}}(z, w)$
 $= \underbrace{(xv(1-v) - y(1-v) - v)}_{=0!} \times \text{rational fraction}(z, w)$

Conclusion $F(z, w) = \sum_{i, j \geq 0} A_{ij} z^i w^j \Rightarrow (1-\lambda z)(1-\mu w) F(z, w)$

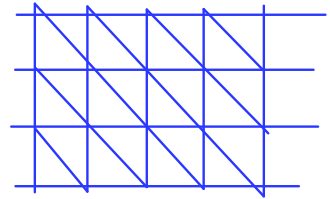
generates $(I - \lambda S) A (I - \mu S^t) \Rightarrow$ same truncated det's
 $\nearrow$ lower T $\nwarrow$ upper T (qed)

20 V

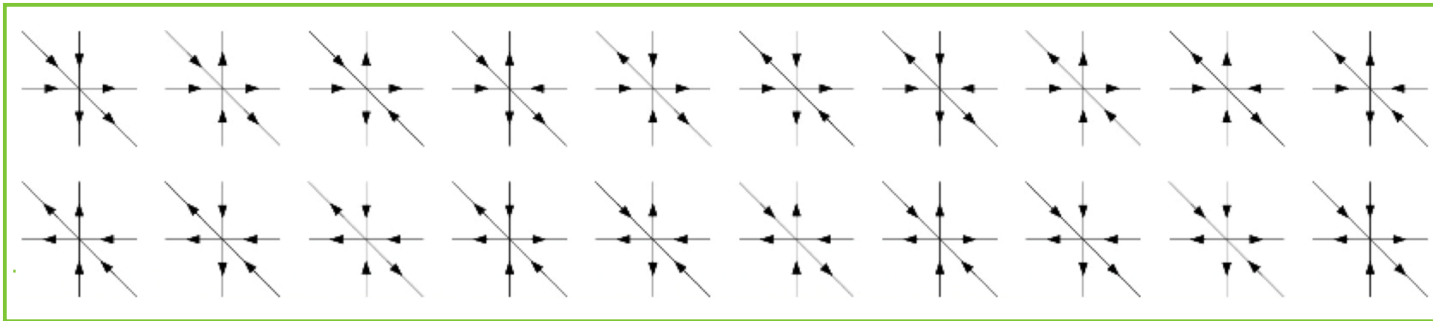
THE TWENTY VERTEX MODEL

20 VERTEX MODEL

- Same idea but triangular lattice

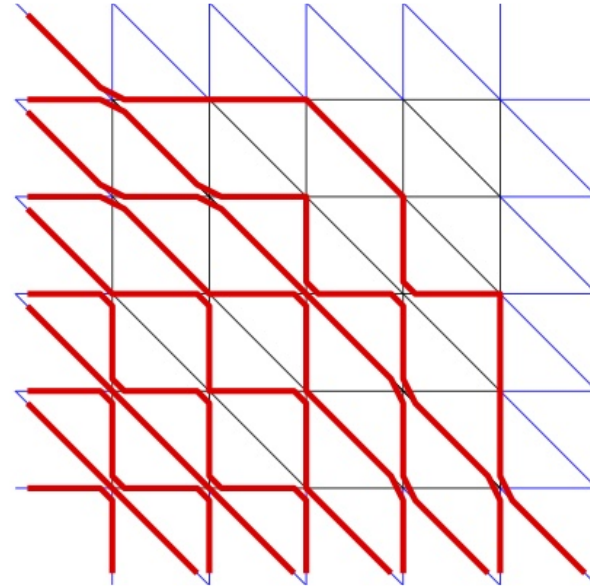
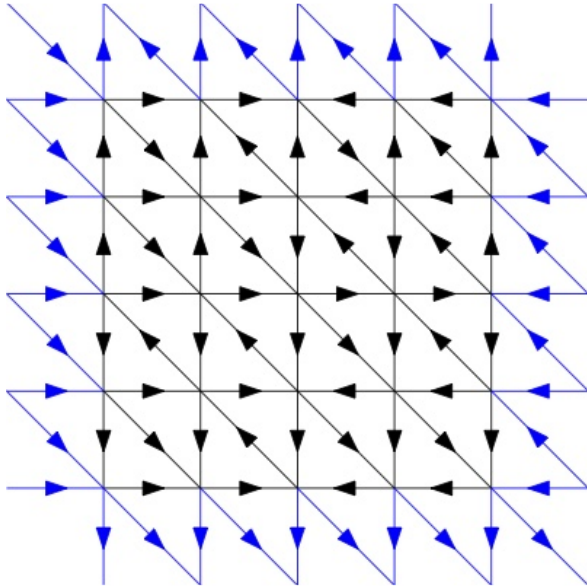


- $\binom{6}{3} = 20$ local environments (ice rule)

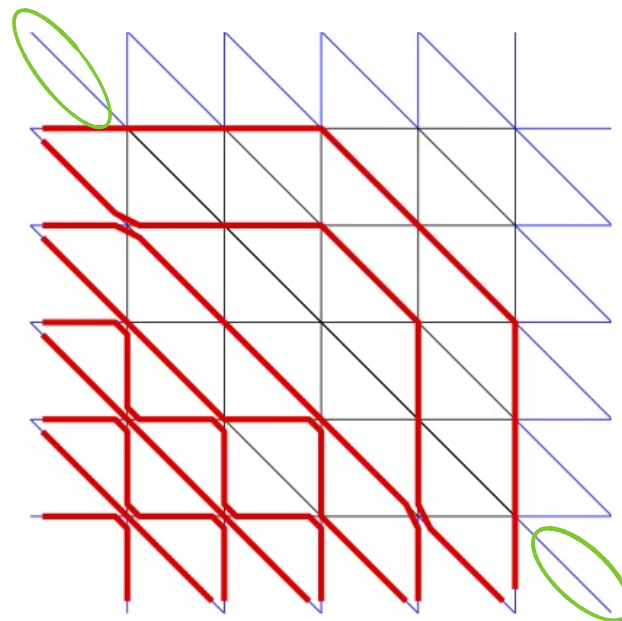
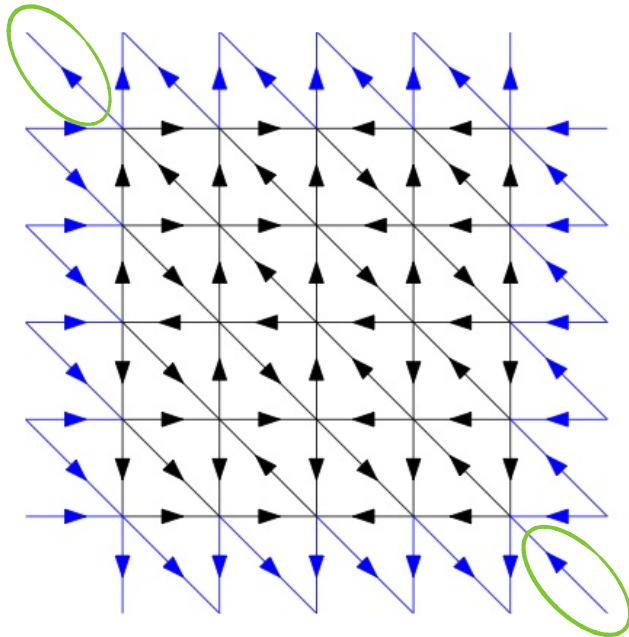


DOMAIN WALL BOUNDARY CONDITIONS

DWBC1



DWBC2



Integrable Weights

- Define 20V weights by resolving triple intersections (cf Yang)

The diagram illustrates the resolution of a triple intersection of three lines labeled z , t , and w . The equation is as follows:

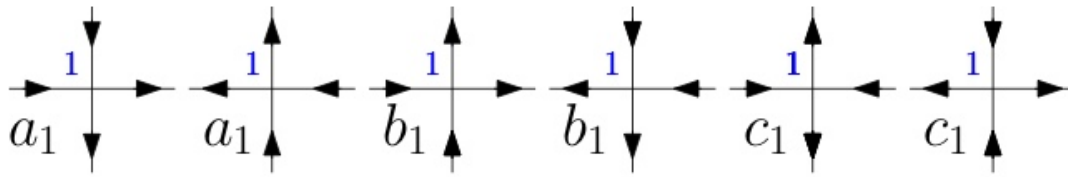
$$\begin{aligned}
 & \text{Triple intersection of } z, t, w \\
 &= \sum_{\alpha, \beta, \gamma} \left(\text{Diagram with points } \alpha, \beta, \gamma \text{ and shaded region} \right) \\
 &= \left(\text{Diagram with point } \alpha \right) \times \left(\text{Diagram with point } \beta \right) \times \left(\text{Diagram with point } \gamma \right)
 \end{aligned}$$

Labels in the diagrams include spectral parameters (a_1, b_1, c_1) , (a_2, b_2, c_2) , and (a_3, b_3, c_3) , and 6V weights $6V_1$, $6V_2$, and $6V_3$.

