

INTEGRABLE COMBINATORICS

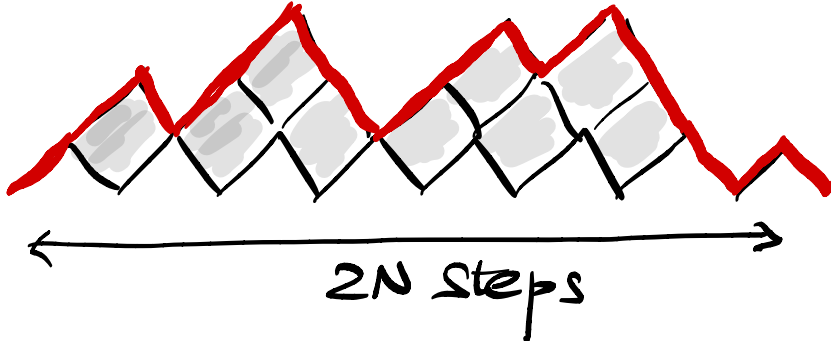
(P. Di Francesco IPhT Saclay / UIUC Math)

- 0. Integrability in combinatorics Structures / Tools
- 1. Lorentzian Gravity Trees / Commuting transfer matrices
- 2. Planar Maps geodesics Trees / discrete integrability
- 3. Cluster Algebra and Dimer Paths / Flat connections
- 4. Into the Non-commutative world NC continued fractions
NC discrete integrability

0. INTEGRABILITY IN COMBINATORICS

- enumerative combinatorics = count objects in classes
- algebraic combinatorics = use algebraic structures

Example: $Z_N = \sum_{\text{paths}}$



weight = $q^{\# \diamond}$

Weyl algebra $\begin{cases} u d = q d u \\ \diamond \end{cases}$ u, d : Weyl generators

Representation: $\text{Vect} \{ |i\rangle, i \in \mathbb{Z}_+ \}$ $\begin{cases} u |n\rangle = |n+1\rangle \\ d |n\rangle = q^{n-1} |n-1\rangle, d |0\rangle = 0 \end{cases}$

$$Z_N = \langle 0 | (d+u)^{2N} | 0 \rangle$$

$$= q\text{-Catalan number} \quad [\text{Carlitz-Riordan '64}]$$

O. INTEGRABILITY IN COMBINATORICS

- enumerative combinatorics = count objects in classes
- algebraic combinatorics = use algebraic structures
- Integrability = existence of sufficiently many symmetries
 - conservation laws that simplify the problem
 - "flat connections"
 - commuting operators to be simultaneously diagonalized.

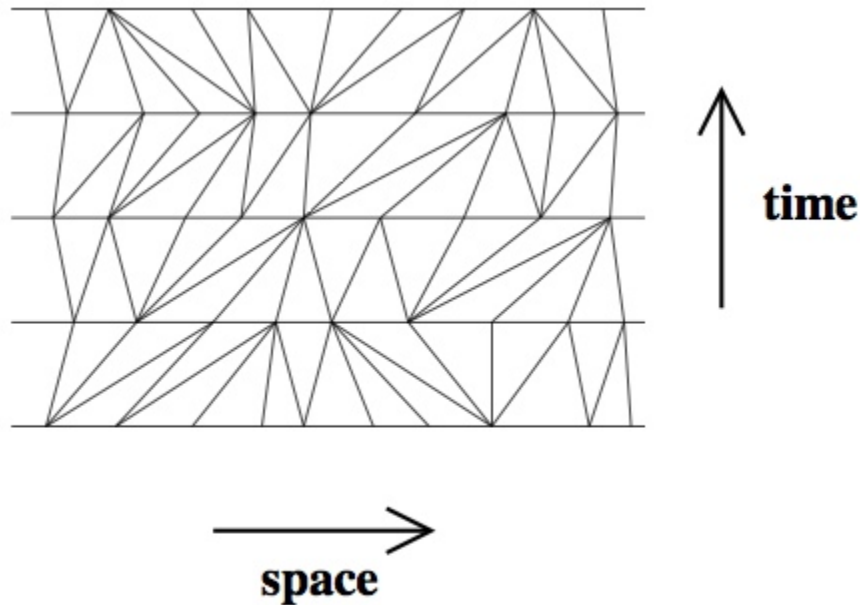
O. INTEGRABILITY IN COMBINATORICS

- OBJECTS :
- Lorentzian Triangulations / Trees (1+1D QG)
 - Planar Maps / Blossom Trees (2D QG)
 - Friezes / Domino Tilings / Paths (Cluster Alg.)
 - Vertex Models / Loop Models (Stat Phys.)

- TOOLS
- Generating functions, Bijections
 - Commuting transfer matrices
 - Networks
 - Yang-Baxter equation $\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}$

1. 1+1-Dimensional Lorentzian Gravity

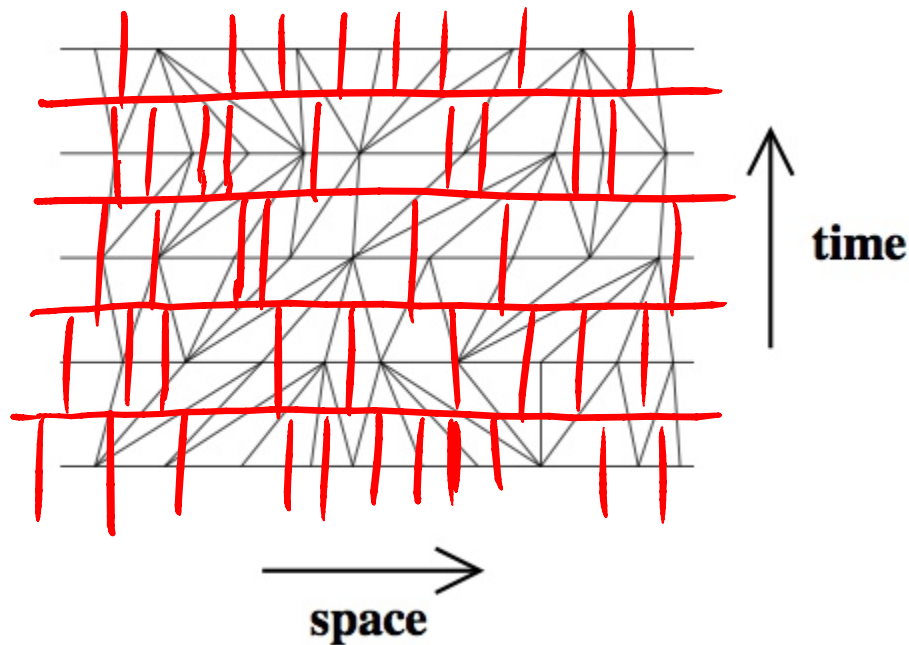
[PDF + Emmanuel Günter + Charlotte Kristjansen '99]
[Ambjorn + Loll + Watabiki + Jurkiewicz + Kristjansen].



Triangulations that are

1. Random in space direction
2. Regular in time direction

1+1 Dimensional Lorentzian Gravity

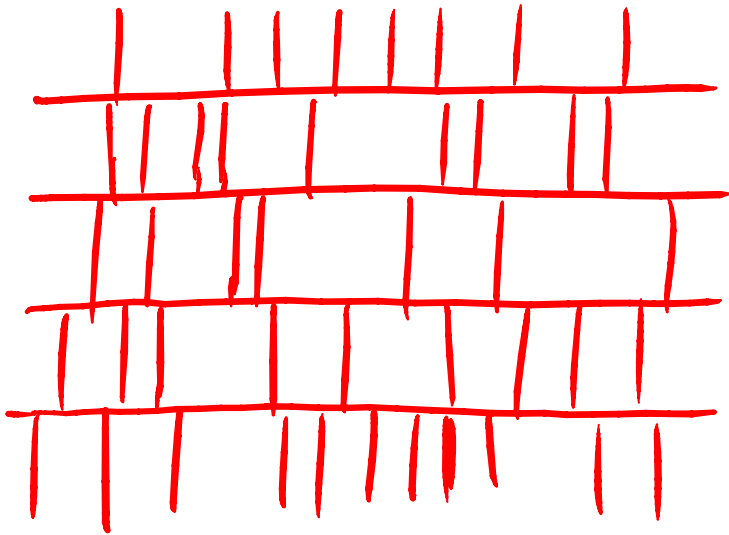


Dual graph
(in red)

Triangulations that are

1. Random in space direction
2. Regular in time direction

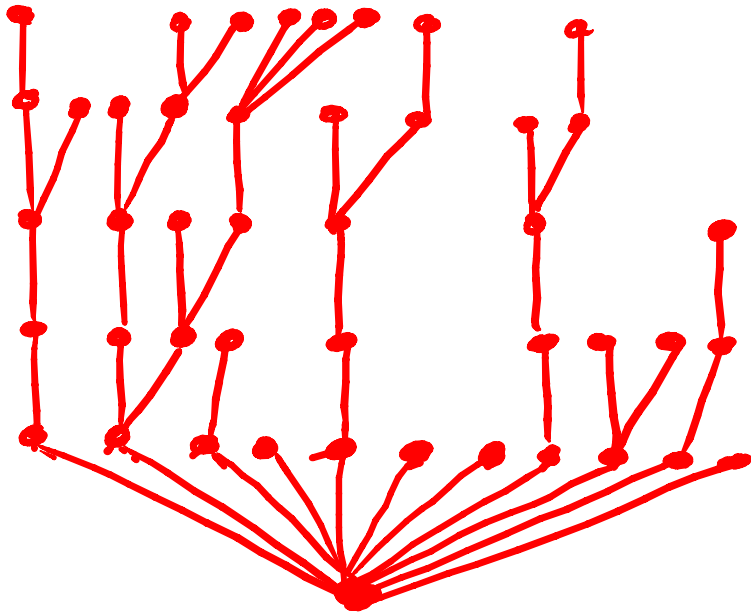
1+1 Dimensional Lorentzian Gravity



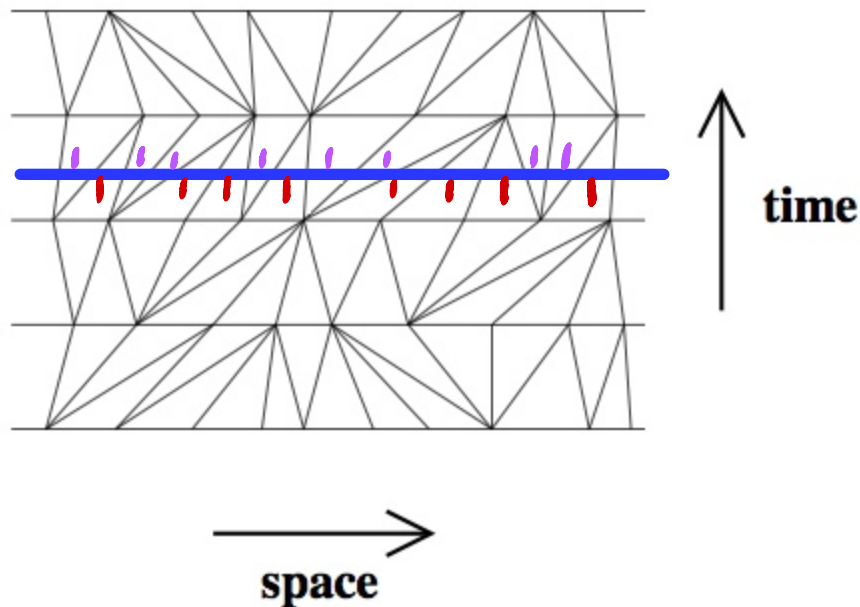
Dual = trivalent graph

→ squeeze ←
(push left)

1+1 Dimensional Lorentzian Gravity



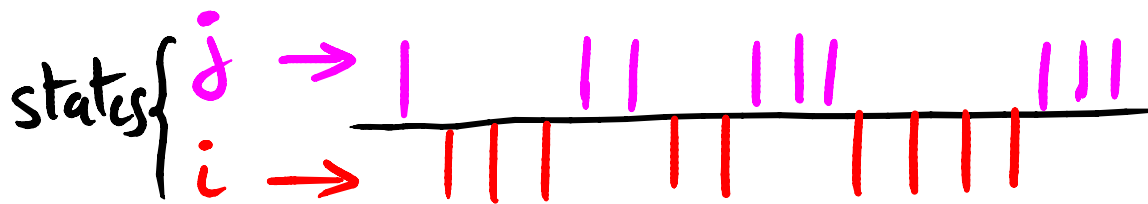
$\equiv$ TREE
(Basic Combinatorial
object)



Triangulations that are

1. Random in space direction
2. Regular in time direction

⇒ TRANSFER MATRIX

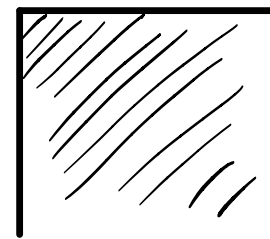


$$T_{ij} = \binom{i+j}{i}$$

$$(i, j \in \mathbb{Z}_+)$$

↑
INFINITE SIZE MATRIX!

- $T =$ infinite (corner) matrix



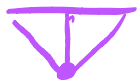
- generating function:

$$f_T(z, w) = \sum_{i, j \geq 0} z^i w^j T_{i, j} = \frac{1}{1 - z - w}$$

- Matrix Product = Convolution of gen. fctns.

$$\begin{aligned} f_{AB}(z, w) &= \sum_{i, j \geq 0} z^i w^j (AB)_{i, j} \\ &= \sum_{i, j, k \geq 0} z^i t^k A_{i, k} w^j t^{-k} B_{k, j} \\ &= \oint_{2i\pi t} \frac{dt}{t} f_A(z, t) f_B(t^{-1}, w) \\ &= f_A * f_B(z, w) \end{aligned}$$

- Vectors $f_v(z) = \sum_{i \geq 0} z^i v_i$ etc.

- Include $\left\{ \begin{array}{l} \text{curvature weight } a \end{array} \right.$   or Π
 $\left\{ \begin{array}{l} \text{area weight } g \end{array} \right.$   or $\top$

Then

$$T_{ij}(g, a) = (ag)^{i+j} \sum_{k=0}^{\min(i,j)} \binom{i}{k} \binom{j}{k} a^{-2k}$$

or Generating Function (of the matrix entries):

$$\sum_{i,j \geq 0} z^i w^j T_{ij}(g, a) = \frac{1}{1 - ga(z+w) - g^2(1-a^2)zw}$$

$$= f_T(z, w)$$

INTEGRABILITY

Thm

$$[T(g, a), T(g', a')] = 0$$

$$\Leftrightarrow \varphi(g, a) = \varphi(g', a') = \text{const.}$$

$$\varphi(g, a) = \frac{1 - g^2(1 - a^2)}{ag}$$

$\Rightarrow$ COMMUTING TRANSFER MATRICES!