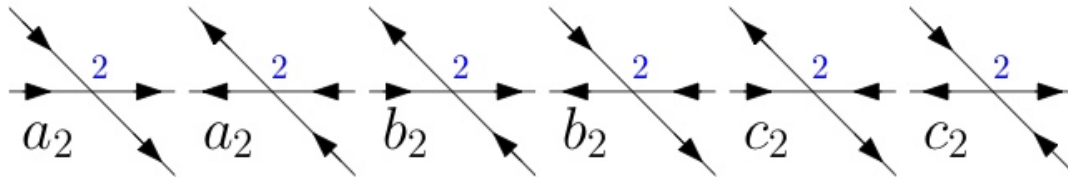
z, t, w = three spectral parameters

Integrable Weights

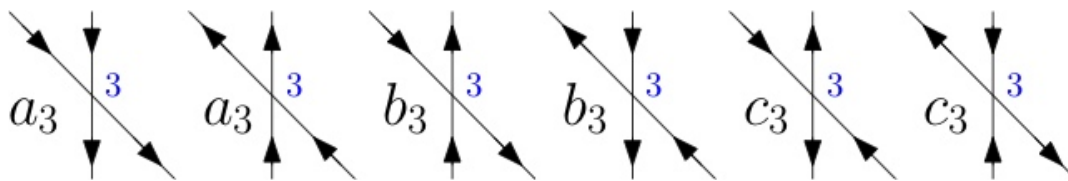
$6V_1$



$6V_2$



$6V_3$



weights of the $6V$
models on the 3
sublattices

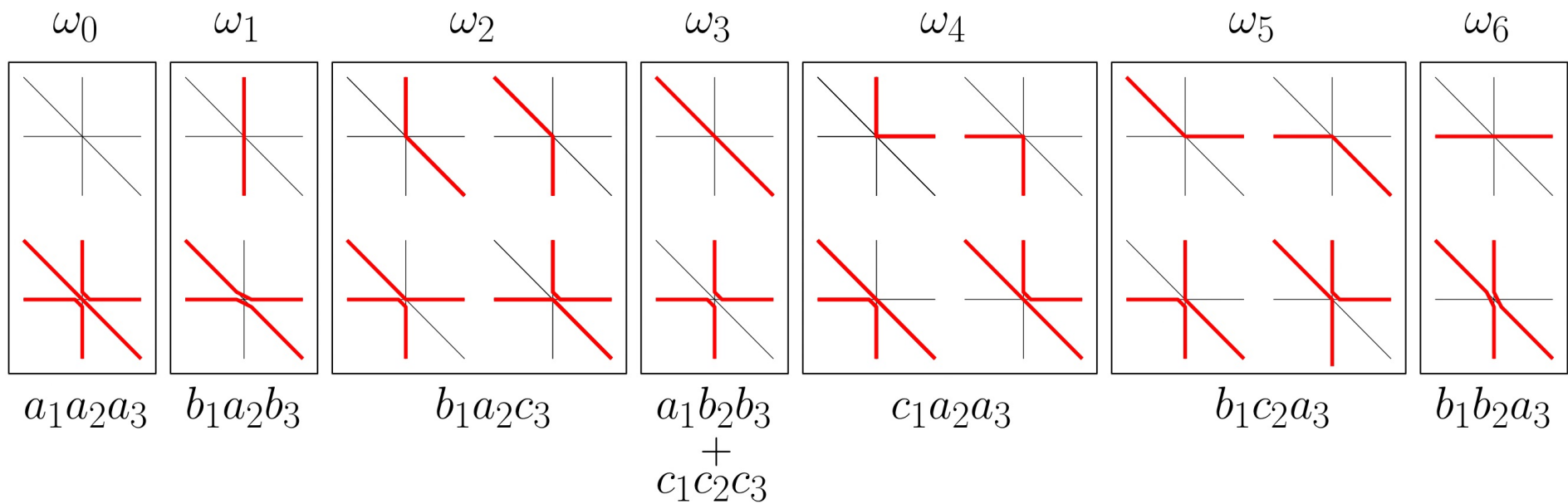
- $20V$ weights are given by sums over inner triangle configs

Example:

$$\begin{array}{c}
 \text{Diagram of a vertex with four incident edges (horizontal, vertical, and two diagonal).} \\
 \text{Diagram of a vertex with four incident edges (horizontal, vertical, and two diagonal).} \\
 \text{Diagram of a vertex with four incident edges (horizontal, vertical, and two diagonal).} \\
 \text{Diagram of a vertex with four incident edges (horizontal, vertical, and two diagonal).}
 \end{array}
 =
 \begin{array}{c}
 \text{Diagram of a vertex with four incident edges (horizontal, vertical, and two diagonal).} \\
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 \text{Diagram of a vertex with four incident edges (horizontal, vertical, and two diagonal).}
 \end{array}
 +
 \begin{array}{c}
 \text{Diagram of a vertex with four incident edges (horizontal, vertical, and two diagonal).} \\
 \text{Diagram of a vertex with four incident edges (horizontal, vertical, and two diagonal).} \\
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 \end{array}
 =
 \begin{array}{c}
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 \text{Diagram of a vertex with four incident edges (horizontal, vertical, and two diagonal).} \\
 \text{Diagram of a vertex with four incident edges (horizontal, vertical, and two diagonal).}
 \end{array}$$

$$w = a_1 b_2 c_3 + c_1 c_2 b_3 = b_1 a_2 c_3$$

- Homogeneous case: 3 parameter family: (z^t, w, q)



$$\omega_0 = \sin(\lambda + \eta) \sin\left(\frac{\lambda + 3\eta + \mu}{2}\right) \sin\left(\frac{\lambda + 3\eta - \mu}{2}\right)$$

$$\omega_1 = \sin(\lambda - \eta) \sin\left(\frac{\lambda - \eta + \mu}{2}\right) \sin\left(\frac{\lambda + 3\eta - \mu}{2}\right)$$

$$\omega_2 = \sin(2\eta) \sin(\lambda - \eta) \sin\left(\frac{\lambda + 3\eta - \mu}{2}\right)$$

$$\omega_3 = \sin(2\eta)^3 + \sin(\lambda + \eta) \sin\left(\frac{\lambda - \eta + \mu}{2}\right) \sin\left(\frac{\lambda - \eta - \mu}{2}\right)$$

$$\omega_4 = \sin(2\eta) \sin\left(\frac{\lambda + 3\eta + \mu}{2}\right) \sin\left(\frac{\lambda + 3\eta - \mu}{2}\right)$$

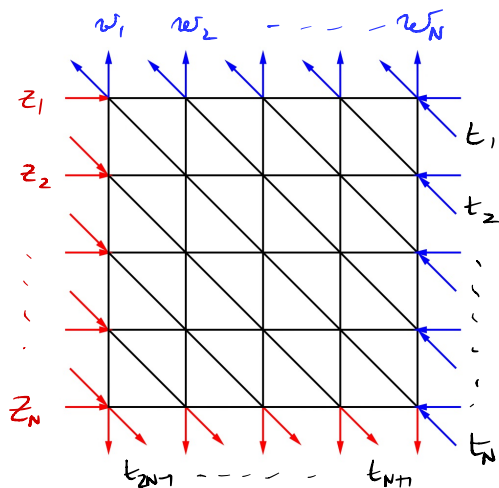
$$\omega_5 = \sin(2\eta) \sin(\lambda - \eta) \sin\left(\frac{\lambda + 3\eta + \mu}{2}\right)$$

$$\omega_6 = \sin(\lambda - \eta) \sin\left(\frac{\lambda + 3\eta + \mu}{2}\right) \sin\left(\frac{\lambda - \eta - \mu}{2}\right),$$

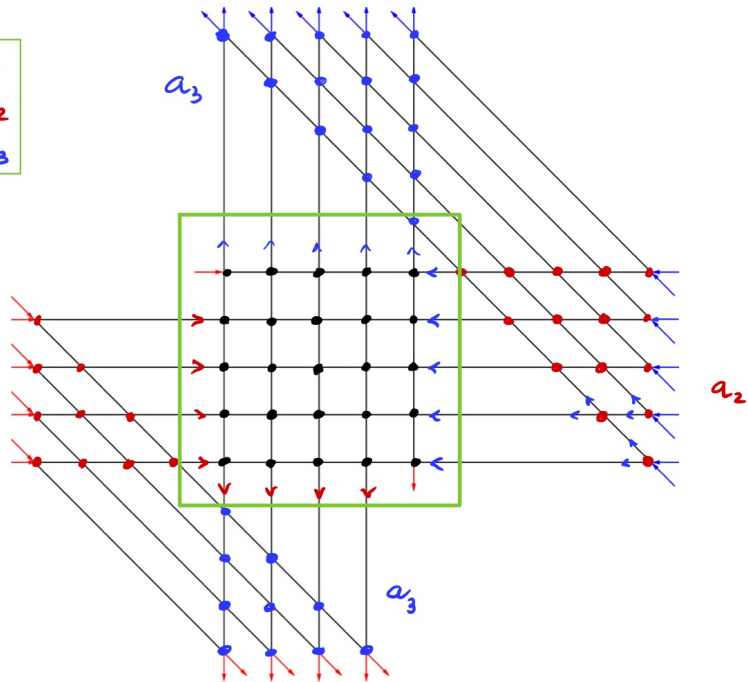
$$\begin{cases} q = e^{i\eta} \\ z = e^{i(\eta + \lambda)} \\ w = e^{-i(\eta + \lambda)} \\ t = e^{i\mu} \end{cases}$$

uniform: $\begin{cases} \eta = \frac{\pi}{8} \\ \lambda = \frac{5\pi}{8} \\ \mu = 0 \end{cases}$

Partition function (Domain-Wall Boundary conditions)



- $6V_1$
- $6V_2$
- $6V_3$



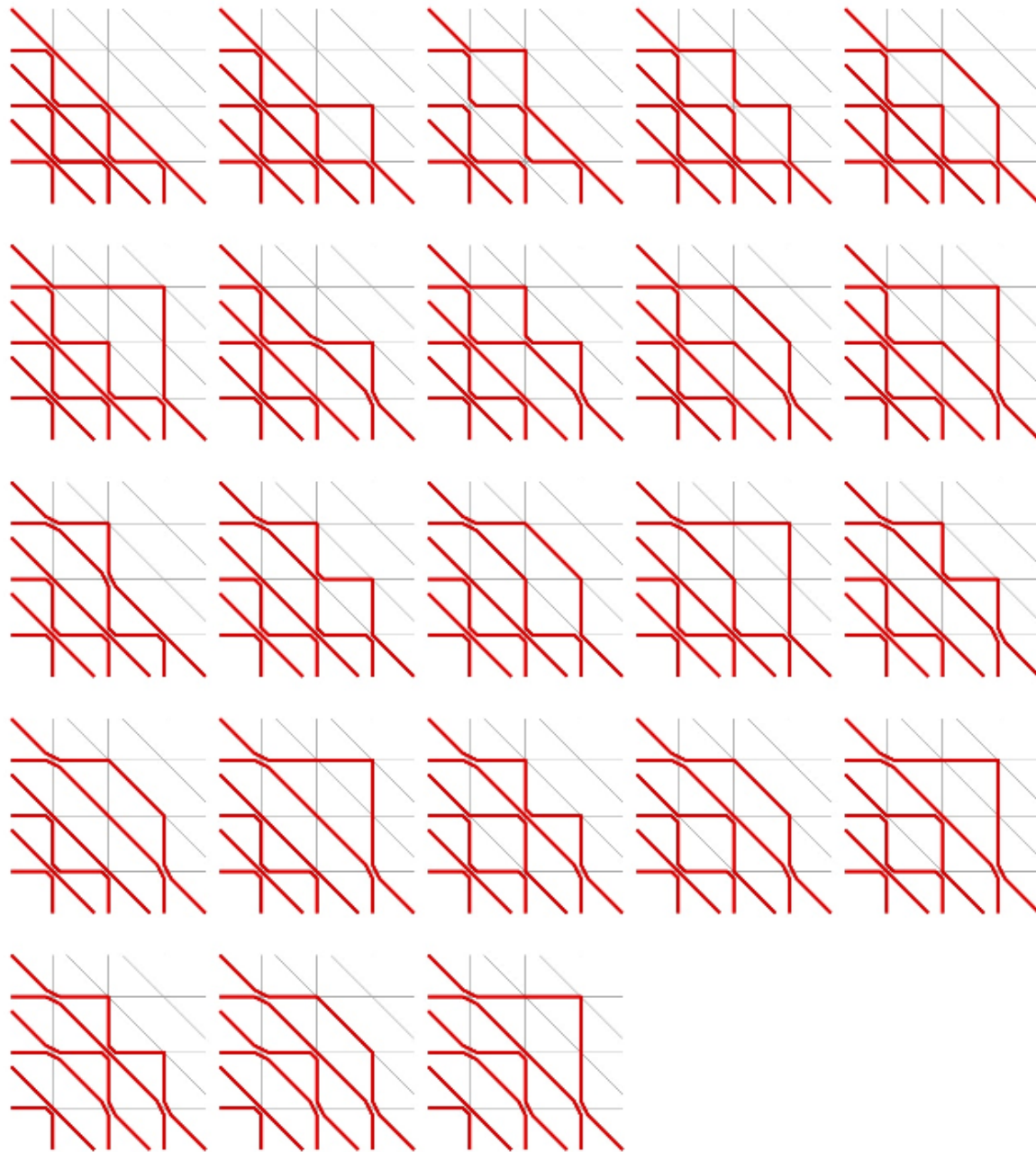
$$Z_{20V} \sim Z_{6V}(a, b, c)$$

$$Z_{\text{uniform } 20V} \leftrightarrow Z(1, \sqrt{2}, 1)_{6V}$$

$$Z_N = 1, 3, \underline{23}, 433, 19705, \dots$$

20v
DWBC1
configurations
 $n=3$

(23)

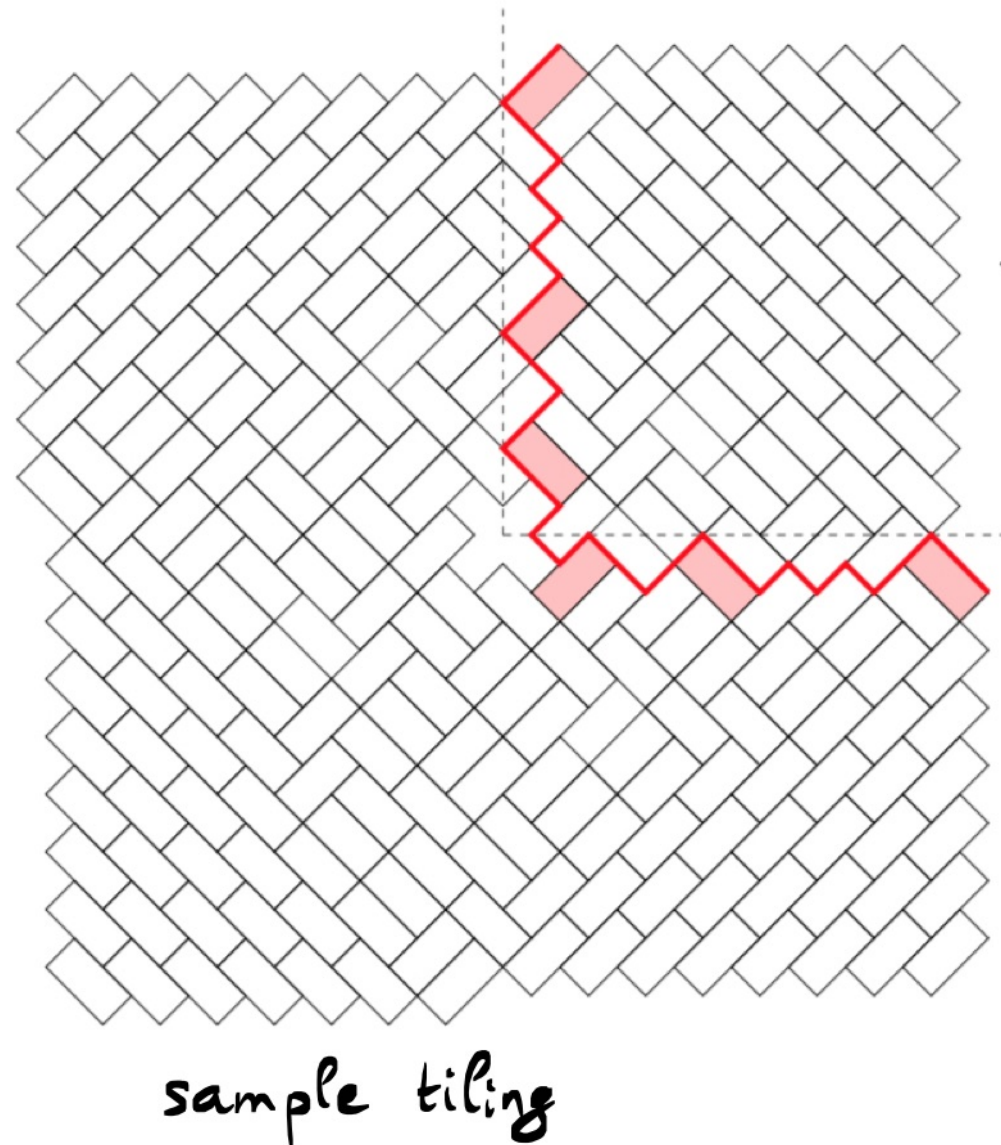
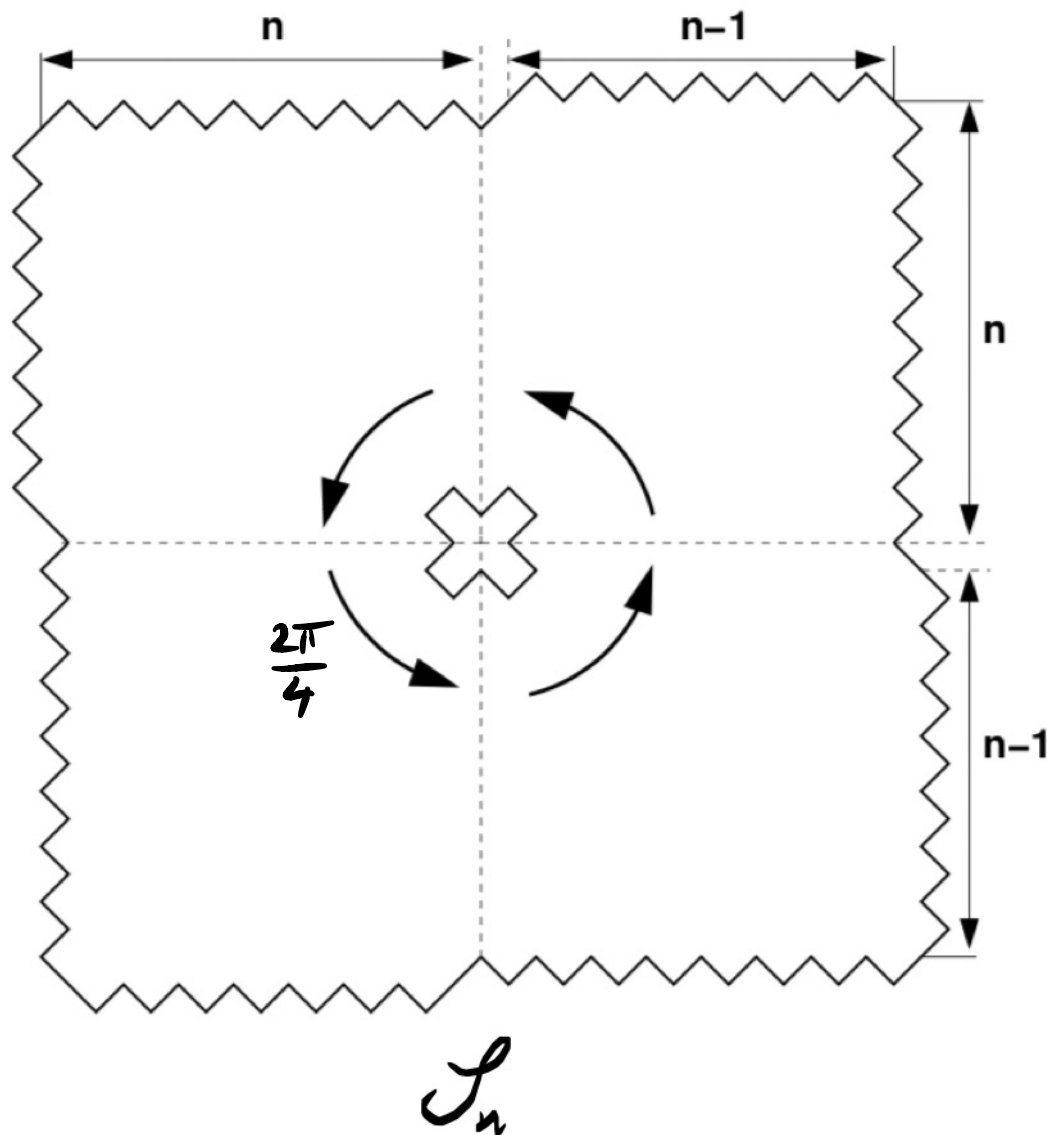


DPP for 20 V ?