⇒ ONE-PARAMETER FAMILY OF COMMUTING TRANSFER MATRICES

set $\varphi(a, g) = q + q^{-1}$; $\lambda = \frac{1 - q^{-1}ag}{1 - qag}$

⇒ for fixed q , $T(a, g) = T_q(\lambda)$ 1-parameter commuting family

$$\oint T_q(\lambda)(z, w) = \frac{1 - \lambda q^2}{(1 - qz)(1 - qw) - \lambda(q - z)(q - w)}$$

$$= \frac{1 - \lambda q^2}{1 - q^2} \frac{\sqrt{1 - q^2}}{1 - qz} \frac{\sqrt{1 - q^2}}{1 - qw} \frac{1}{1 - \lambda \frac{q - z}{1 - qz} \frac{q - w}{1 - qw}}$$

⇒ ONE-PARAMETER FAMILY OF COMMUTING TRANSFER MATRICES

set $\varphi(a, g) = q + q^{-1}$; $\lambda = \frac{1 - \bar{q}^{-1} a g}{1 - q a g}$

⇒ for fixed q , $T(a, g) = T_q(\lambda)$ 1-parameter commuting family

$$f_{T_q(\lambda)}(z, w) = \frac{\frac{1 - \lambda q^2}{1 - q^2} \frac{\sqrt{1 - q^2}}{1 - qz} \frac{\sqrt{1 - q^2}}{1 - qw}}{1 - \lambda \frac{q - z}{1 - qz} \frac{q - w}{1 - qw}} \quad 1$$

$$f_{T_q(\lambda)}(z, w) = \sum_{m \geq 0} \underbrace{\frac{1 - \lambda q^2}{1 - q^2} \lambda^m}_{m\text{-th eigenvalue}} \underbrace{\frac{\sqrt{1 - q^2}}{(1 - qz)^{m+1}} (q - z)^m}_{f_{\sqrt{(m)}}(z)} \underbrace{\frac{\sqrt{1 - q^2}}{(1 - qw)^{m+1}} (q - w)^m}_{f_{\sqrt{(m)}}(w)}$$

SPECTRAL DECOMPOSITION!

SOLUTION

Partition function for Triangulations w i lower and j upper triangles and N time slices

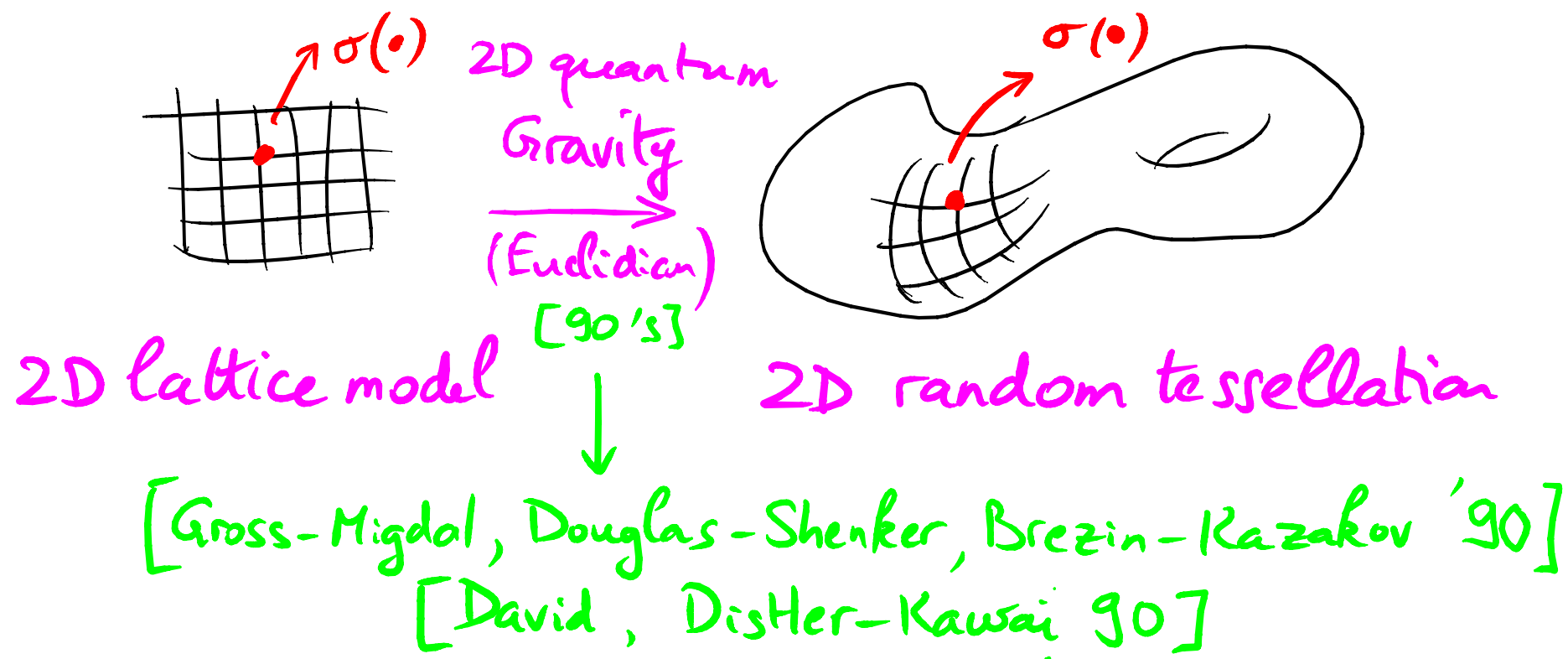
$$Z_{ij} = \text{[Diagram of a triangulated cylinder with } i \text{ lower and } j \text{ upper triangles and } N \text{ time slices]} = \sum_{m \geq 0} v_i^{(m)} \left(\frac{1 - \lambda q^2}{1 - q^2} \lambda^m \right)^N v_j^{(m)}$$

$$\text{or } \sum_{i,j} Z_{ij} z^i w^j = \frac{1 - q^2}{(1 - qz)(1 - qw)} \frac{1}{1 - \left(\frac{1 - \lambda q^2}{1 - q^2} \lambda \right)^N \left(\frac{q - z}{1 - qz} \frac{q - w}{1 - qw} \right)}$$

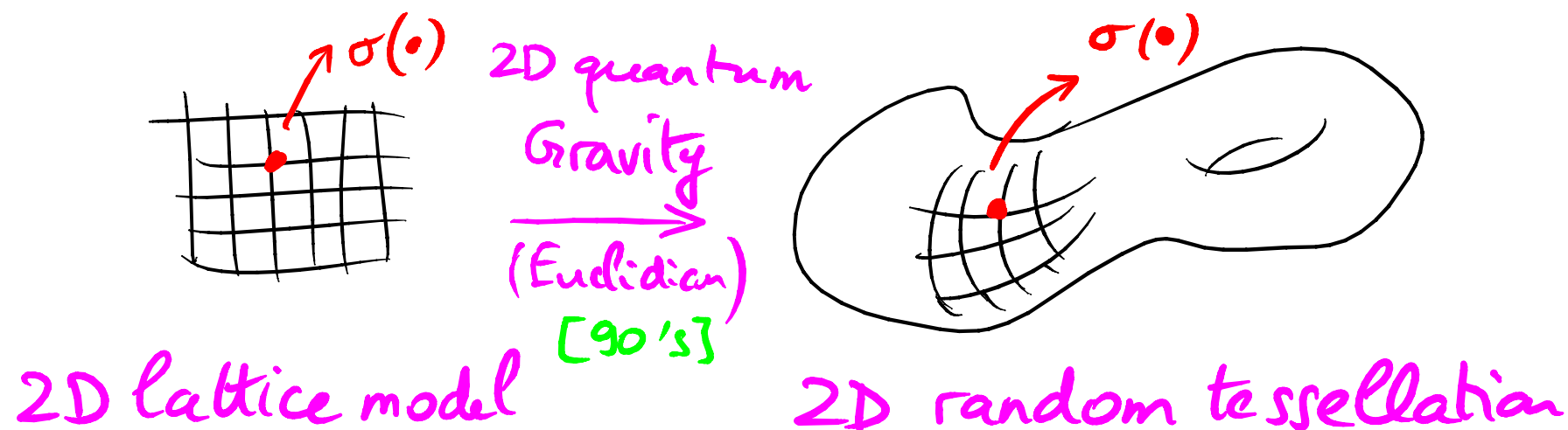
→ continuum limit, scaling functions, etc.

$$(g, a) \mapsto \left(\frac{1 - g^2(1 - a^2)}{ga} = q + q^{-1}, \quad \frac{1 - q^{-1}ag}{1 - qag} = \lambda \right)$$

2. PLANAR MAPS & GEODESICS

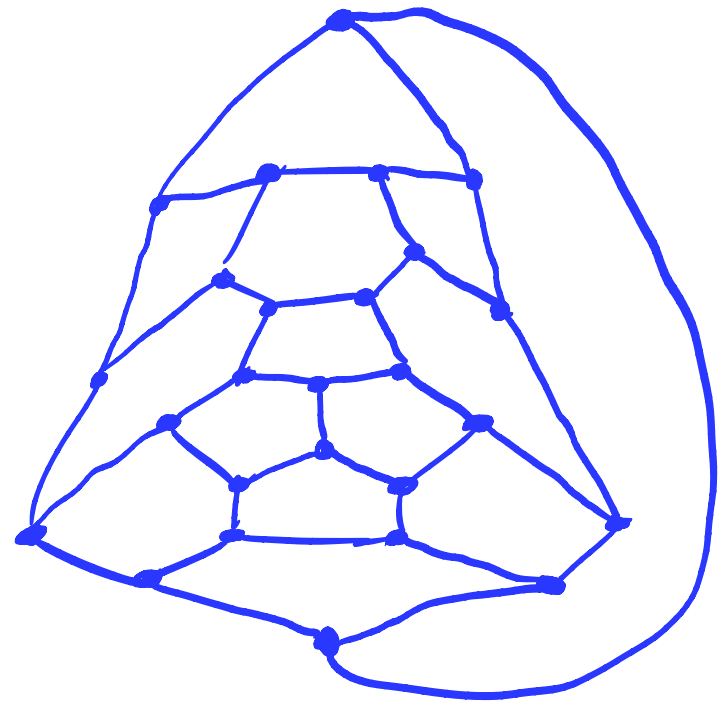
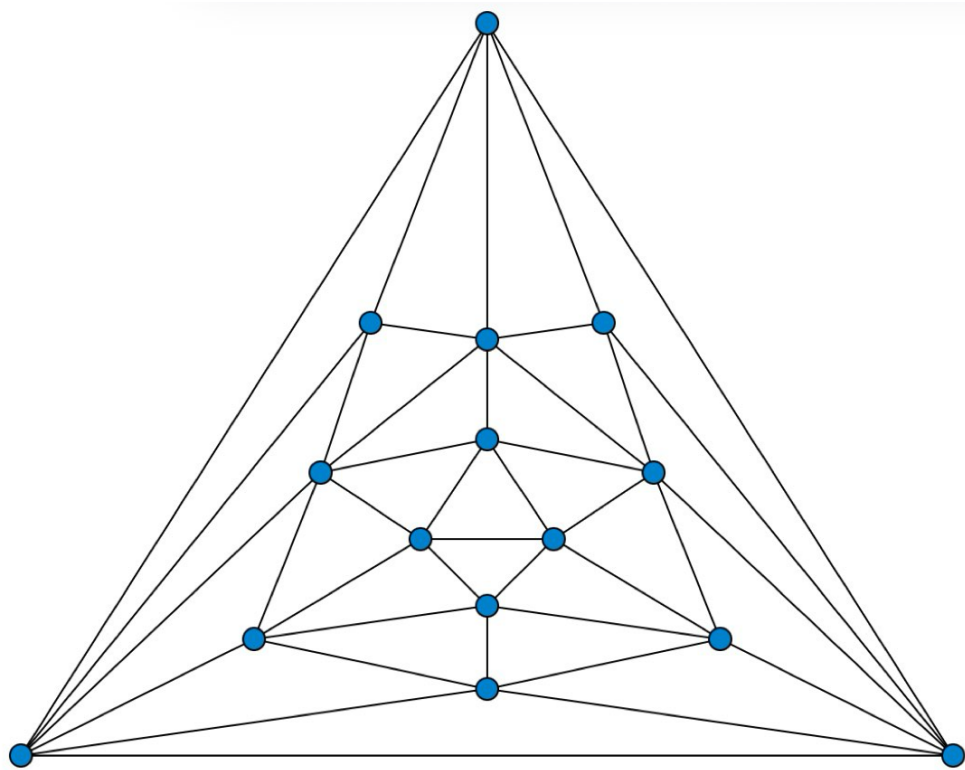


2. PLANAR MAPS & GEODESICS



- Matrix models, exact solutions [90',00's]
- Integrability
- Correlations at fixed geodesic distance?
[DF + Guiller 02]

Example: A random triangulation of the sphere



Planar Triangulation $\leftrightarrow$ Cubic map
(Trivalent)

- Model for discrete random surfaces



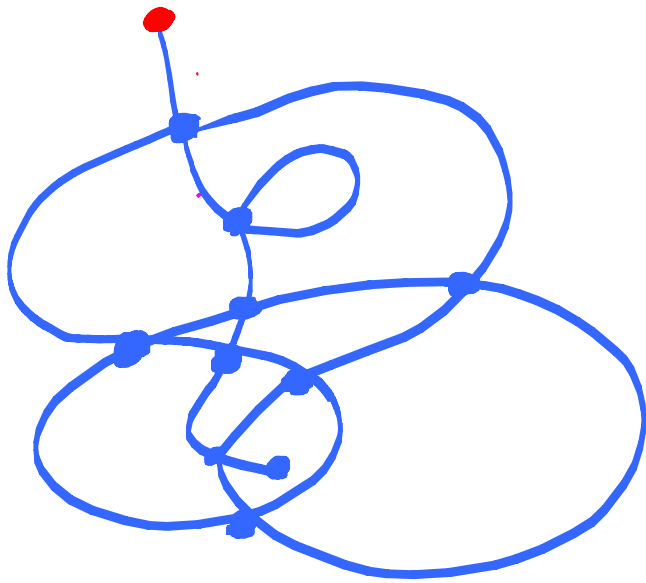
Planar map

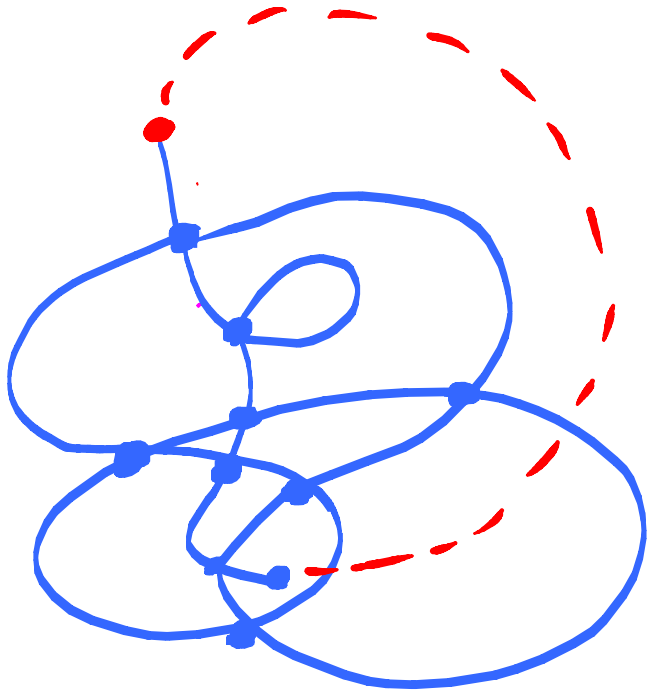
- tetravalent
- 2 legs

counting w/weight

g / vertex

(i.e. $g^{\text{Area}(\text{dualmap})}$)



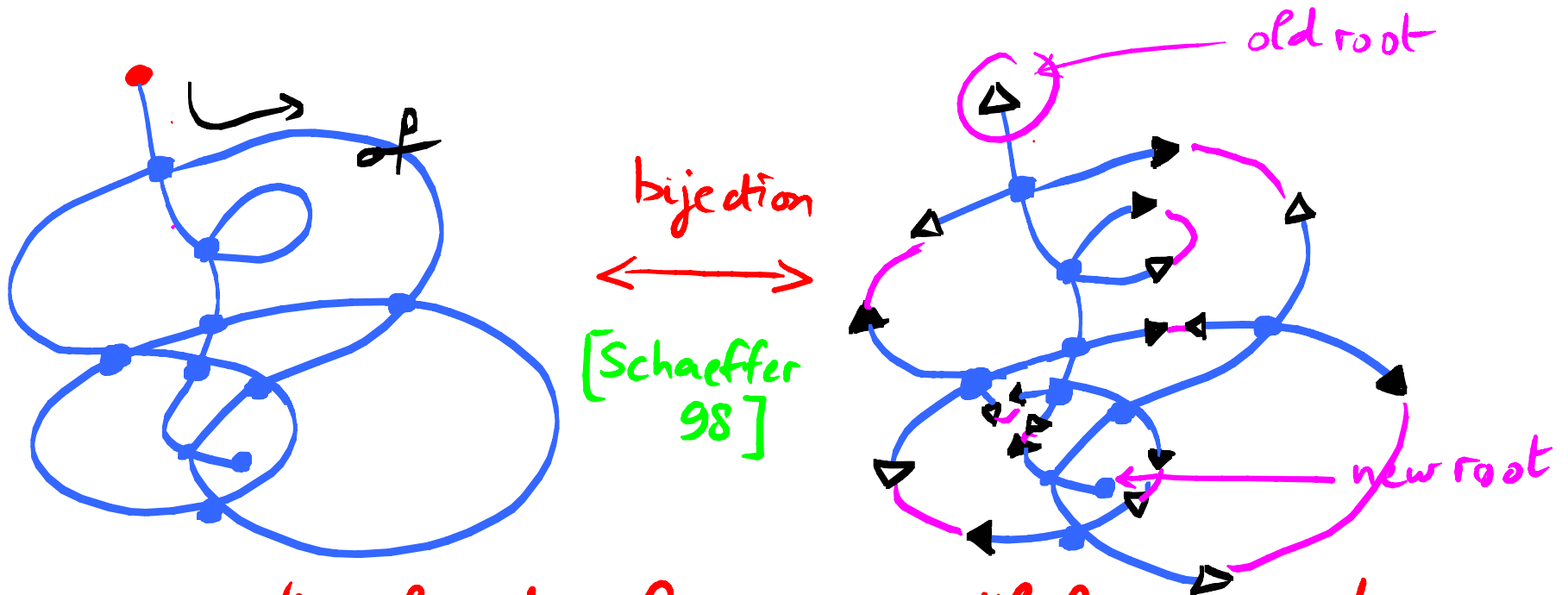


- intrinsic geometry

Planar map

- tetravalent
- 2 legs
- geodesic distance between the legs = 2 here

MAP - TREE BIJECTION



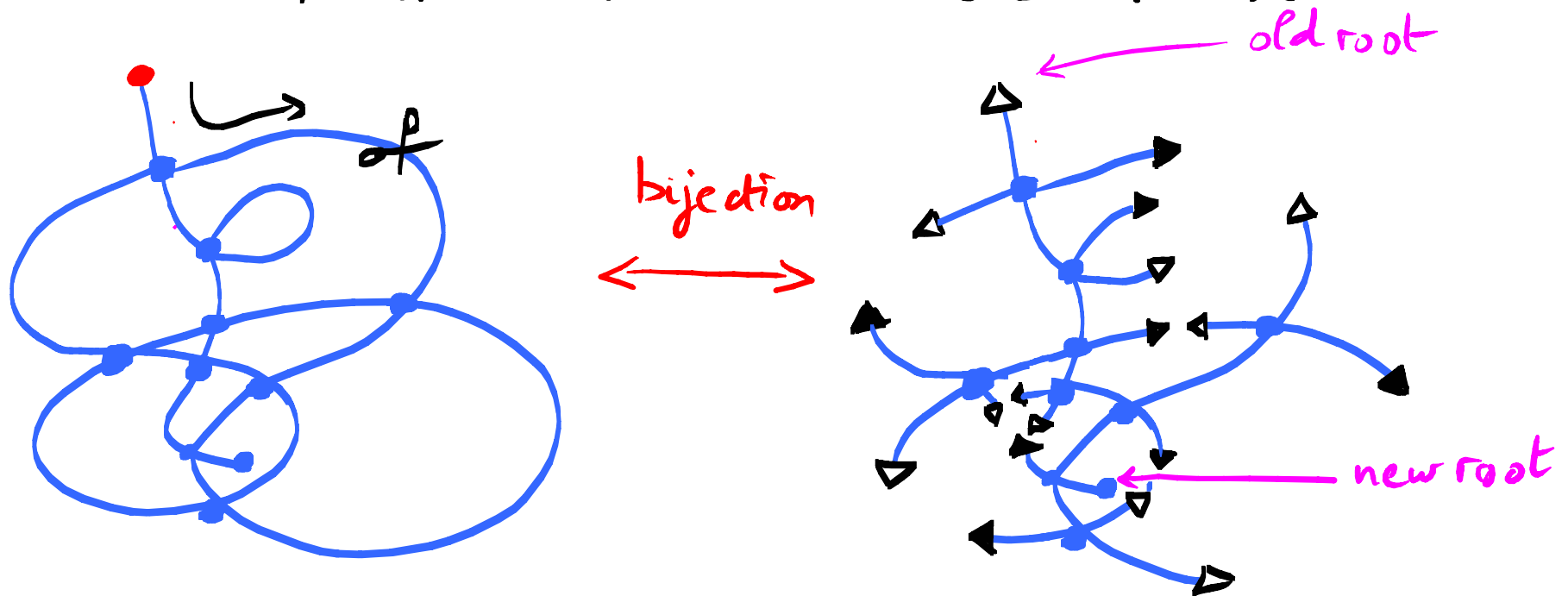
R_n = generating function for maps with legs at distance $\leq n$

Properties: (1) one more $\downarrow$ than $\downarrow$ (conservation law)

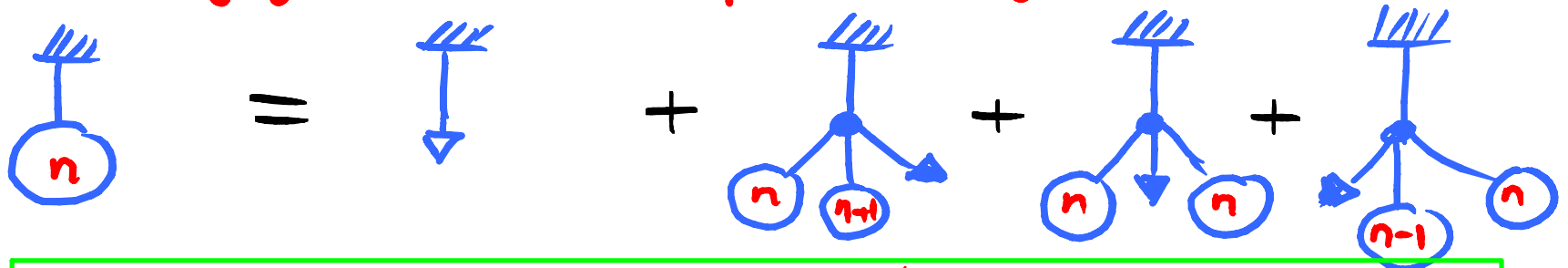
(2) at most n $\left\{ \begin{array}{l} \text{---} \end{array} \right.$ $\downarrow$ $\leftarrow$ weight/vertex

$= R_n(q)$

MAP - TREE BIJECTION



R_n = generating function for maps with legs at distance $\leq n$



$$R_n = 1 + g R_n (R_{n+1} + R_n + R_{n-1})$$

initial data: $R_{-1} = 0$ $R_\infty = R$, sol'n of $R = 1 + 3g R^2$

INTEGRABILITY & SOLUTION

$$0 = R_n - 1 - g R_n (\dot{R}_{n+1} + R_n + R_{n-1}) (*) \quad (R_{-1}=0, R_{\infty}=R)$$

• This is a discrete integrable equation ($n = \text{time}$)

$$\exists \varphi(x, y) = xy(1 - g(x+y)) - x - y \text{ such that}$$
$$\varphi(R_n, R_{n+1}) - \varphi(R_{n-1}, R_n) = (R_{n+1} - R_{n-1})x (*)$$

(DISCRETE CLASSICAL INTEGRABILITY)

INTEGRABILITY & SOLUTION

$$0 = R_n - 1 - g R_n (R_{n+1} + R_n + R_{n-1}) (*) \quad (R_{-1}=0, R_{\infty}=R)$$

- This is a discrete integrable equation ($n = \text{time}$)

$$\exists \varphi(x, y) = xy(1 - g(x+y)) - x - y \text{ such that}$$

$$\varphi(R_n, R_{n+1}) - \varphi(R_{n-1}, R_n) = (R_{n+1} - R_{n-1}) \times (*) = 0!$$

- Exact soliton solution [DF, Boultier, Guilleter 05-07]

$$R_n = R \frac{1 - x^{n+1}}{1 - x^{n+2}} \frac{1 - x^{n+4}}{1 - x^{n+3}}$$

$$R = 1 + 3g R^2$$

$$x + \frac{1}{x} + 4 = \frac{1}{g} R, \quad |x| < 1.$$

- $d_F = 4$

- scaling functions

- continuum limit / Proba

[Le Gall, Micromont, DF, Boultier]
Guilleter

3. A, T-SYSTEM (Frieze patterns)

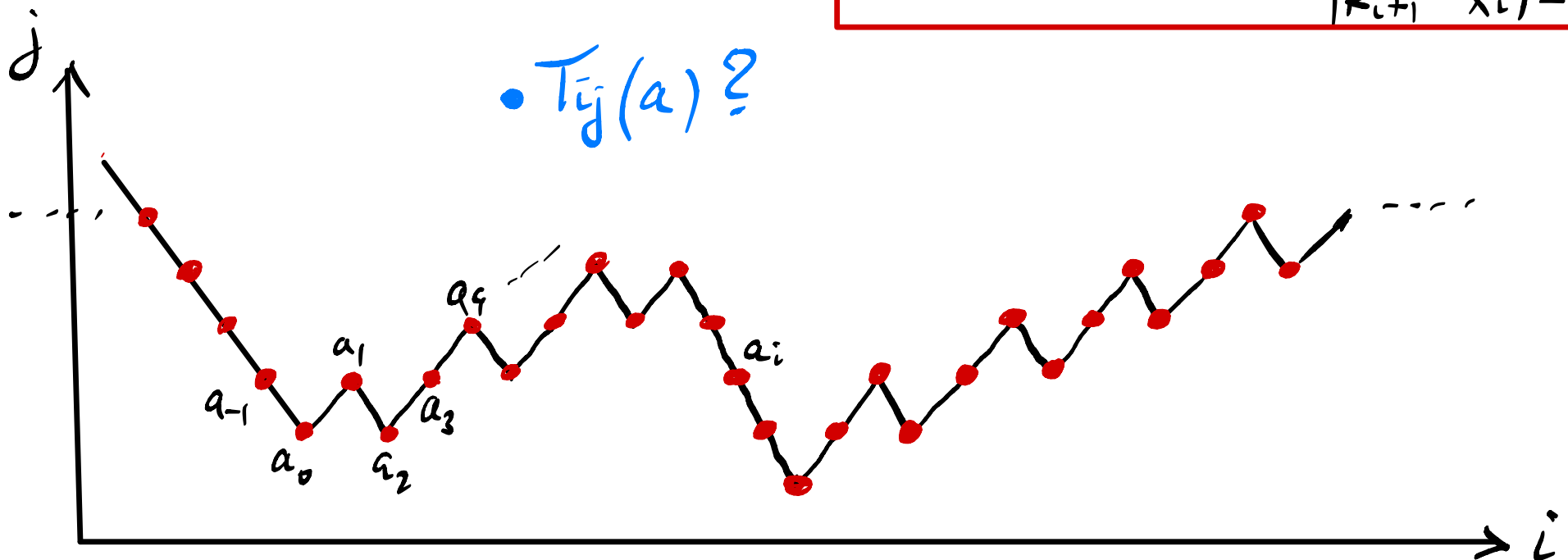
$$T_{ij+1} T_{\bar{i}, \bar{j}-1} = T_{\bar{i}+1, \bar{j}} T_{\bar{i}-1, \bar{j}} + 1 \quad \begin{cases} (i, j) \in \mathbb{Z}^2 \\ \bar{i} + \bar{j} = 0 [2] \end{cases}$$

• 1+1D recursion relation : $\bar{i}$ = space $\bar{j}$ = time

• initial data assignment :

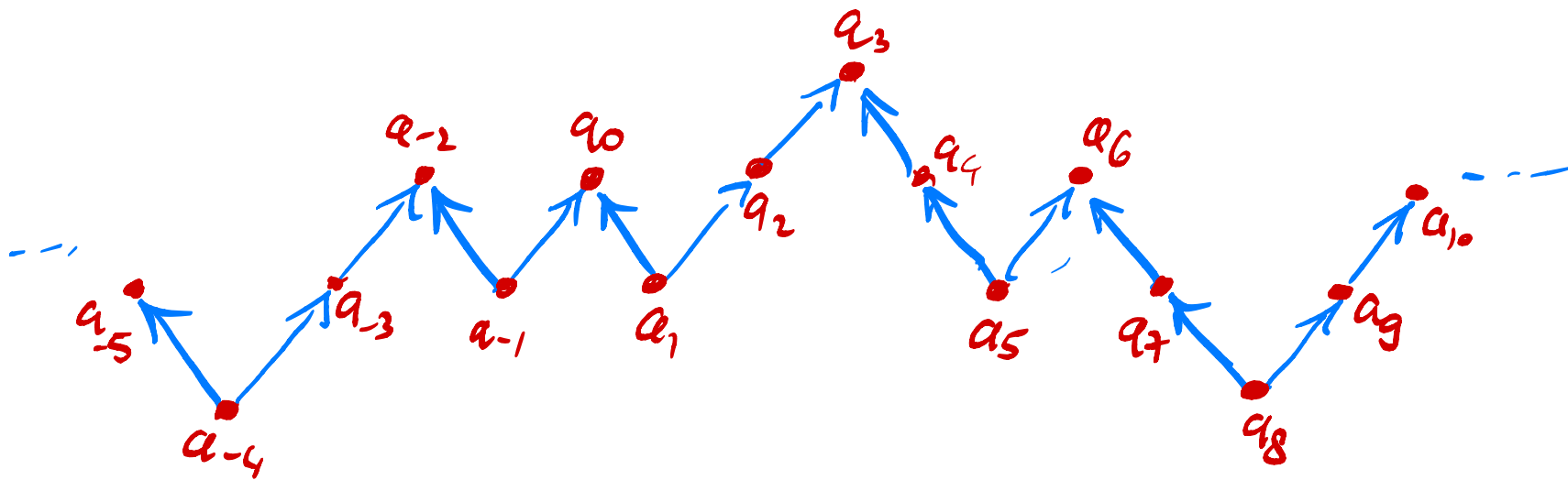
$$T_{i, k_i} = a_i ; k_i = \text{broken line} \\ |k_{i+1} - k_i| = 1$$

• $T_{ij}(a)$?