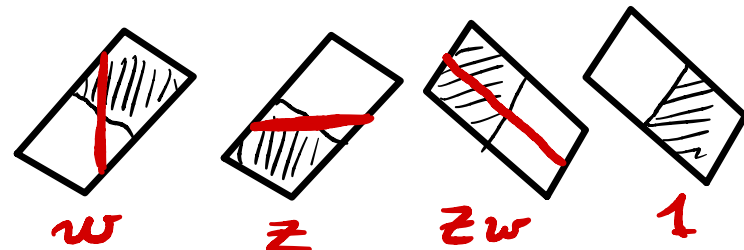


HASDT

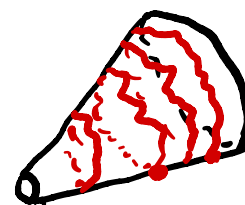
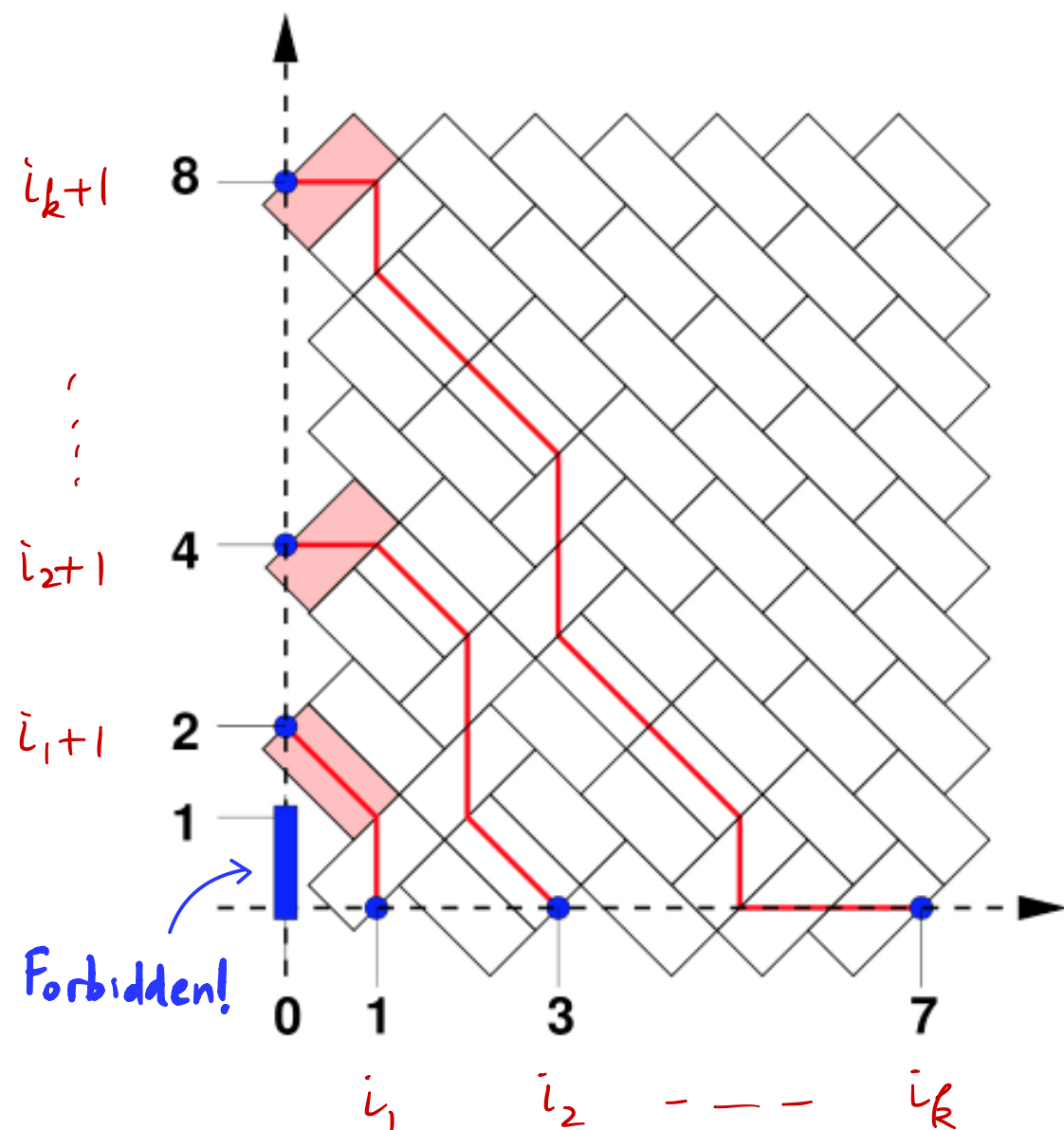
Domino Tilings of The Holy Aztec Square with quarter-turn symmetry.



Counting Configurations



- Non-intersecting Schröder paths w fixed ends
- first step cannot be $|$
- start and ends identified (cone).



$$Z_n = 1, 3, \textcircled{23}, 433, 19705, \dots$$

Counting Configurations : Gessel-Viennot!

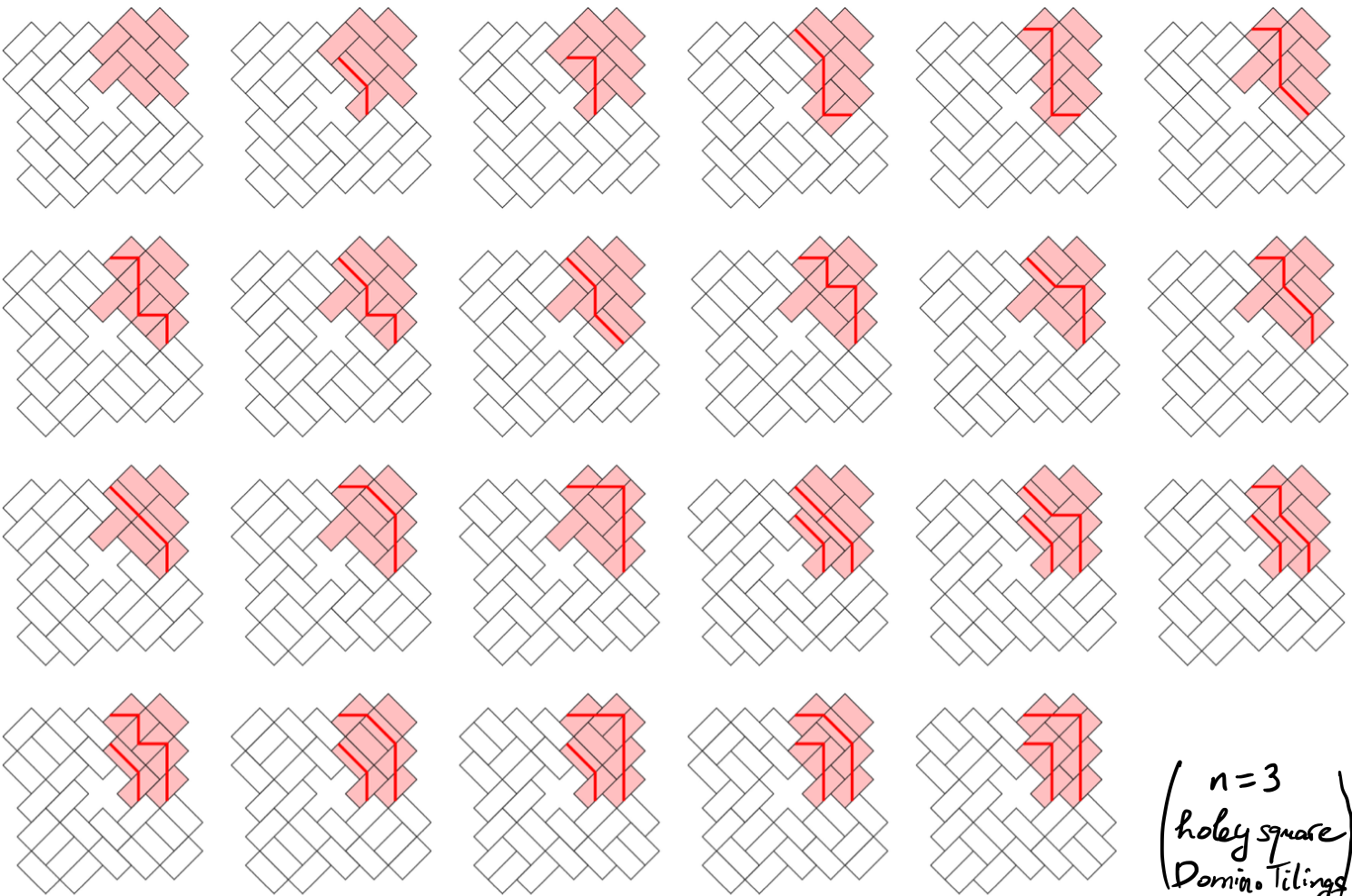
Thm [PDF-Guiller 19]

$$T_4(\mathcal{J}_n) = \det_{1 \leq i, j \leq n} \left(\left\{ \frac{1}{1-zw} + \frac{2z}{(1-z)(1-z-w-zw)} \right\}_{z^i w^j} \right)$$