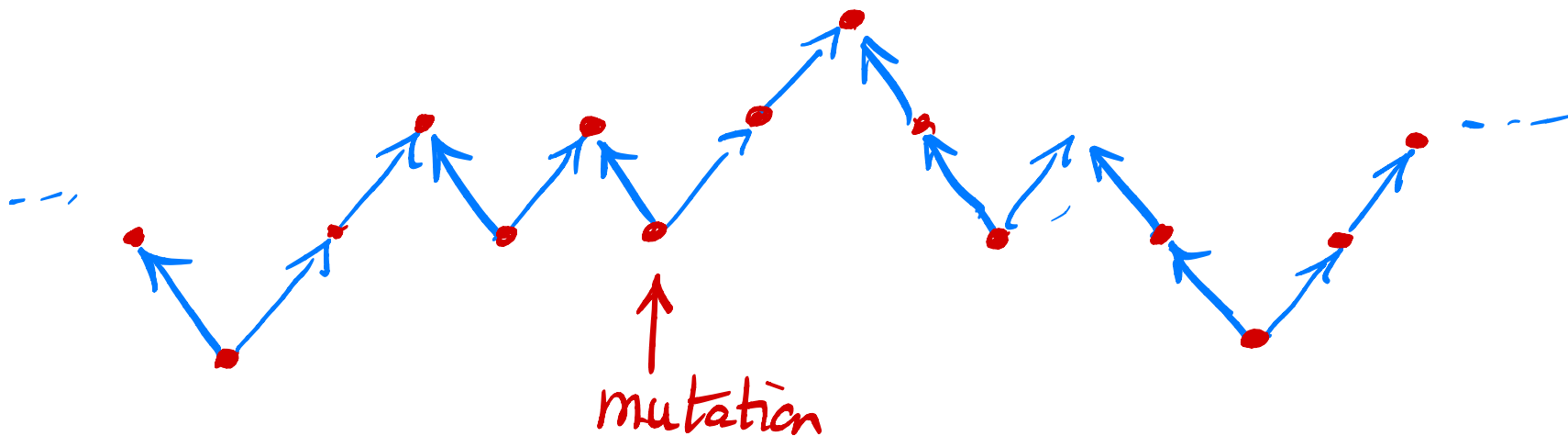
CLUSTER ALGEBRA FORMULATION

- initial broken line $\rightarrow$ infinite quiver Q + variable at each vertex
($a_i, i \in \mathbb{Z}$)



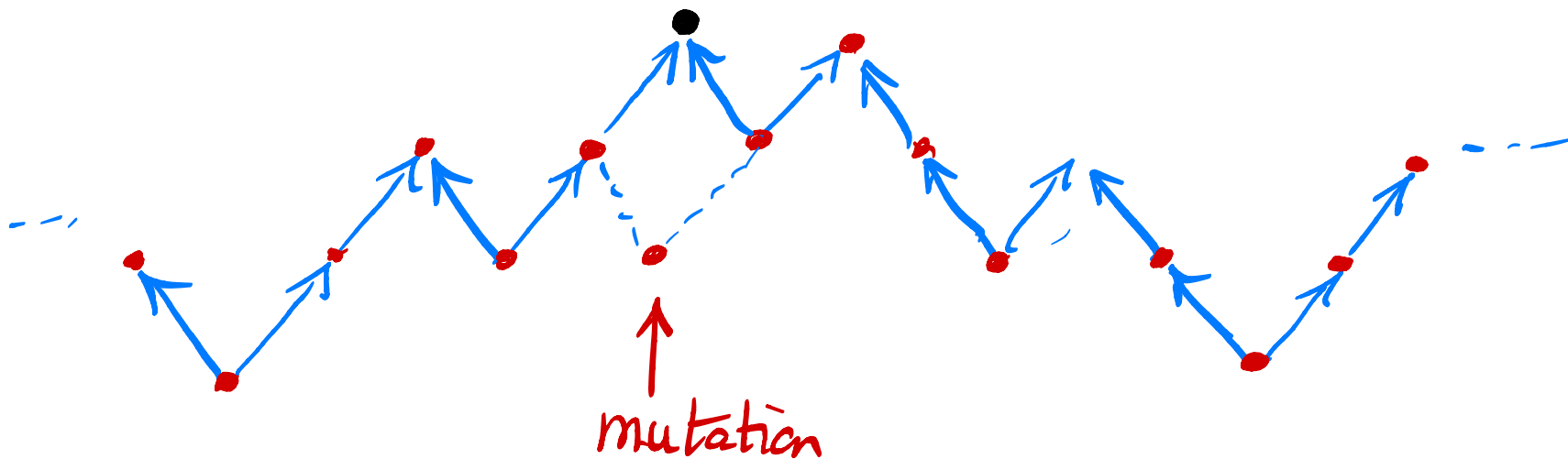
CLUSTER ALGEBRA FORMULATION

- initial broken line $\rightarrow$ infinite quiver Q + variable / vertex
- applying the equation locally $\rightarrow$ mutation of Q / variable



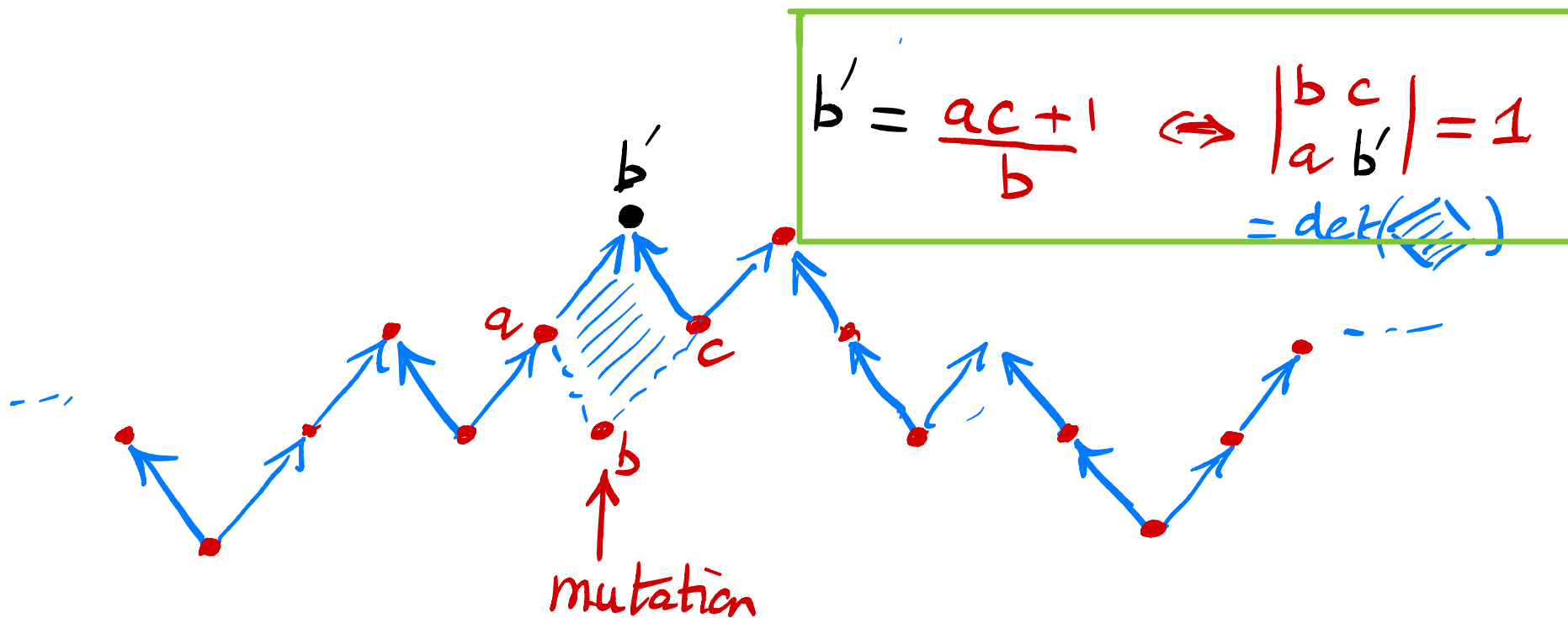
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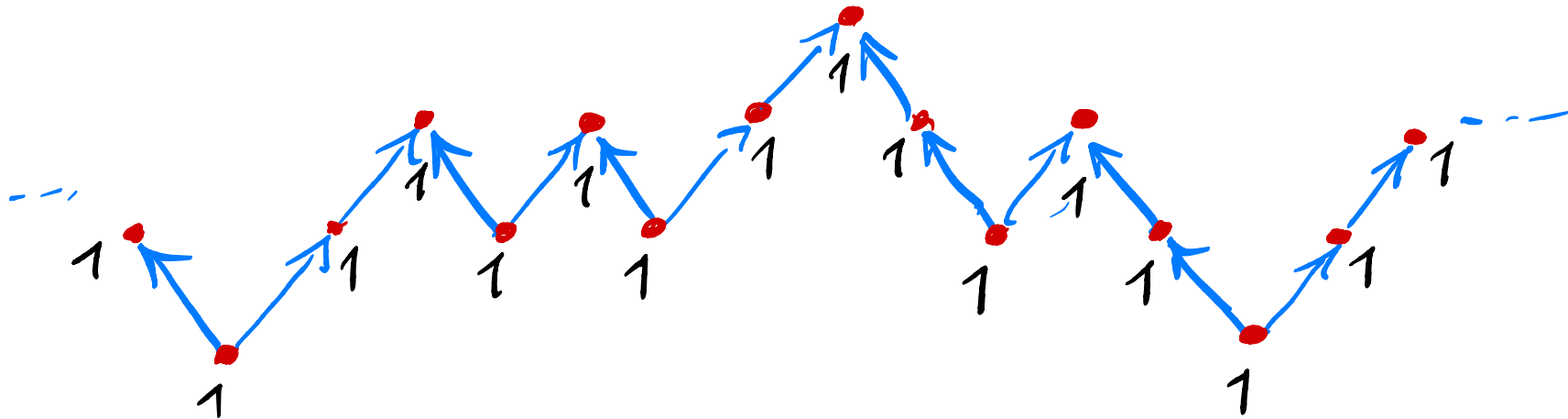
CLUSTER ALGEBRA FORMULATION

- initial broken line $\rightarrow$ infinite quiver Q + variable / vertex
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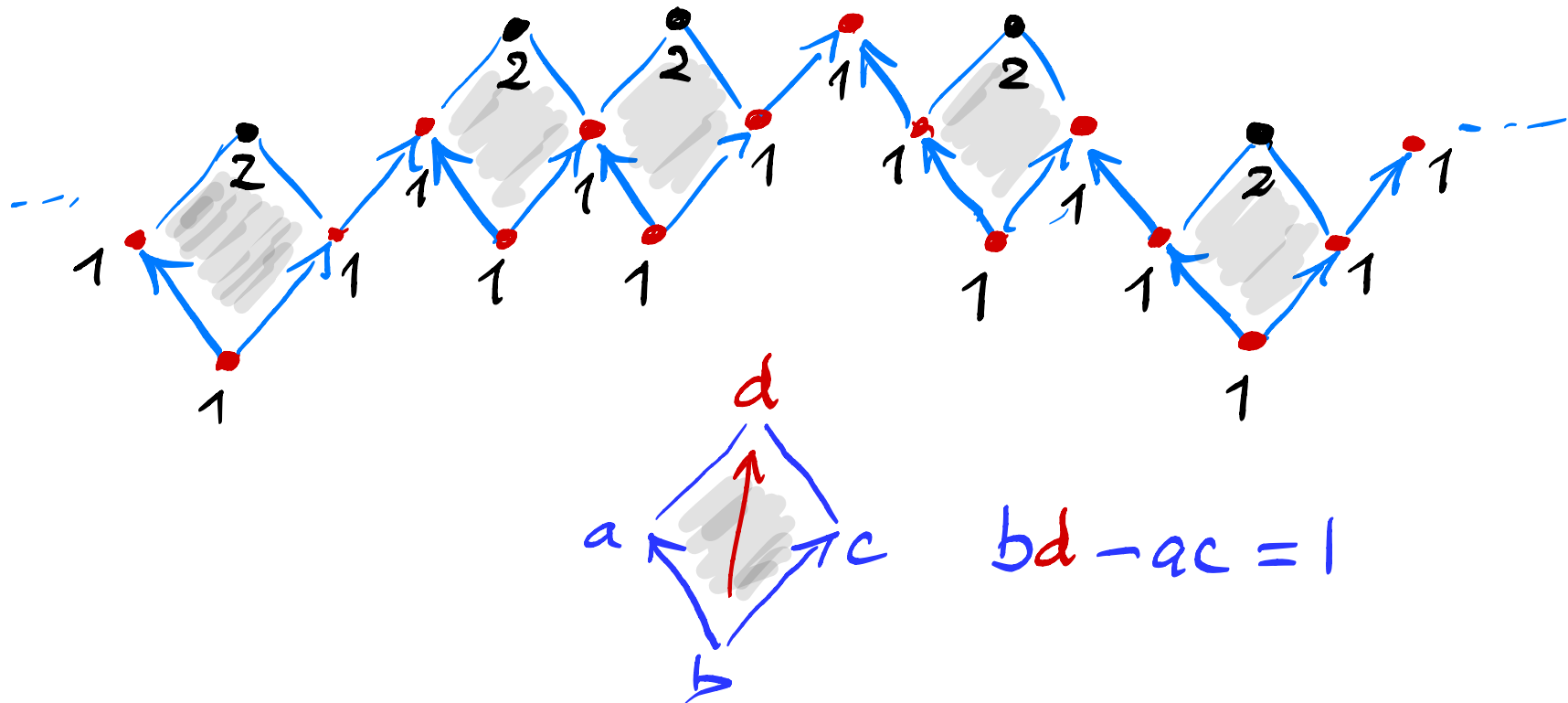


$\rightarrow$ reptation of the broken line from initial data to vertex $(i, j) \rightarrow T_{ij}(a) = \text{Laurent Pol}(\{a_i\}_{i \in \mathbb{Z}})$

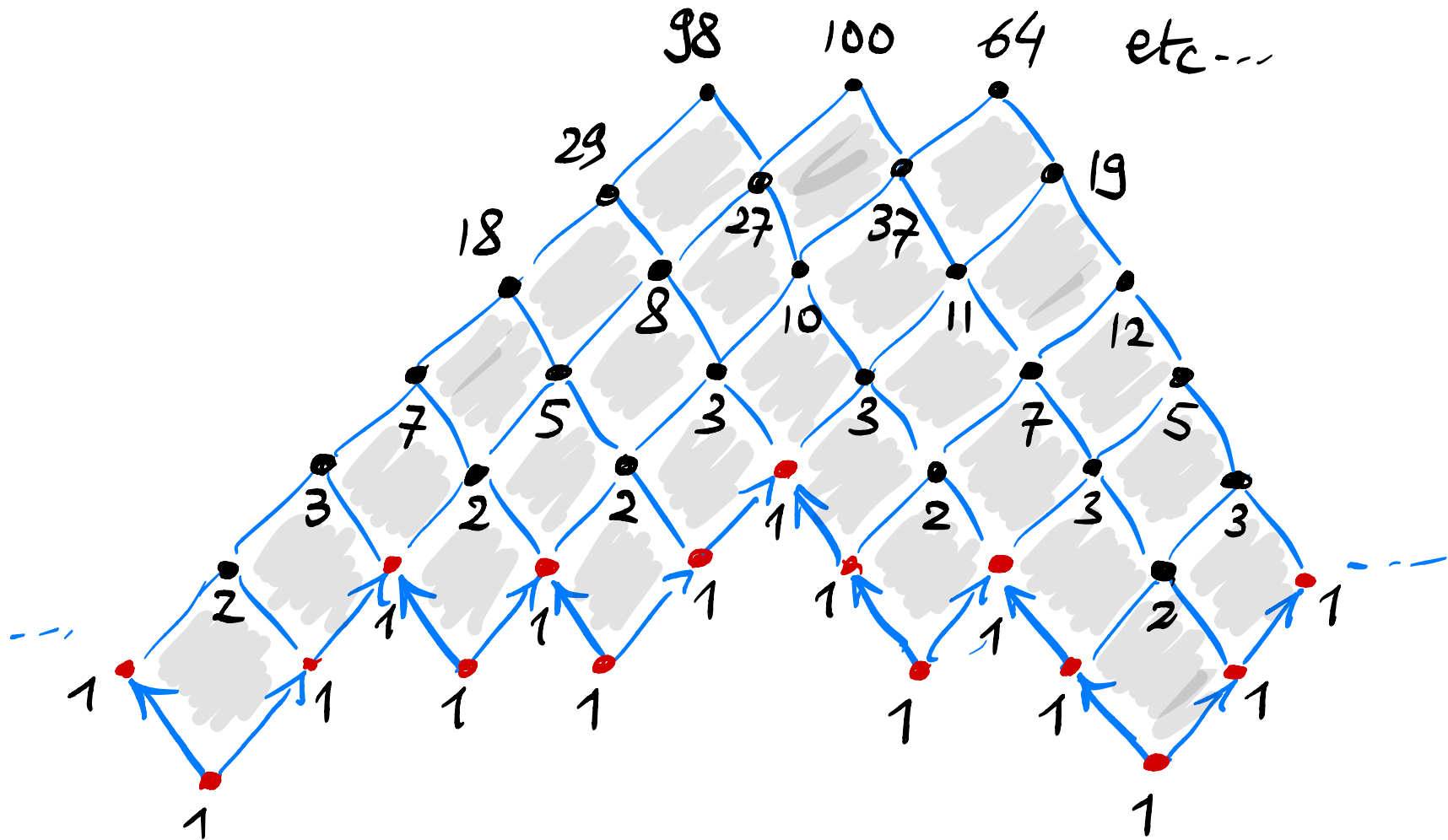
INTEGRABILITY / COMBINATORIAL SOLUTION



INTEGRABILITY / COMBINATORIAL SOLUTION

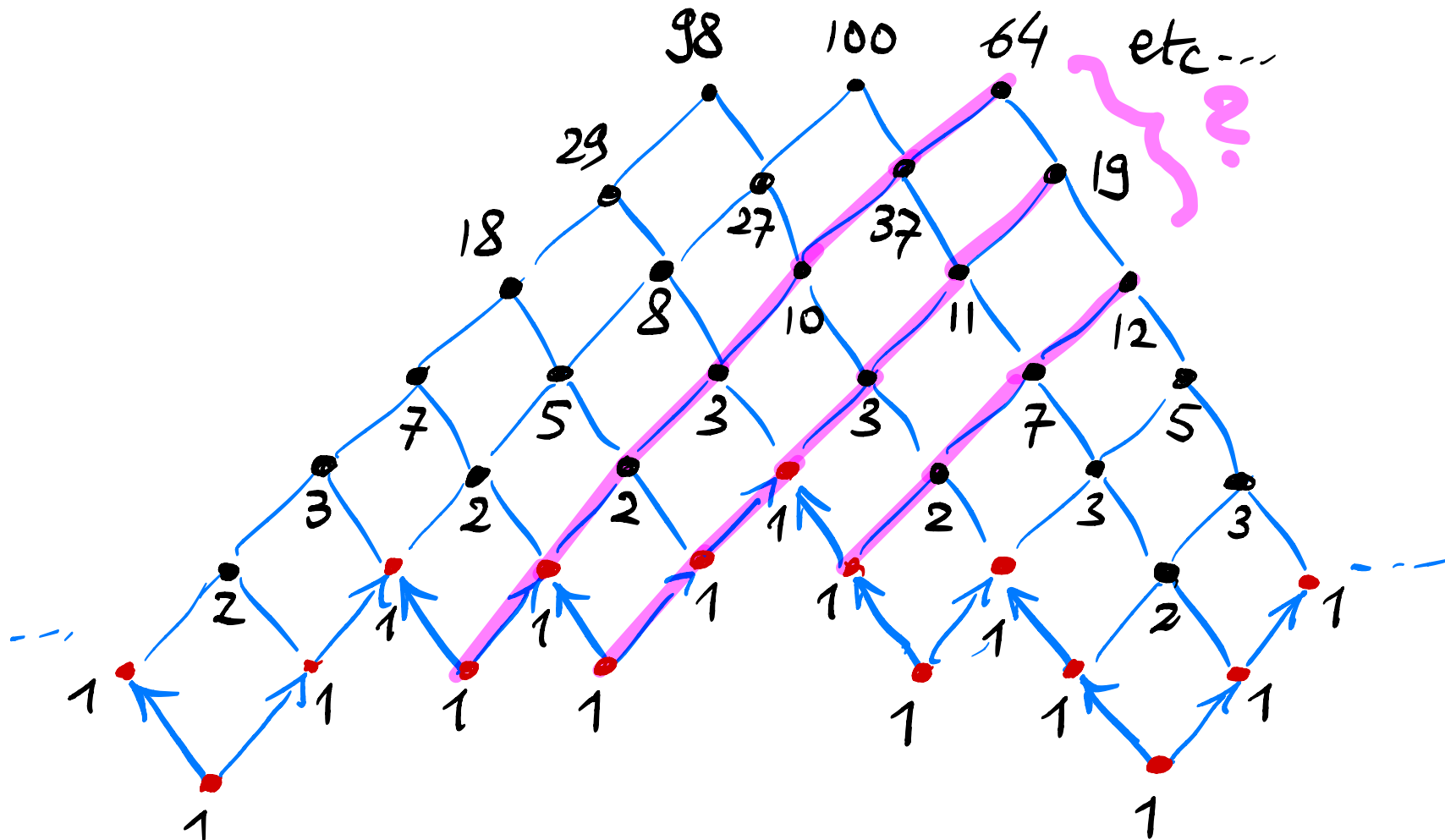


INTEGRABILITY / COMBINATORIAL SOLUTION



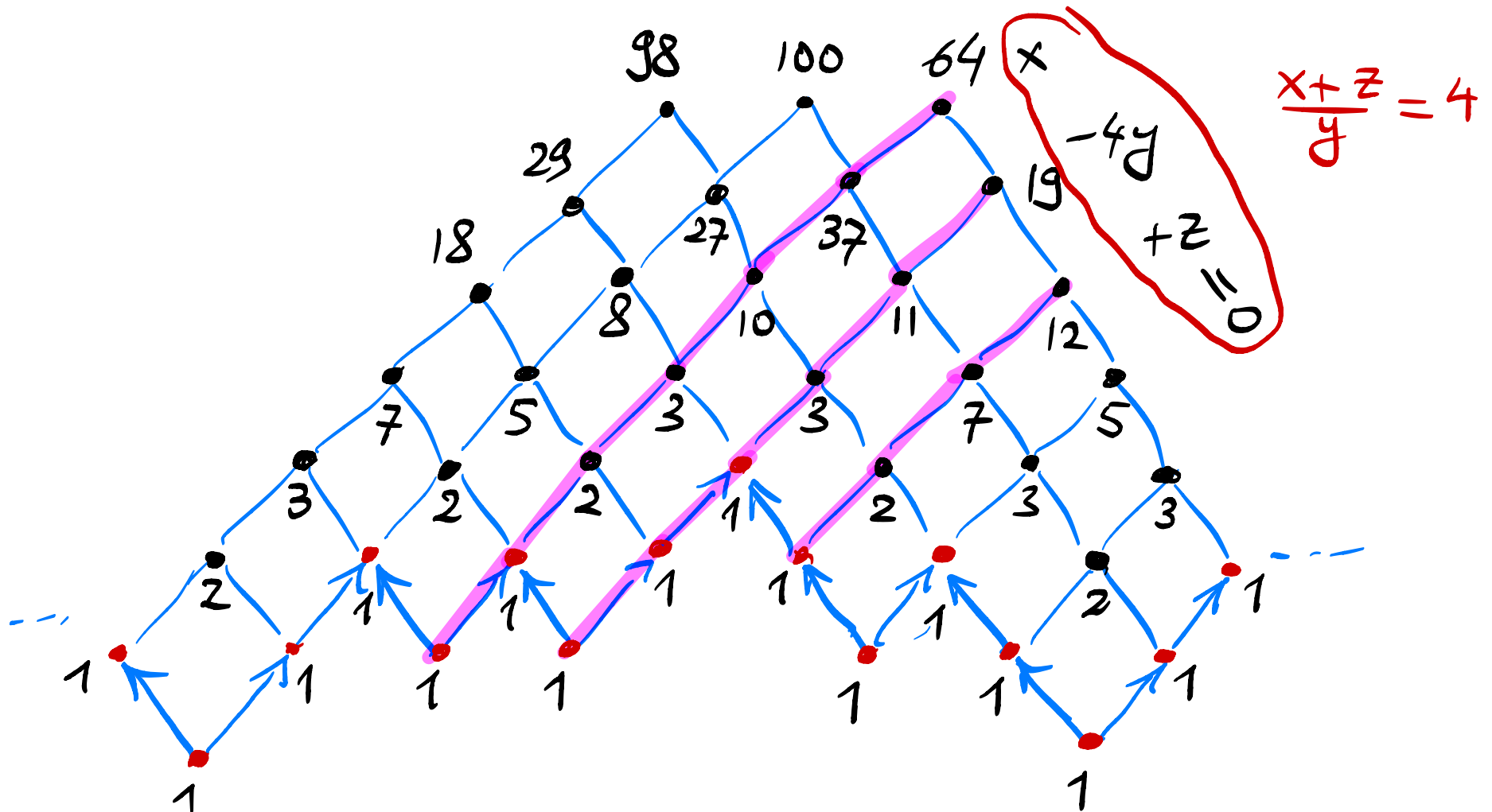
Integers only despite divisions $\leftrightarrow$ Laurent Phenomenon
of Cluster Algebra

INTEGRABILITY / COMBINATORIAL SOLUTION



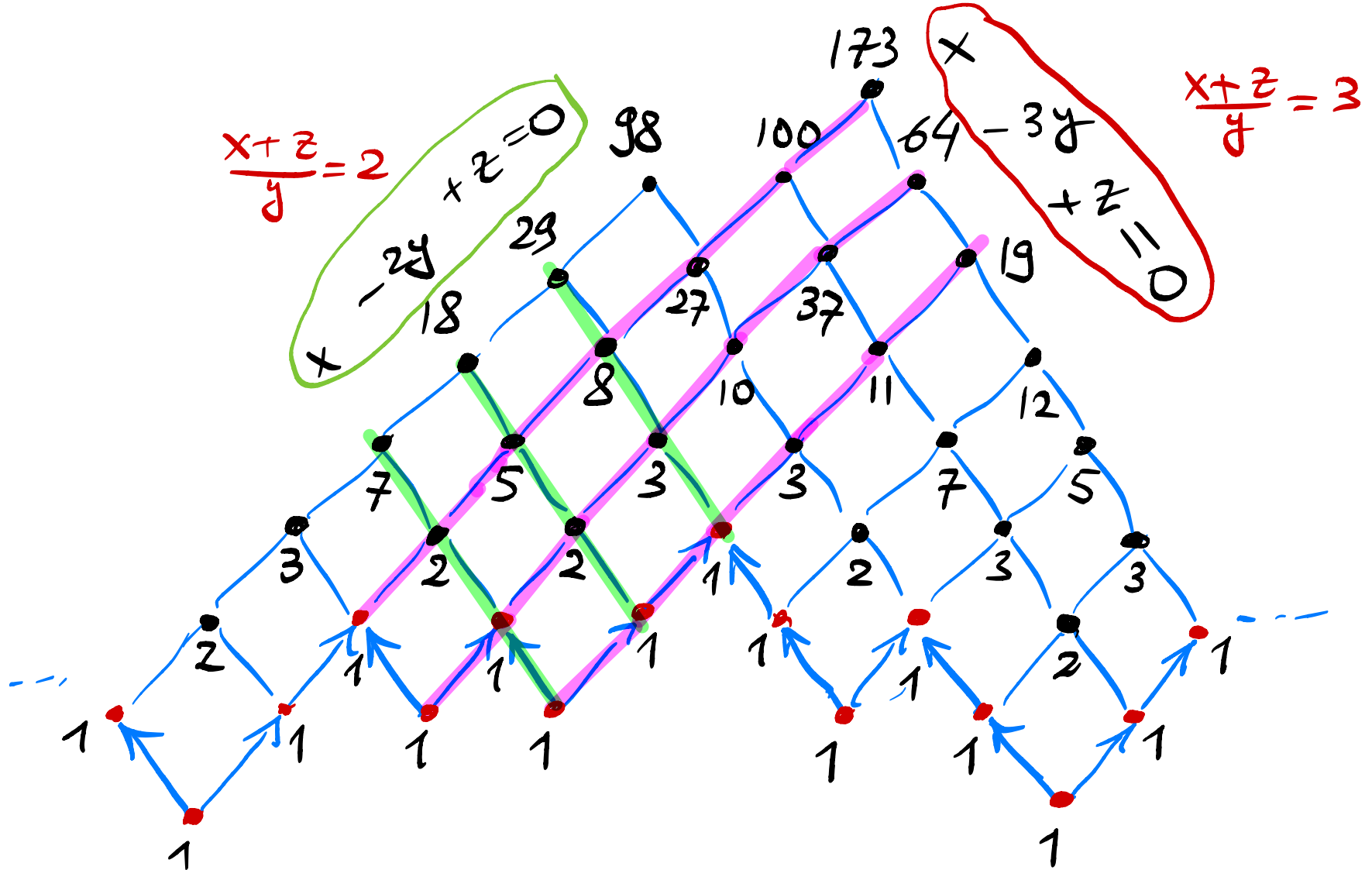
your turn!