Proof: (Cauchy) $\det(\text{Id} + M) = \sum_{i_1 < \dots < i_k} |M_{i_1 \dots i_k}^{i_1 \dots i_k}|$
 (Binet) ↓ ↓ ↓
 (Gessel-Viennot)

$$T_4(\mathcal{J}_n) = 1, 3, \textcircled{23}, 433, 19705, 2151843, \dots$$

Ex: $n=3 \quad \det \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 4 & 8 & 12 \end{pmatrix} \right] = \textcircled{23}$ Domino Tiling configurations →



$n=3$
 hole square
 Domino Tilings

THM

[DF+Grütter 19]

$$\begin{aligned} Z_{20V}(\text{uniform}) &= Z_{6V}(1, \sqrt{2}, 1) \\ &= Z_{\text{HASDT}}(\text{uniform}) \end{aligned}$$

Proof same technique as ASM-DPP

$$f_{\text{HASDT}} = \frac{1}{1-zw} + \frac{2z}{(1-z)(1-z-w-zw)}$$

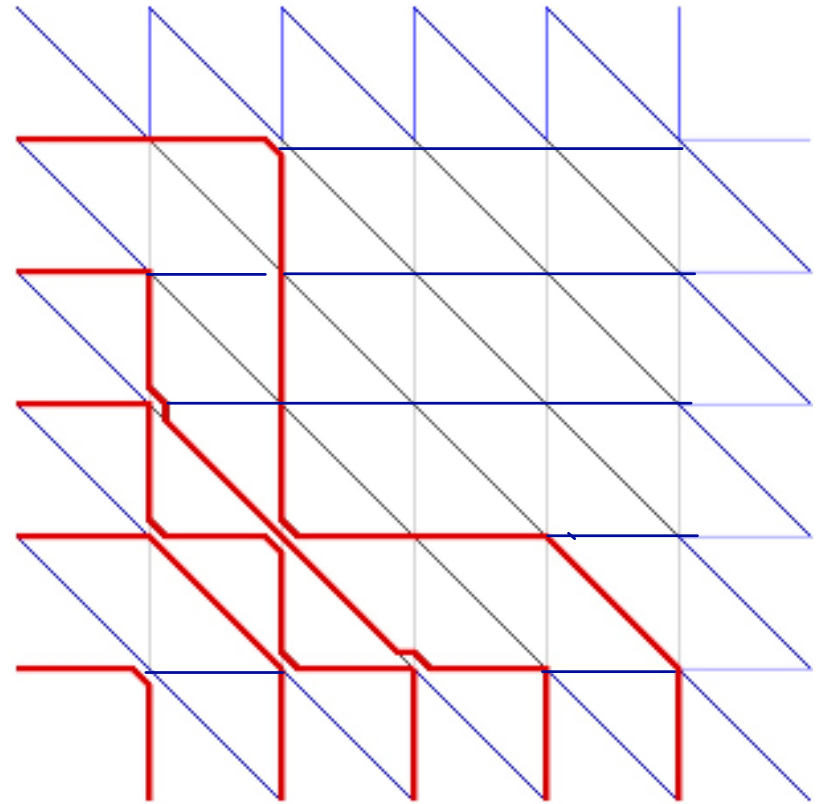
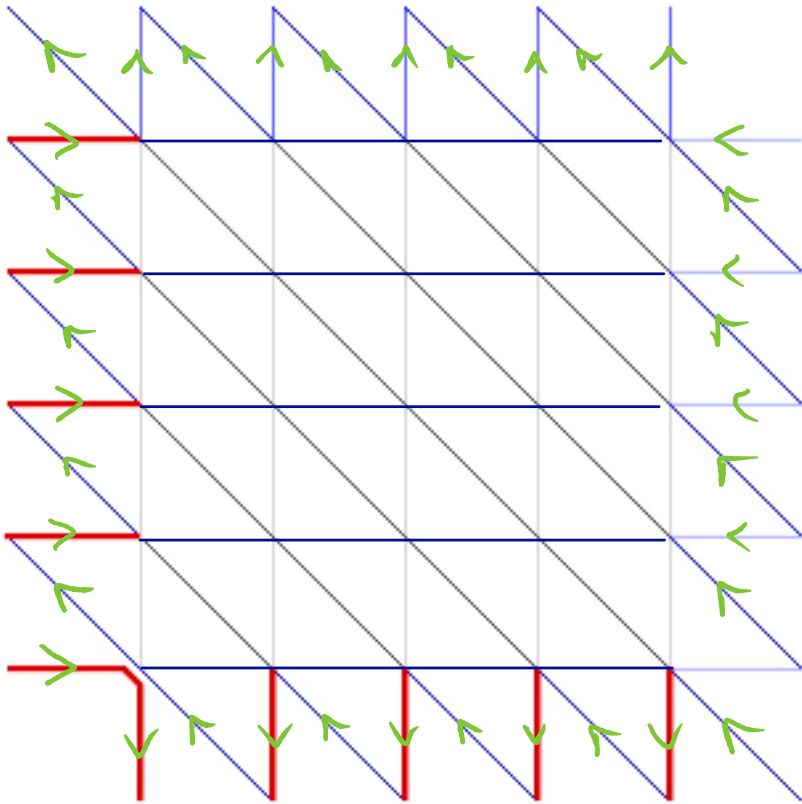
$$f_{6V}(1, \sqrt{2}, 1) = \frac{i}{1-zw} + \frac{1-i}{1-z-w-zw} \quad \begin{pmatrix} x=1 & y=2 \\ r=1-i \end{pmatrix}$$

Proof:

$$(1+iz)(1-w) f_{6V}(1, \sqrt{2}, 1) = (1-z)(1-iw) f_{\text{HASDT}}$$

+ refinements

OTHER DWBC FOR THE 20V MODEL



DWBC 3

1, 3, 29, 901, 89893, ...

The DWBC 3 Conjecture

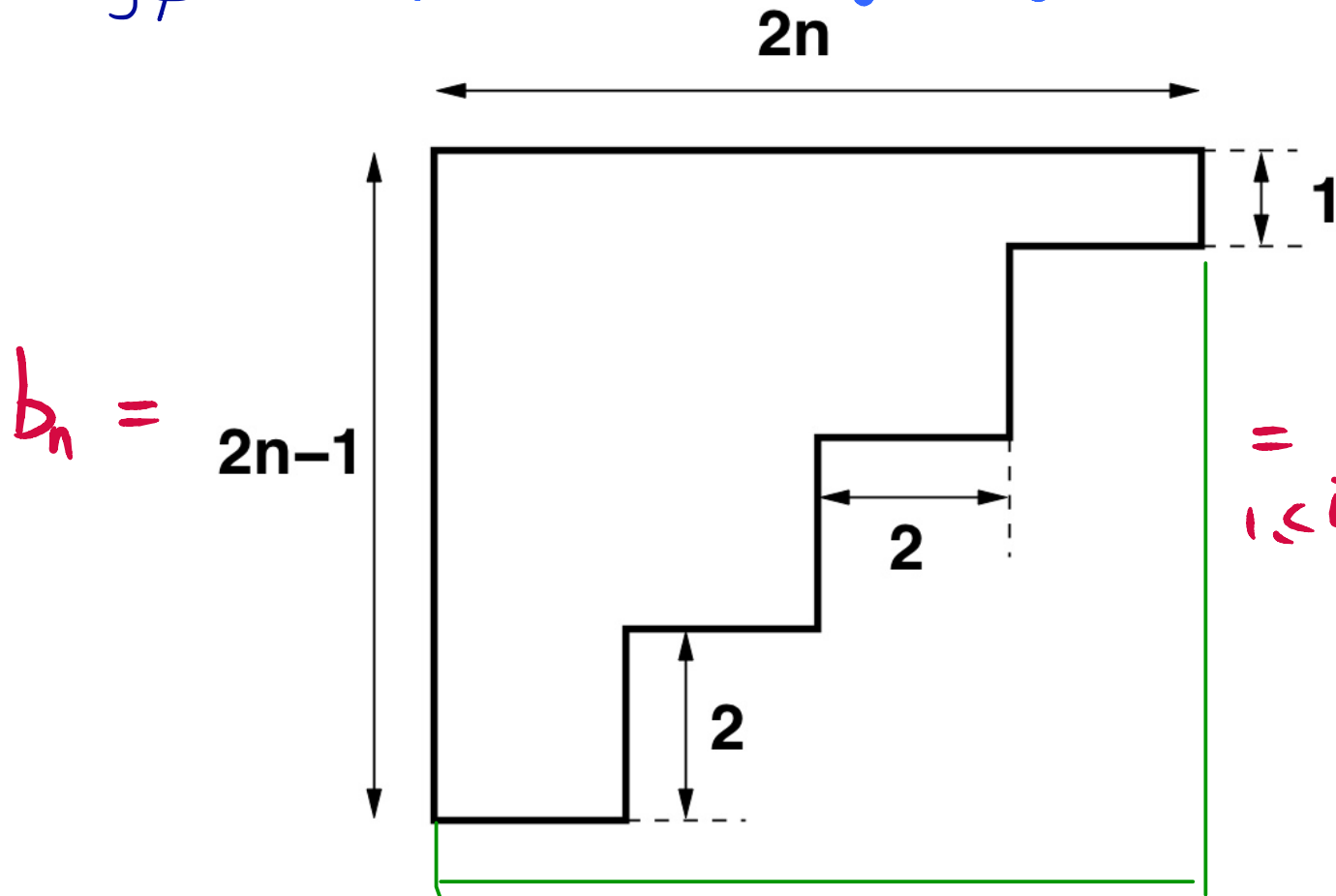
[OEIS for B_n] $\rightarrow$ Domino Tilings of a $2n \times 2n$ square

$$= 2^n b_n^2$$

$$b_n = 1, 3, 29, 901, \dots$$

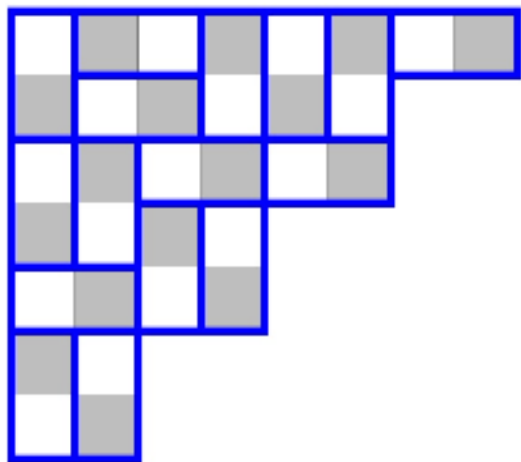
[Temperley-Fisher 61]