INTEGRABILITY / COMBINATORIAL SOLUTION



your turn!

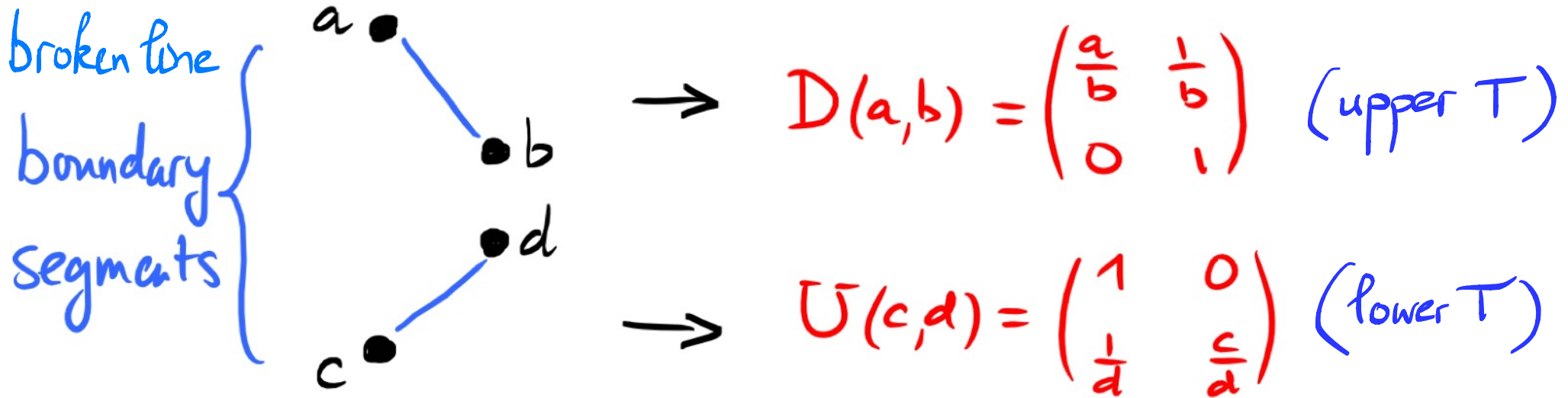
INTEGRABILITY / COMBINATORIAL SOLUTION



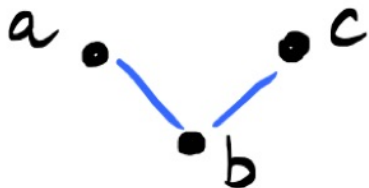
$\Rightarrow$ INFINITELY MANY CONSERVATION LAWS
(DETERMINED BY INITIAL BROKEN LINE)

SOLUTION : CONSTRUCT A FLAT CONNECTION

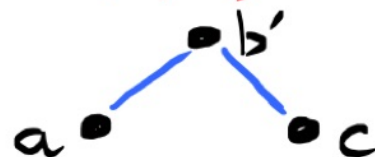
MATRIX REPRESENTATION:



$$D(a,b)U(b,c)$$

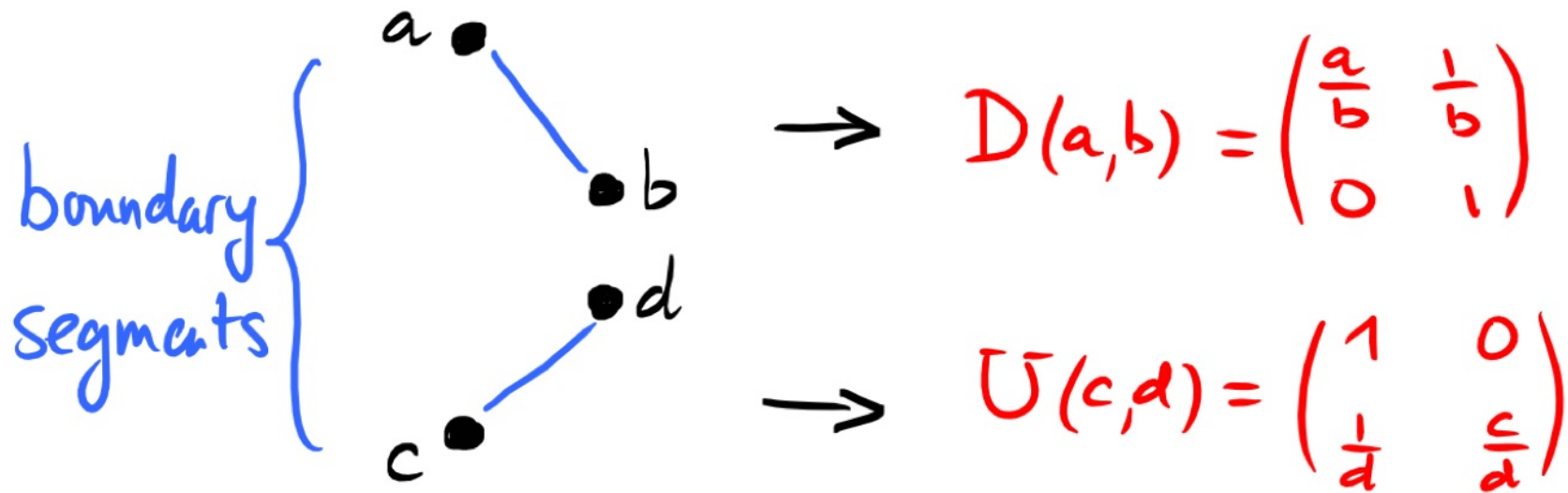


$$U(a,b')D(b',c)$$

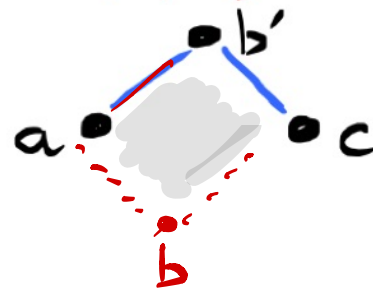
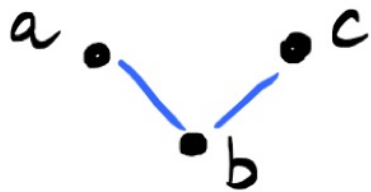


SOLUTION : CONSTRUCT A FLAT CONNECTION

MATRIX REPRESENTATION:

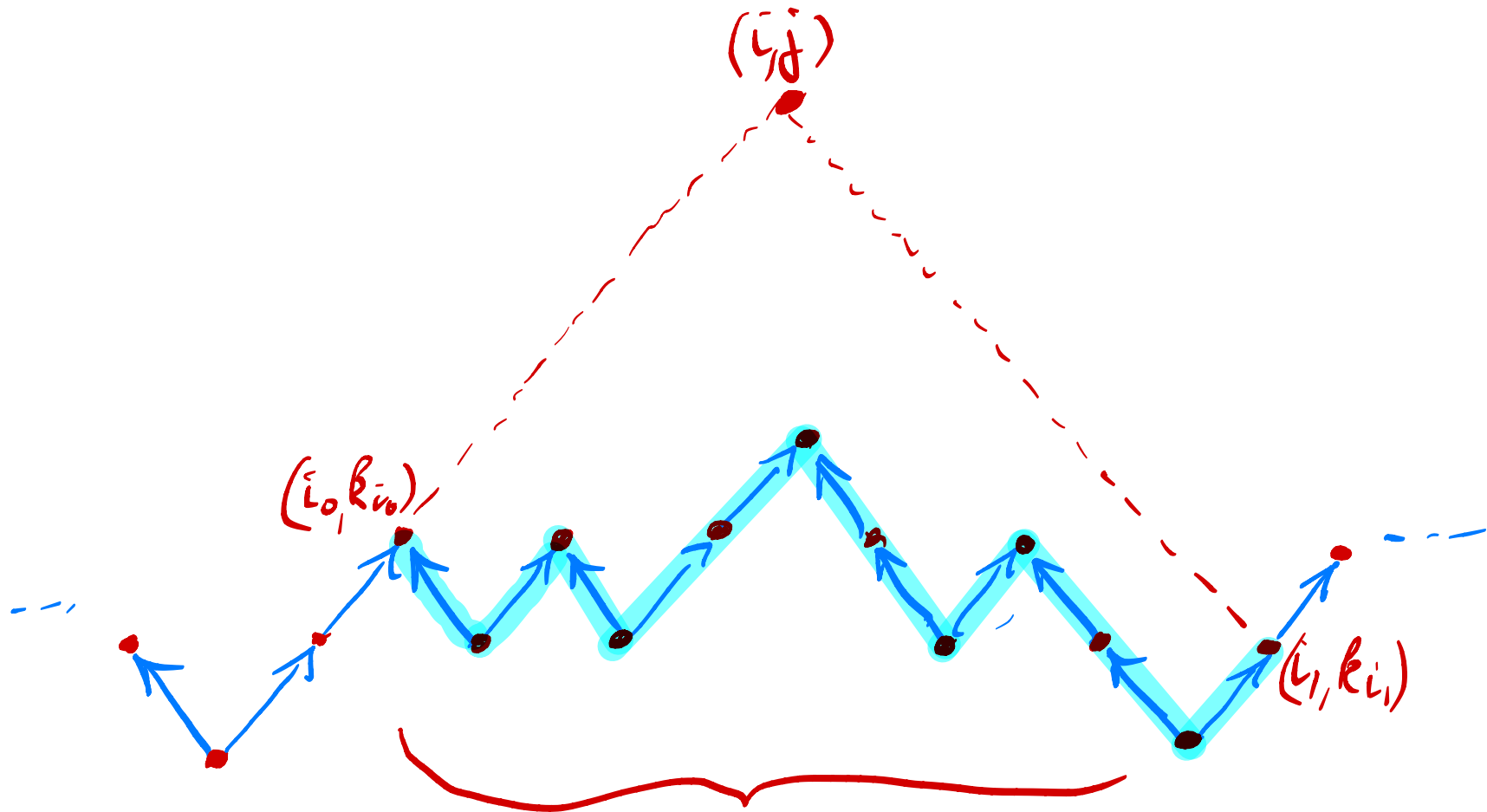


$$D(a, b)U(b, c) = U(a, b')D(b', c) \Leftrightarrow bb' = ac + 1$$



$$\Rightarrow \text{MUTATION} \Leftrightarrow \text{REFACTORIZATION} \quad DU = U'D'$$

SOLUTION

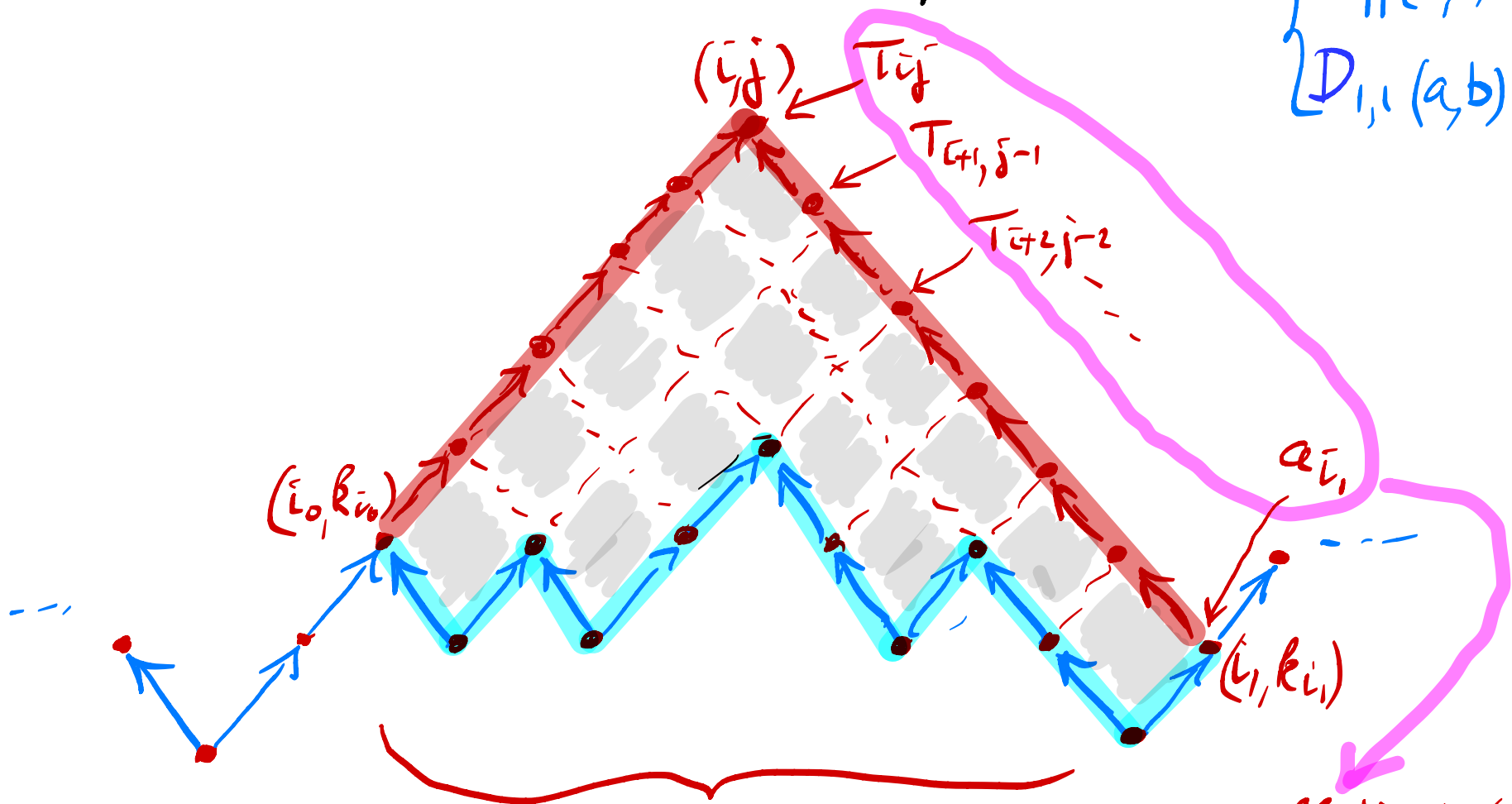


$$\text{Thm} \quad T_{ij} = \left(\prod_{i=i_0}^{i_j-1} \left\{ \begin{array}{c} D(a_i, a_{i+1}) \\ U(a_i, a_{i+1}) \end{array} \right\} \right)_{1,1} \cdot a_{i_j}$$

CONSERVED UNDER MUTATION

PROOF

$$\begin{cases} U_{1,1}(a,b) = 1 \\ D_{1,1}(a,b) = \frac{a}{b} \end{cases}$$



$$\prod_{i=i_0}^{i-1} \left\{ \begin{matrix} D(a_i, a_{i+1}) \\ U(a_i, a_{i+1}) \end{matrix} \right\}_{1,1} \cdot a_i =$$

blue path

$$\left\{ \left(\prod_{i_0}^{i-1} U_{1,1} \right) \left(\prod_{i_0}^{i-1} D_{1,1} \right) \right\} \cdot a_i = T_{ij}!$$

$(1,1)$ element = 1 lower triang $\begin{pmatrix} \Delta \\ \circ \end{pmatrix}$ upper triang $\begin{pmatrix} \circ \\ \nabla \end{pmatrix}$

red path

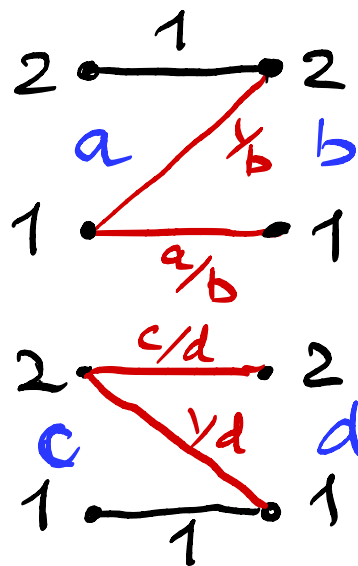
cancellation in $(1,1)$ element

SOLUTION: NETWORK FORMULATION

- The formula gives $T_{ij}(a_i, b_i)$ as an explicit Laurent polynomial w/ coefficients in $\mathbb{Z}_+$.
- reformulation as network partition function.

$$D(a, b) = \begin{matrix} & 1 & 2 \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} a & \frac{1}{b} \\ 0 & 1 \end{pmatrix} \end{matrix}$$

$$U(c, d) = \begin{matrix} & 1 & 2 \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} 1 & 0 \\ \frac{1}{d} & \frac{c}{d} \end{pmatrix} \end{matrix}$$



face label

Matrix Product

Concatenation

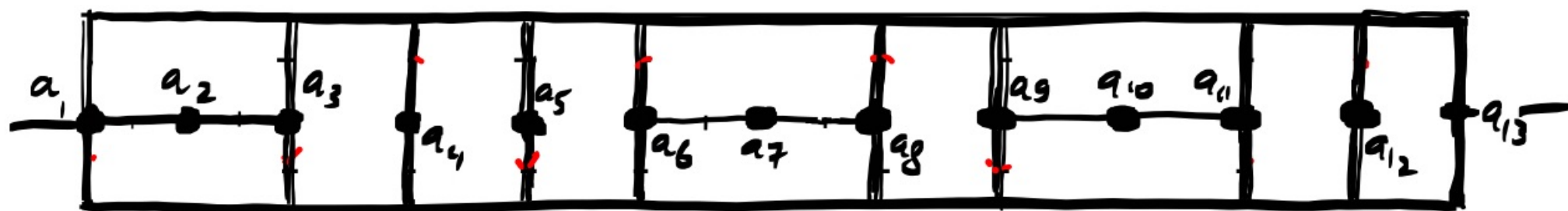
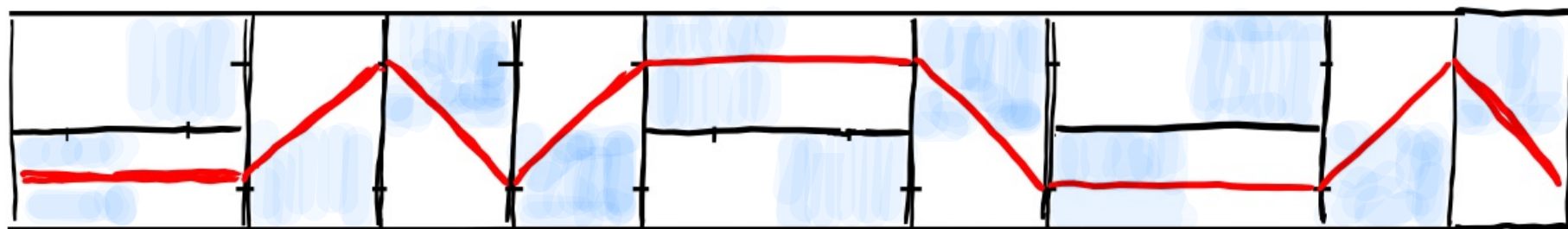
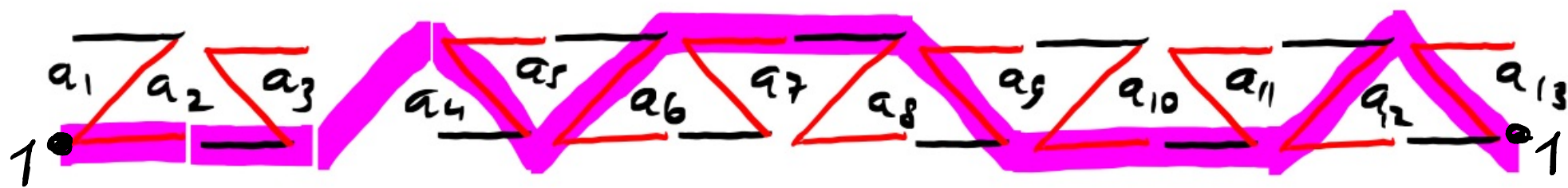
Ex "flat" broken line



$$\Rightarrow T_{ij} = \sum_{\text{paths } 1 \rightarrow 1} \left(\begin{matrix} & 2 \\ \begin{matrix} 1 \\ i_0 \end{matrix} & \begin{pmatrix} \text{network diagram} \end{pmatrix} \end{matrix} \right) \times a_{i_1}$$

(weight of steps)

FROM PATHS TO DOMINO TILINGS

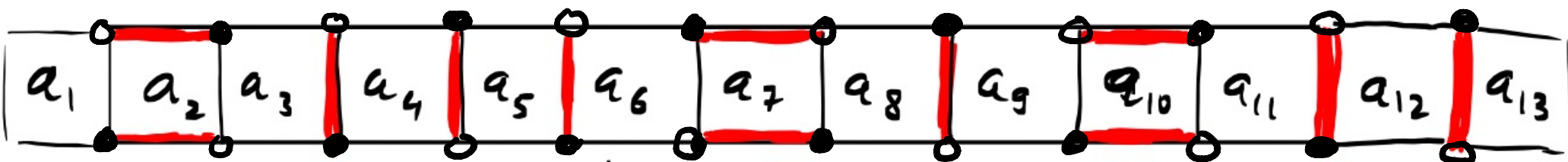
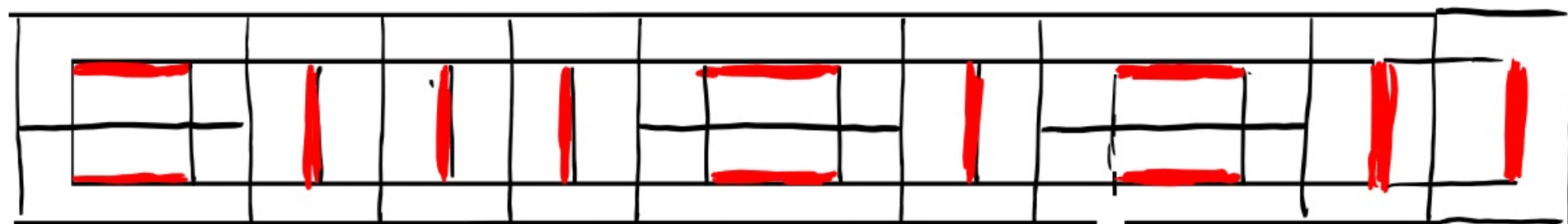
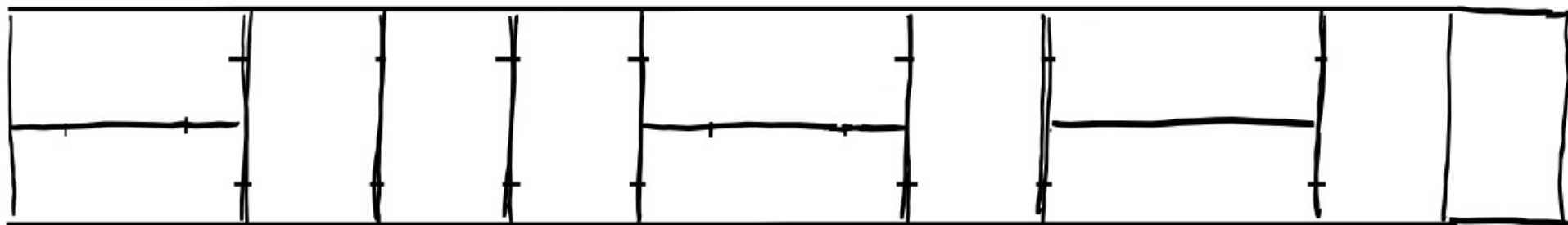


$$\text{Weight} = \prod a_i^{N_i - 3}$$

$\uparrow$ original initial data

$\rightarrow$ degree of vertex i

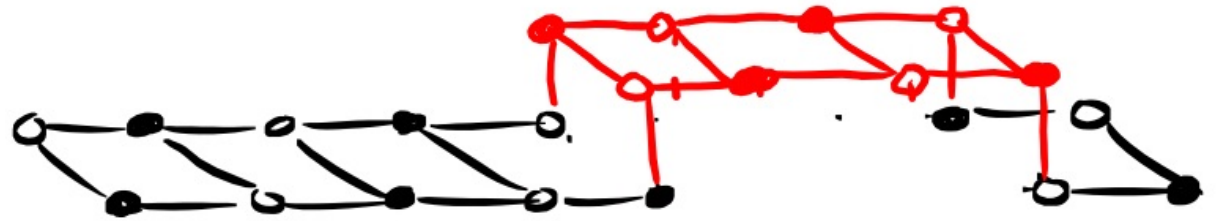
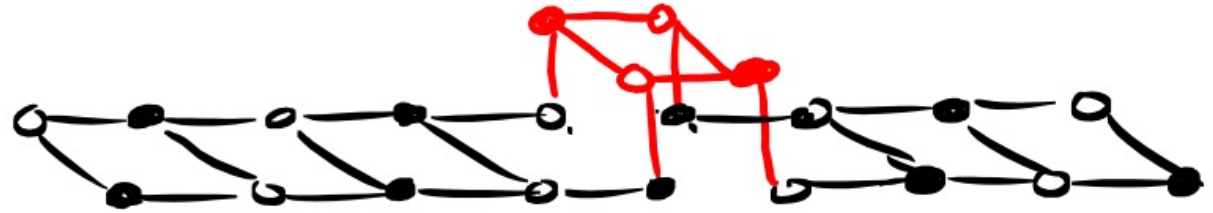
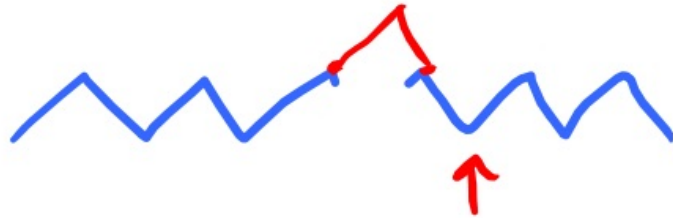
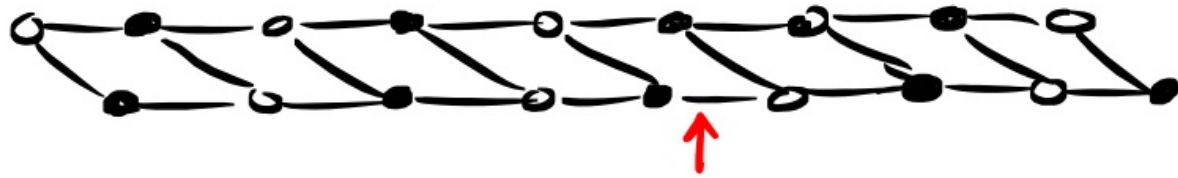
From Dominoes to Dimers