[Pachter]₉₇ proof of integrality of b_n found a Domino Tiling interpretation

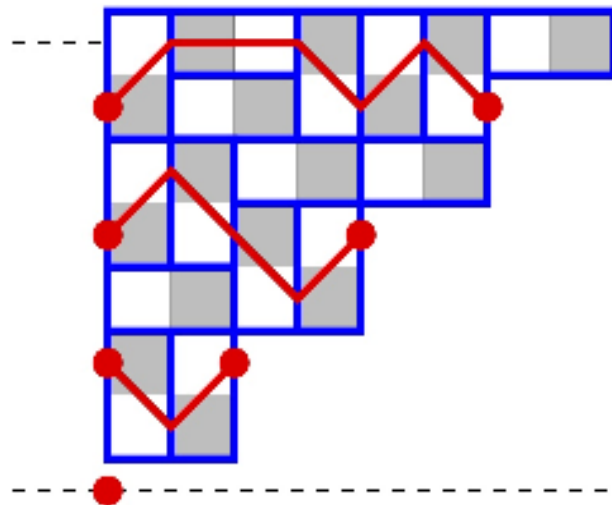


$$b_n = \prod_{1 \leq i < j \leq n} \left(4 \cos^2 \frac{\pi i}{2n+1} + 4 \cos^2 \frac{\pi j}{2n+1} \right)$$

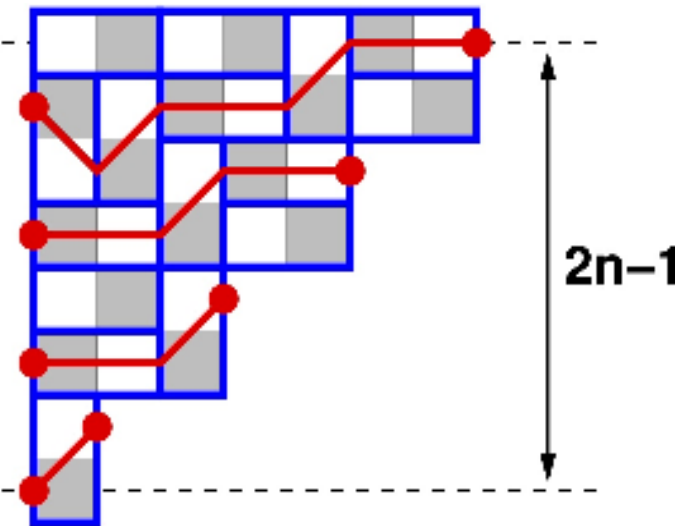
Counting the domino tilings of Patenker's triangle



(a)



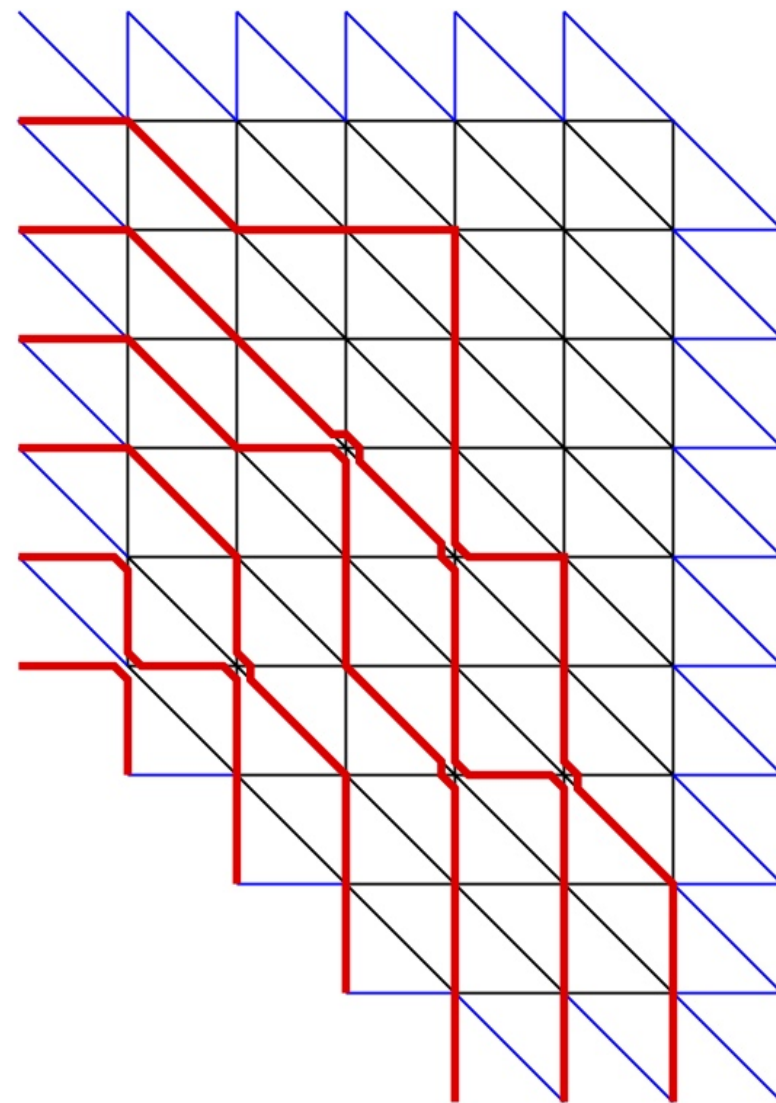
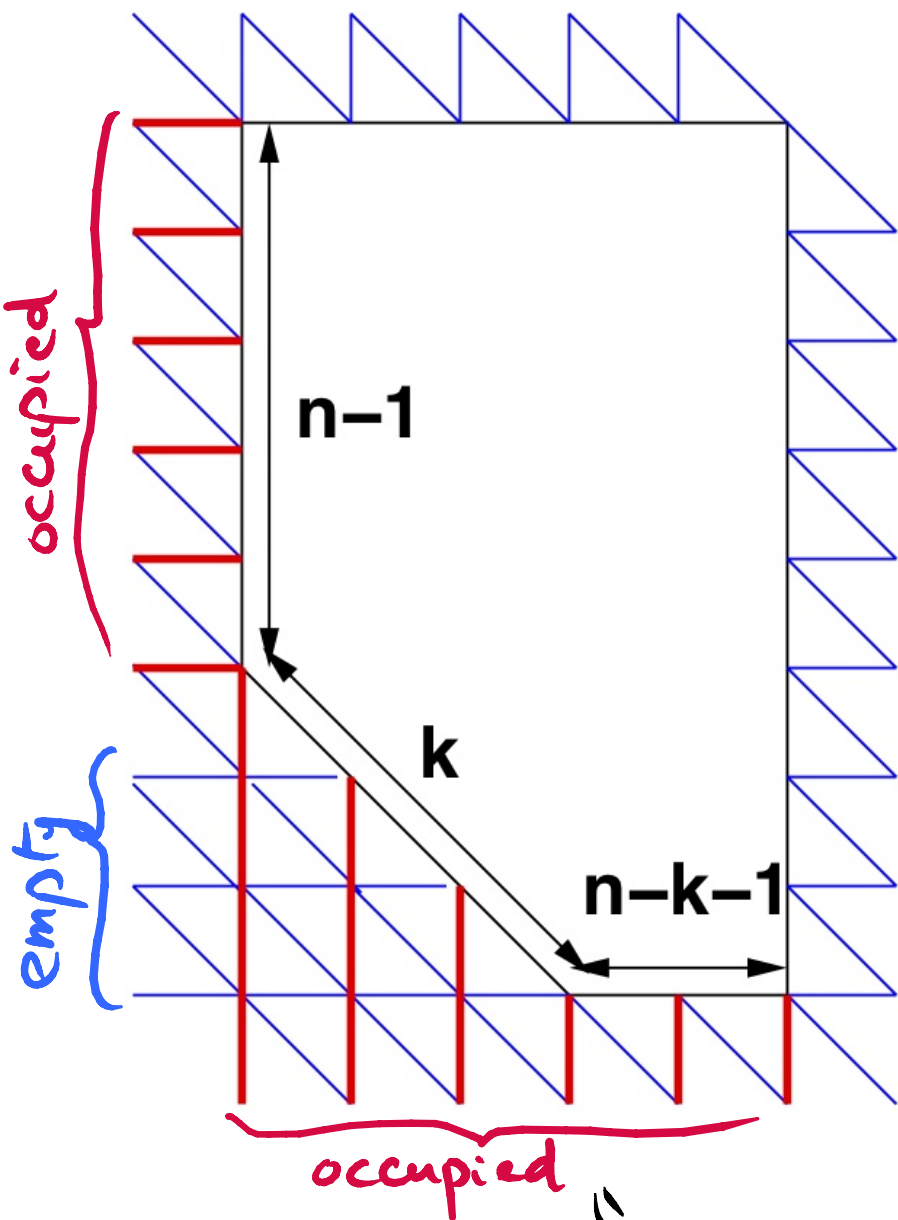
(b)



(c)