Weight = $\prod a_i$ ^{1-Ni}

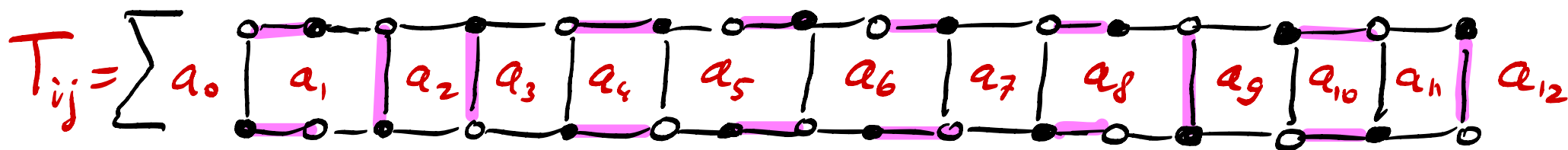
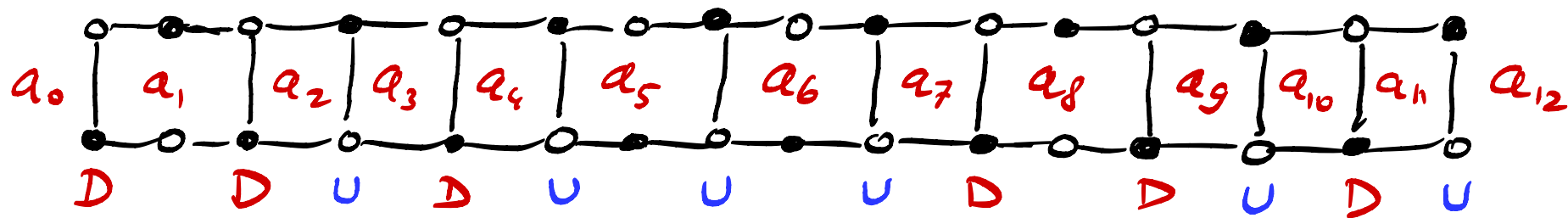
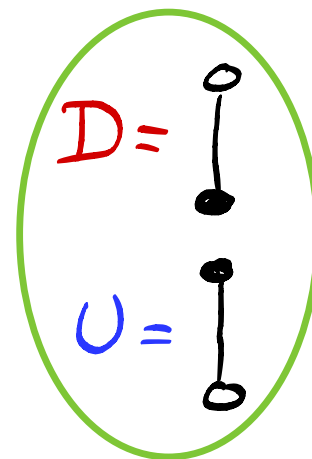
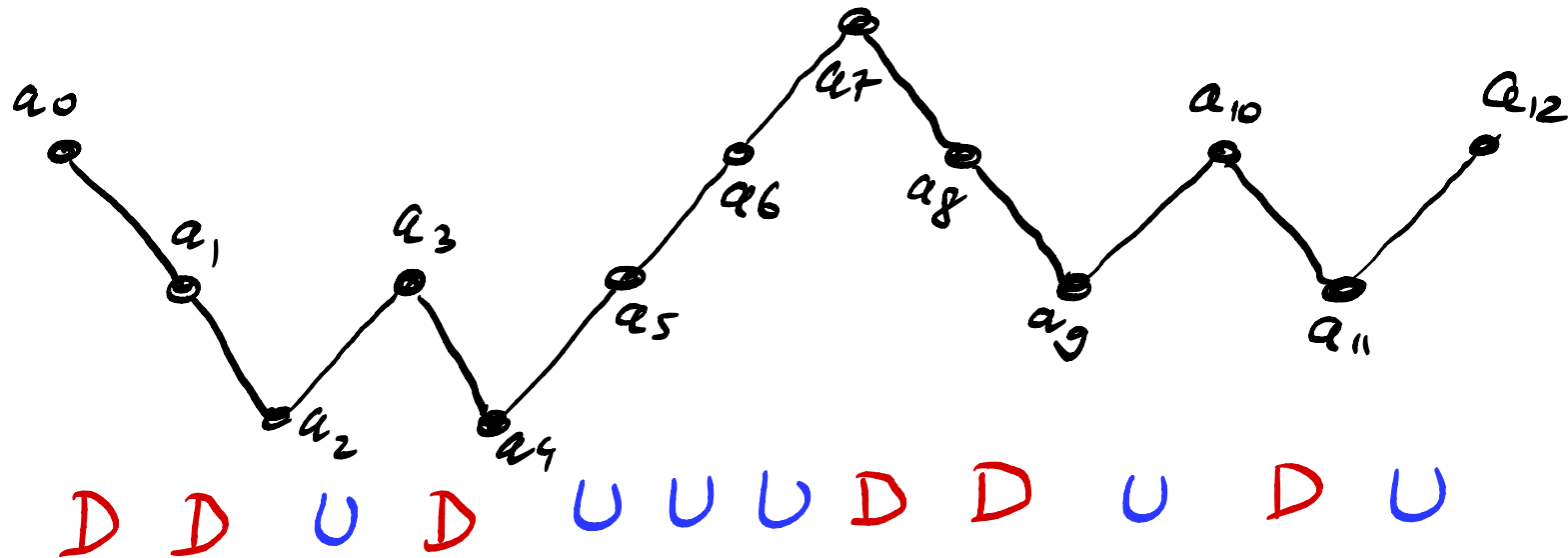
↖ # dimers on square i
($N_i \in \{0, 1, 2\}$)

MUTATIONS ?



etc.

GENERAL CASE



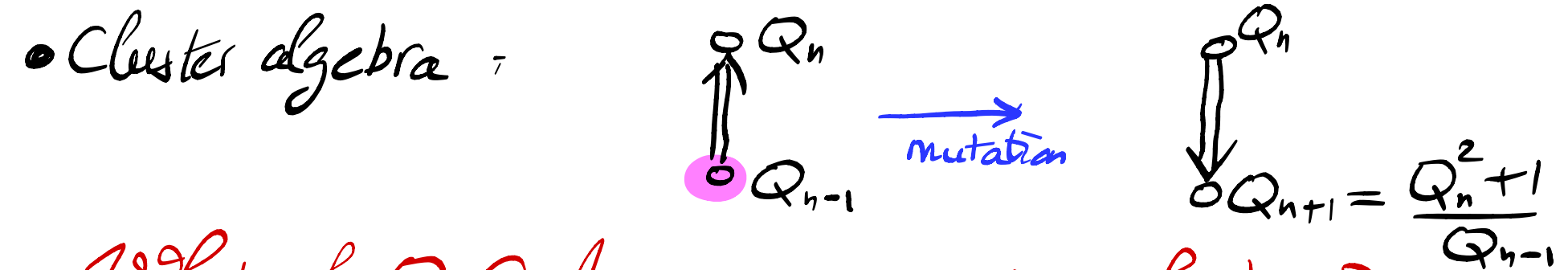
dimer
configs

$$\text{wt}(\text{square}) = a_i^{1-N_i} \quad \text{wt}(\text{hexagon}) = a_i^{2-N_i}$$

4. INTO THE NON-COMMUTATIVE WORLD

- Simplest case of A_1 -T-system :  (2-periodic)

A_1 -Q-system $Q_{n+1} Q_{n-1} = Q_n^2 + 1$ initial data (Q_0, Q_1)



What if $Q_n \in \mathcal{A}$ a non-commutative algebra?

NC- A_1 -Q-syst. $Q_{n+1} Q_n^{-1} Q_{n-1} = Q_n + Q_n^{-1}$ + initial data (Q_0, Q_1)

Thm This is a discrete non-commutative integrable eqn.

Conserved quantities?

2 CONSERVATION LAWS

① $C_n := Q_n^{-1} Q_{n-1} Q_n Q_{n-1}^{-1}$ (group commutator)

Thm $C_n = C_{n+1}$ ($= C$, independent of n) (*)

Proof: $Q_{n+1} Q_n^{-1} Q_{n-1} = \cancel{Q_{n+1}} C_n \cancel{Q_{n-1}} Q_n^{-1} = (Q_n^2 + 1) \cancel{Q_n^{-1}}$
 $= \cancel{Q_n^{-1}} \cancel{Q_{n+1}} C_{n+1} \cancel{Q_{n-1}} = \cancel{Q_n^{-1}} (Q_n^2 + 1)$

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 $= \cancel{Q_n^{-1}} \cancel{Q_{n+1}} C_{n+1} \cancel{Q_{n-1}} = \cancel{Q_n^{-1}} (Q_n^2 + 1)$

② $K_n = Q_n^{-1} Q_{n-1} + Q_{n+1} Q_n^{-1} = (Q_{n+1} + C Q_{n-1}) Q_n^{-1}$

Thm $K_n = K_{n+1}$ ($= K$, independent of n) (**)

Proof: $\cancel{Q_{n+1}} K_n \cancel{Q_n} = Q_n^2 + 1 + Q_{n+1}^2$
 $\cancel{Q_{n+1}} K_{n+1} \cancel{Q_n} = Q_n^2 + Q_{n+1}^2 + 1$

(*) and (**) $\Rightarrow$ left linear recursion

$Q_{n+1} - K Q_n + C Q_{n-1} = 0$

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$$\text{NB: } C = \underbrace{Q_1^{-1} Q_0}_{y_3} \underbrace{Q_1 Q_0^{-1}}_{y_1}; \quad K = \underbrace{Q_1 Q_0^{-1}}_{y_1} + \underbrace{Q_1^{-1} Q_0^{-1}}_{y_2} + \underbrace{Q_1^{-1} Q_0}_{y_3}$$

$$\text{set: } F(t) := \sum_{n \geq 0} t^n Q_n \Rightarrow (1 - Kt + Ct^2) F(t) = Q_0 + (Q_1 - KQ_0)t$$

$$\begin{aligned} \Rightarrow F(t) Q_0^{-1} &= \left(1 - (y_1 + y_2 + y_3)t + y_3 y_1 t^2 \right)^{-1} \left(1 - (y_2 + y_3)t \right) \\ &= \left(1 - (1 - y_3 t)^{-1} y_2 t \right)^{-1} y_1 t \end{aligned}$$

NON-COMMUTATIVE CONTINUED FRACTION!

$$\left(\text{classically: } \frac{1}{1 - \frac{t y_1}{1 - \frac{t y_2}{1 - t y_3}}} \right)$$

$$Q_{n+1} - K Q_n + C Q_{n-1} = 0$$

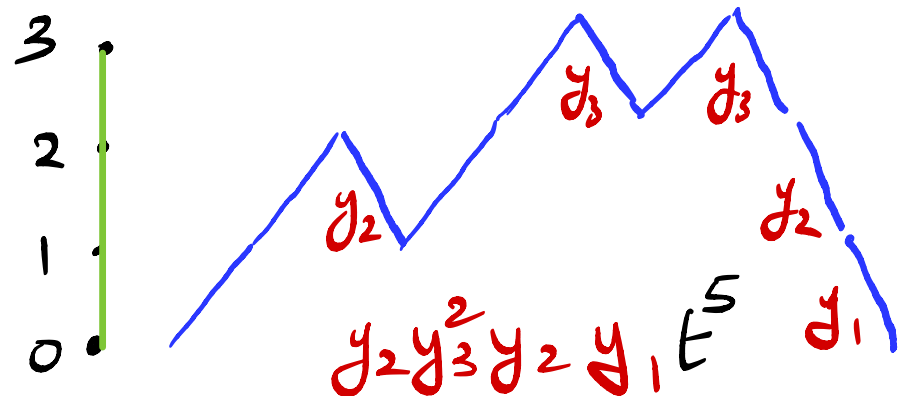
NB: $C = \underbrace{Q_1^{-1} Q_0}_{y_3} \underbrace{Q_1 Q_0^{-1}}_{y_1}; K = \underbrace{Q_1 Q_0^{-1}}_{y_1} + \underbrace{Q_1^{-1} Q_0^{-1}}_{y_2} + \underbrace{Q_1^{-1} Q_0}_{y_3}$

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$$\begin{aligned} \Rightarrow F(t) Q_0^{-1} &= (1 - (y_1 + y_2 + y_3)t + y_3 y_1 t^2)^{-1} (1 - (y_2 + y_3)t) \\ &= (1 - (1 - y_3 t)^{-1} y_2 t)^{-1} y_1 t)^{-1} \end{aligned}$$

NON-COMMUTATIVE CONTINUED FRACTION!
= PATHS!

$$F(t) Q_0^{-1} = \sum_{\substack{\text{paths} \\ 0 \rightarrow 0 \\ n \in \{0, 1, 2, 3\}}} \prod_{i \text{ descents}} \prod_{i \rightarrow i-1} (y_i t)$$



AN INTERESTING SUBCASE:

THE QUANTUM Q-SYSTEM for A₁

- C is central in $\mathcal{A}$: set $C = q$

$$\begin{aligned} q Q_{n+1} Q_{n-1} &= Q_n^2 - 1 \\ Q_n Q_{n+1} &= q Q_{n+1} Q_n \end{aligned} \quad (*)$$

- functional representation: $Q_n =$ difference operators acting on space of functions $f(x_1, x_2)$
$$\begin{cases} T_1 f = f(q x_1, x_2) \\ T_2 f = f(x_1, q x_2) \end{cases}$$

$$\text{Thm } Q_n = \frac{x_1^{n+1}}{x_1 - x_2} T_1 + \frac{x_2^{n+1}}{x_2 - x_1} T_2 \text{ obey } (*)$$

$$C = q$$

$$K = x_1 + x_2$$

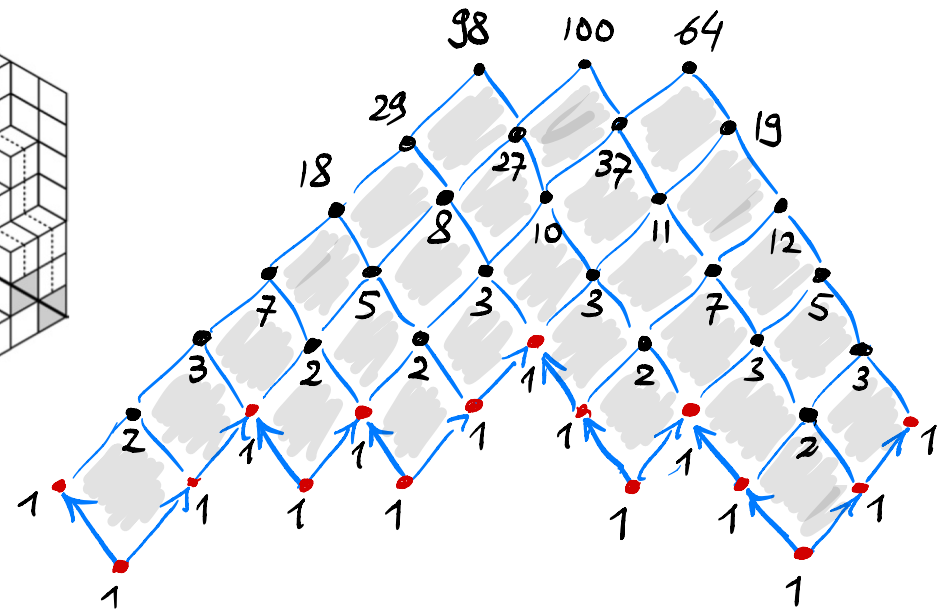
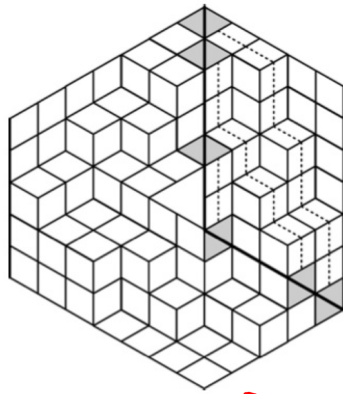
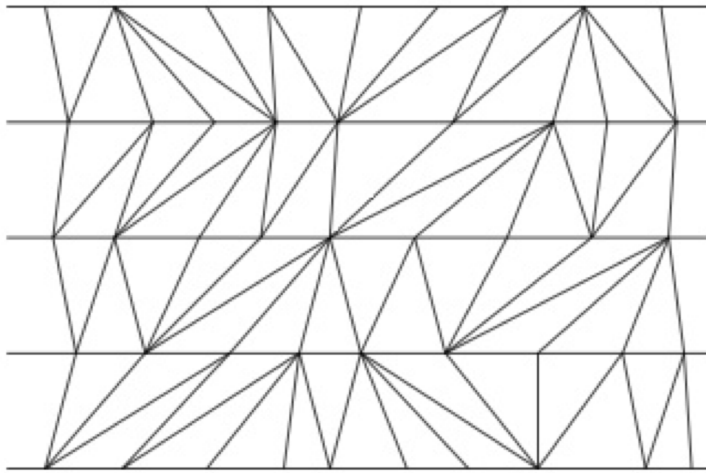
Whittaker operators

CONCLUSION =

- Q_n is a NC Laurent polynomial of (Q_0, Q_1) with non-negative integer coefficients.
- each NC monomial $\equiv$ path on $[0, 3]$ from $0 \rightarrow 0$.
- NON-COMMUTATIVE COMBINATORICS
 $\Leftrightarrow$ BIJECTIONS!
- NC INTEGRABILITY = NEW WORLD
why 2 conservation laws? (only 1 in classical)

5. SUMMARY

- Integrability / Path-Tree formulations
 "integrable combinatorics" } continuous
 } discrete
- Cluster Algebra / Paths
also dimer models / cluster integrable systems Brane tilings
 Quiver gauge theory
 [Conchikov-Kenyon 13]
- Combinatorics survive beyond quantization
 - because of path _(dimer) formulations
 - quantum cluster algebra
 quantum determinant
 - Non commutative cluster algebra



$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

merci!