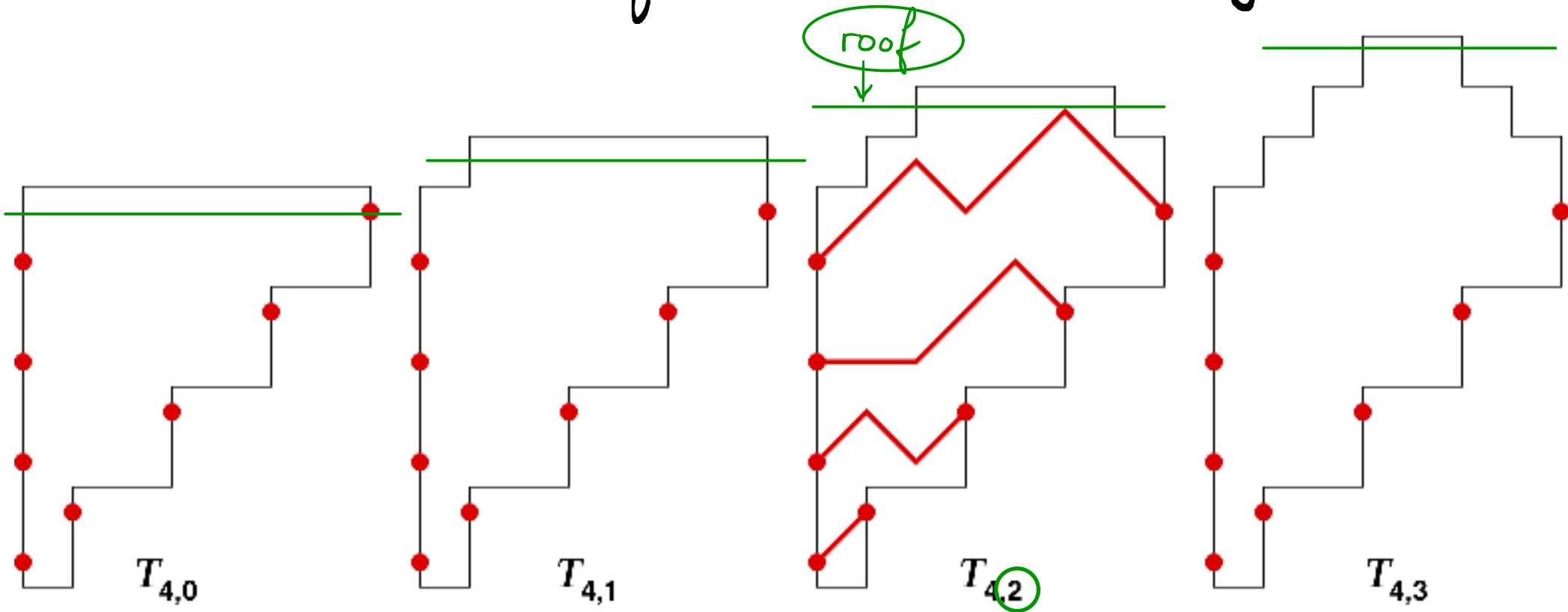
use Gessel-Viennot for Schröder paths under a roof

Result = 1, 3, 29, 901, 89893, 28793575, ...



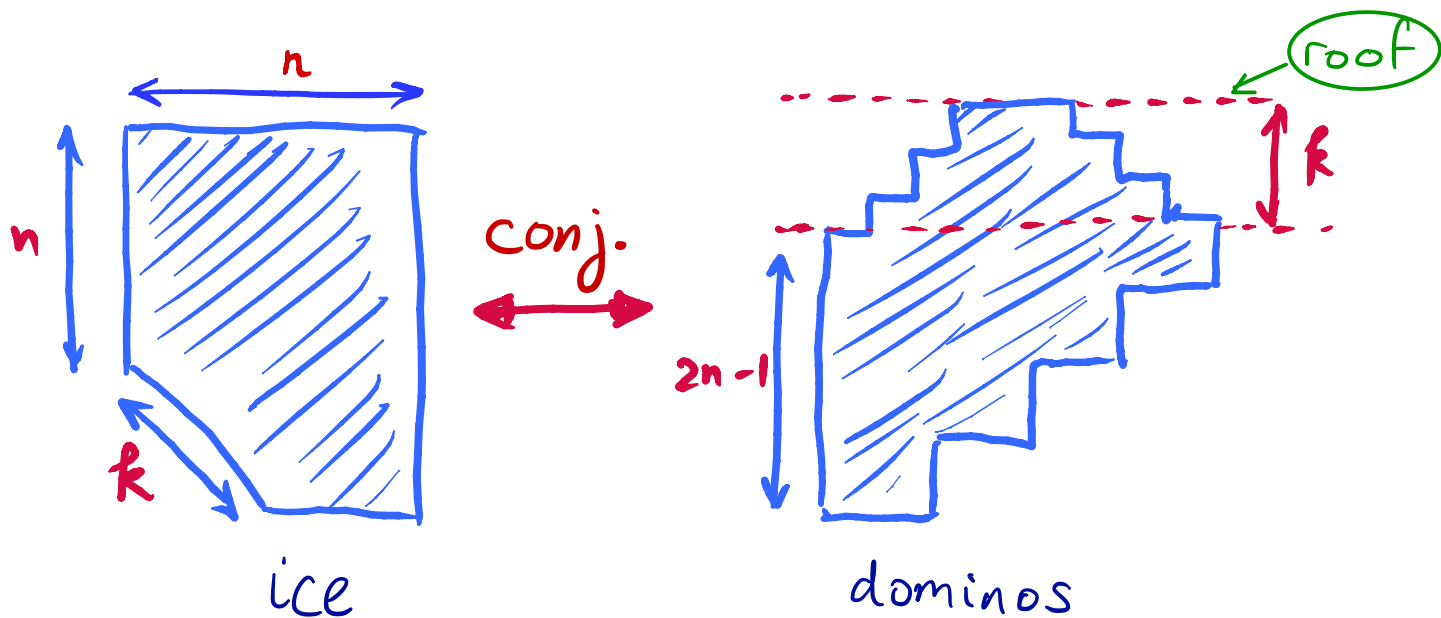
"Pentagon of triangular ice"

"raise the roof" above Packer's triangle

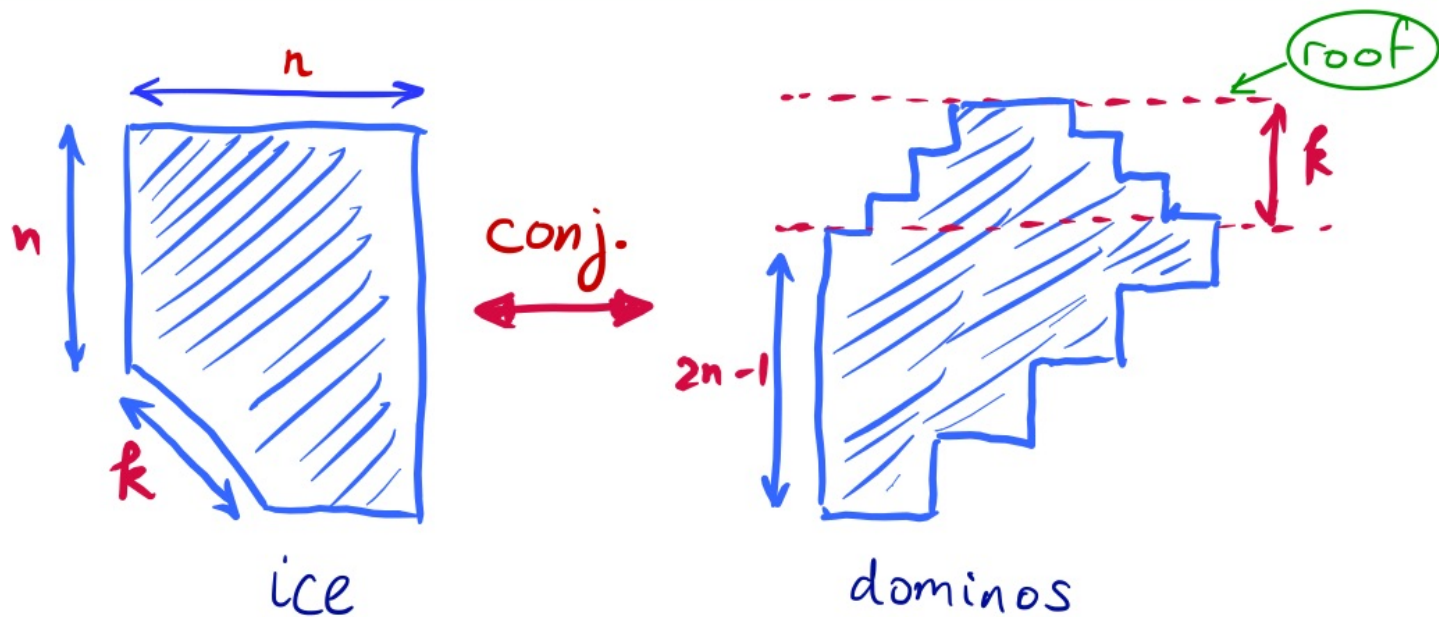


Conjecture still open, except for "no roof"

Conjecture [PDF+E. Guilleter 19] The number of configurations of triangular ice in a pentagon w/ DNBC3 is equal to that of domino tilings of Patchett's raised triangle

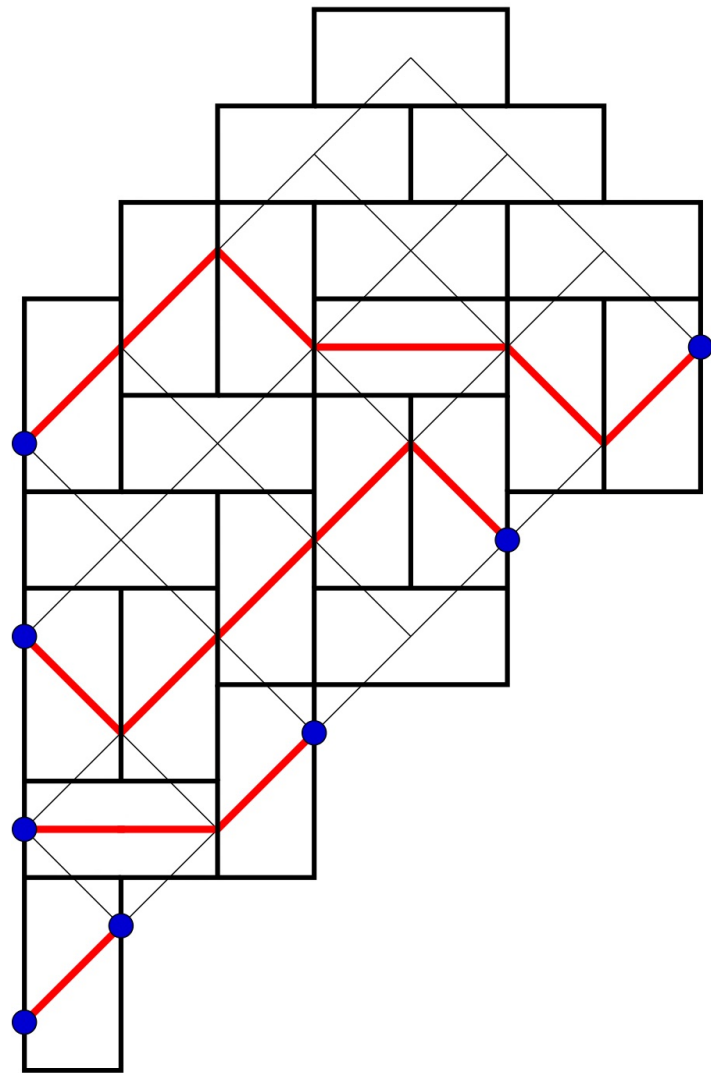
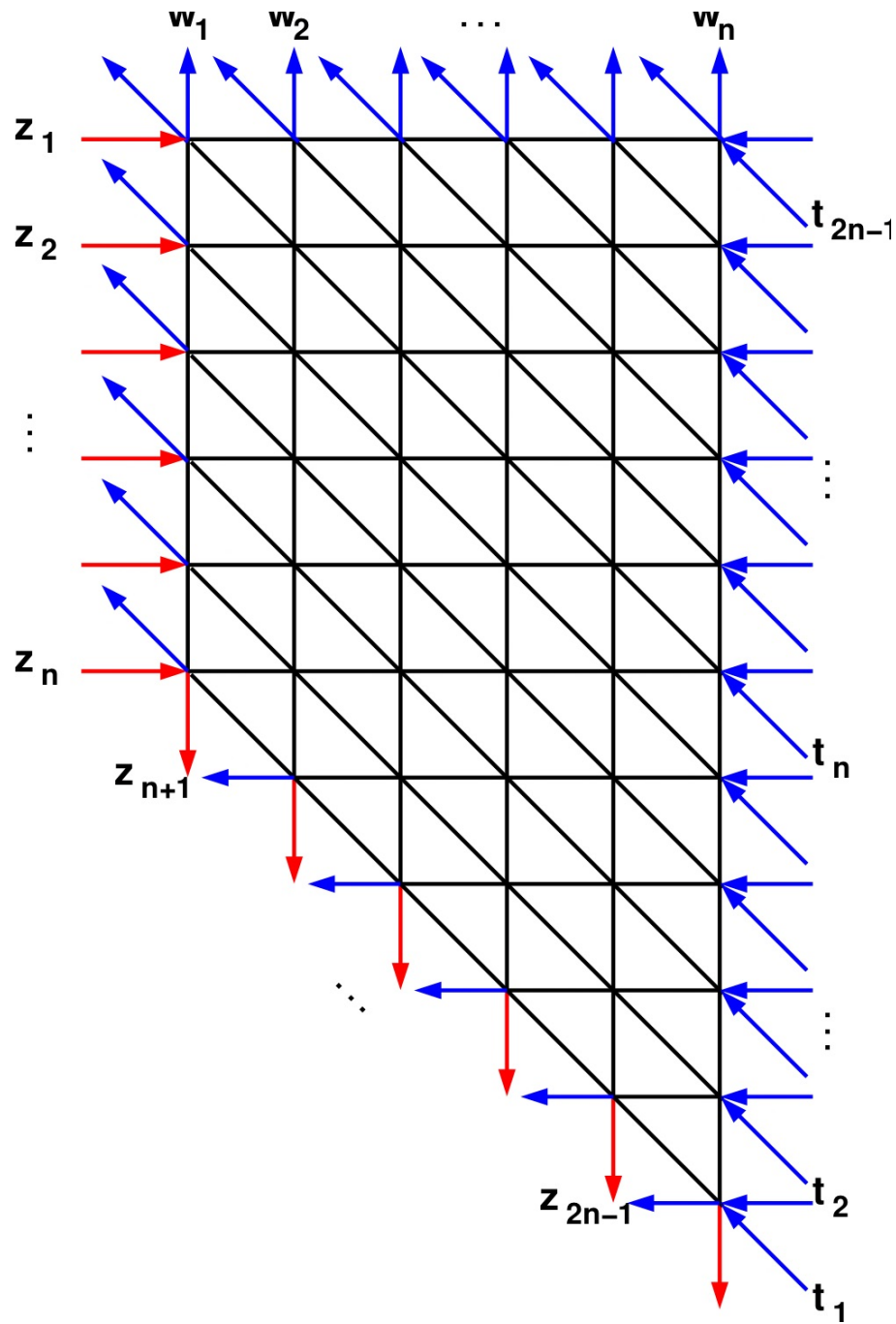


Conjecture [PDF+E. Guittier 19] The number of configurations of triangular ice in a pentagon w/ DNBC3 is equal to that of domino tilings of Patcher's raised triangle



Conjecture still open, except for "no roof" case

20V DWBC 3 $\leftrightarrow$ DT of Pachter Triangle [DF21]



DT of AztecTriangle_n

20V-DWBC 3 on Q_n

$$Z_n = 1, 4, 60, 3328, \dots$$

Thm [Di Francesco 2021]

$$Z_{Q_n}^{20V-DWBC3} = Z_{AT_n}^{DT} = \det_{\substack{\alpha \leq i, j \leq n-1}} \left(\frac{1+u}{(1-uv)^2 - v(1+u)^2} \middle| u^i v^j \right)$$

Conjecture (Now Thm [Koutschan 21] [Corteel-Krattenthaler 23])

$$Z_{Q_n}^{20V-DWBC3} = 2^{\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} \frac{(4j+2)!}{(n+2j+1)!}$$

$$= 1, 4, 60, 3328, 678912, \dots$$

+ refinements see also: [CIUCU '21] [Krattenthaler '21]
 + combinatorial proof [Koutschan-Krattenthaler-Salazar '24]
 [Byun, CIUCU '25]

CONCLUSION

- Integrability and Combinatorics
exact enumeration
free fermions vs interacting fermions

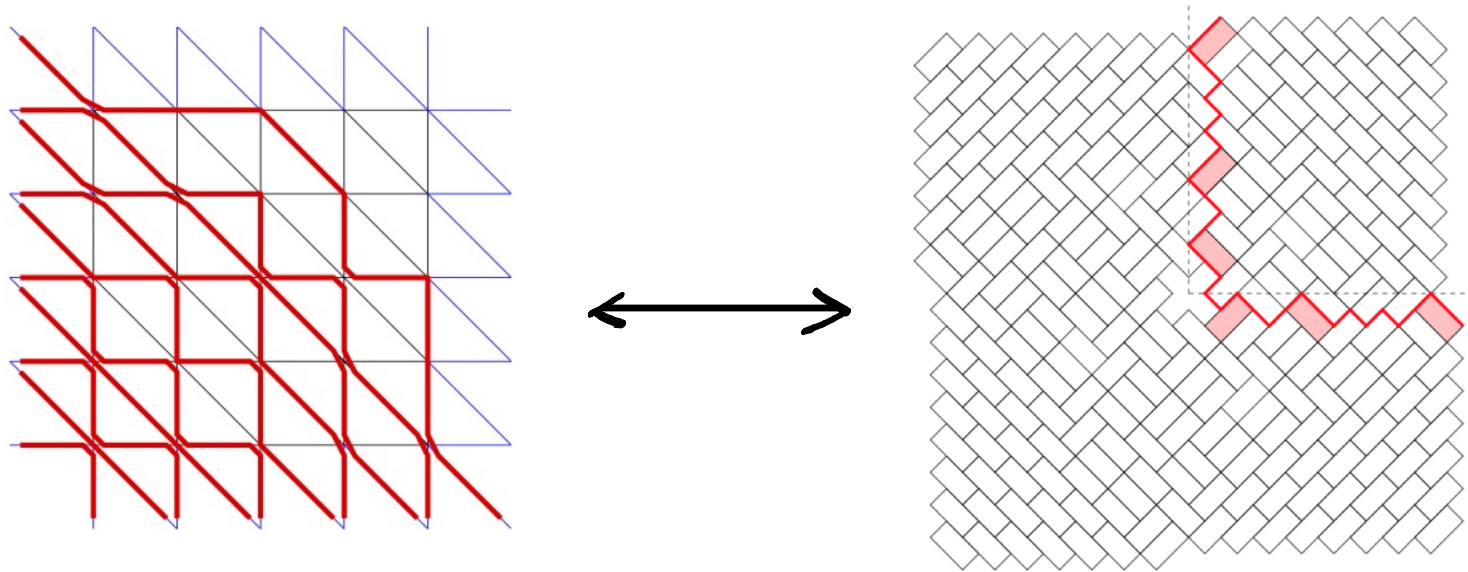
- Non-bijections

6V - DPP - TSSCPP

20V - HAQSDT - ATDT

20V triangle - ?

} However,
related limit
shapes!
(à suivre...)



Ceci n'est pas une bijection.