Limit Shape of Random Lozenge Tilings of a Hexagon with q-Volume Weights

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Chaire de la Vallée Poussin 2025

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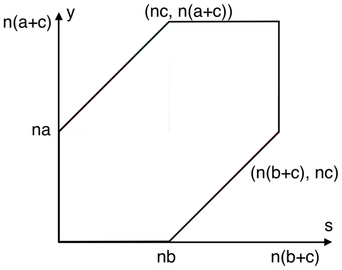
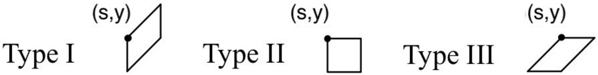
Lozenge Tilings in a Hexagon





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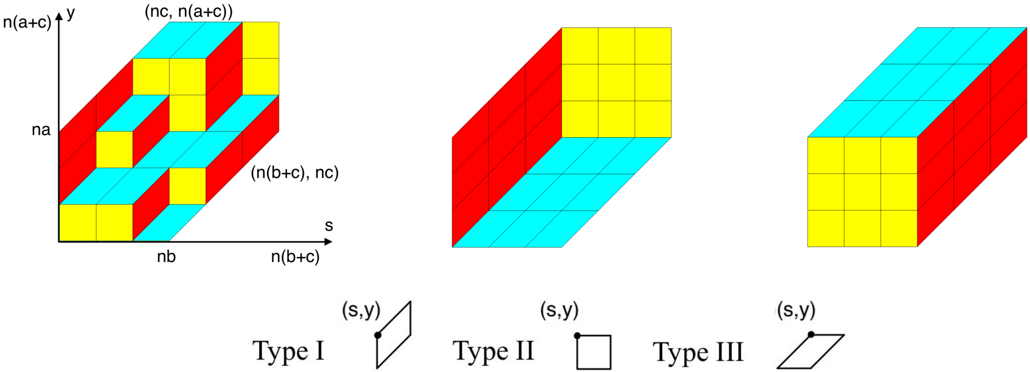
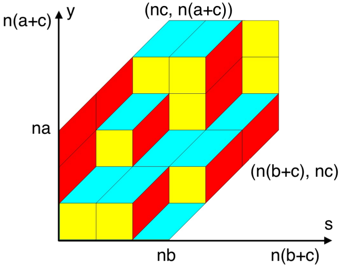
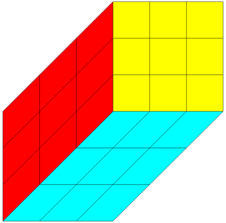
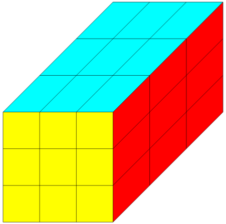


Counting tiling configurations





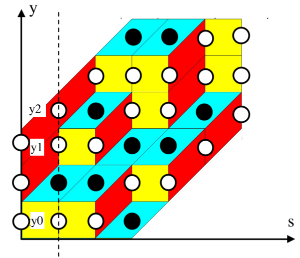


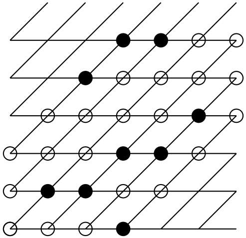
Theorem (MacMahon (1896))

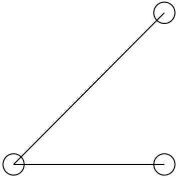
The number of tiling configurations equals

$$\det\binom{nb+nc}{nc+i-j}_{i,j=1}^{na} = \prod_{i=1}^{na}\prod_{j=1}^{nb}\prod_{k=1}^{nc}\frac{i+j+k-1}{i+j+k-2}. \tag{1}$$

Lindström–Gessel–Viennot Theorem(LGV):







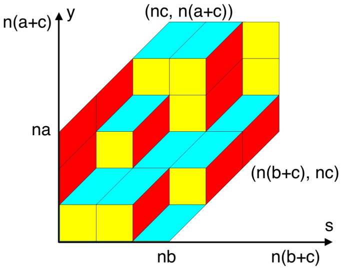
- Tiling = non-intersecting paths on a triangular lattice
- Counting number of paths: $h_{j,i} := \#\{\text{paths} : (0, j) \to (nb + nc, nc + i)\}$.
- A binomial formula

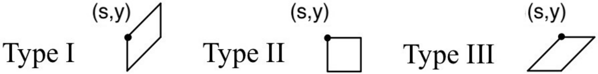
$$h_{j,i} = \binom{nc + nb}{nc + i - j}. \tag{2}$$

- LGV: number of non-intersecting paths is given by (proof by Leibniz formula)

$$\det \left(h_{j,i}\right)_{i,j=1}^{na} = \sum_{\sigma} sgn(\sigma) h_{1,\sigma(1)} \cdots h_{na,\sigma(na)}. \tag{3}$$

Randomisation





- Weight of each type: $\mathfrak{w}(\lozenge) = \mathfrak{w}(\square) = 1$
- Probability of a configuration $\mathcal{T}$ is

$$\mathbb{P}(\mathcal{T}) \sim \prod_{\lozenge \in \mathcal{T}} \mathfrak{w}(\lozenge). \tag{4}$$

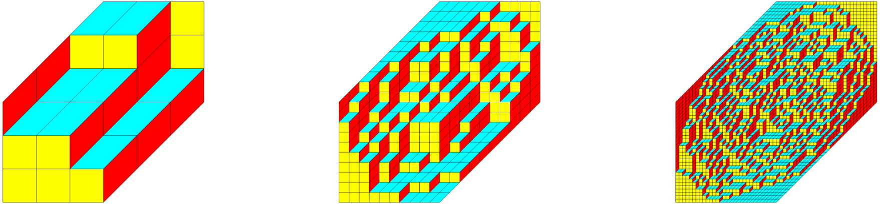
Large n

- Uniform: $\mathfrak{w}(\lozenge) = 1$

(s, y) grows linearly

$$y = nx \qquad s = nt \tag{5}$$

Large n behaviour for $n = 3, 10, 30$

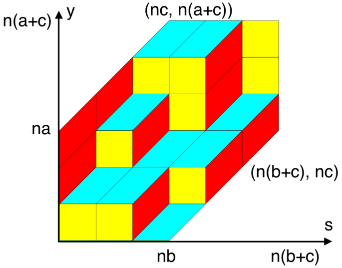


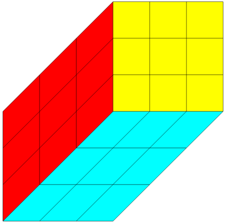
q-volume weights

- q-Volume: $\mathfrak{w}(\lozenge(s,y)) = \mathbf{q}^{y}$

Probability of a configuration $\mathcal{T}$ is

$$\mathbb{P}(\mathcal{T}) \sim \prod_{\lozenge \in \mathcal{T}} \mathfrak{w}(\lozenge) \sim \mathbf{q}^{volume}. \tag{6}$$





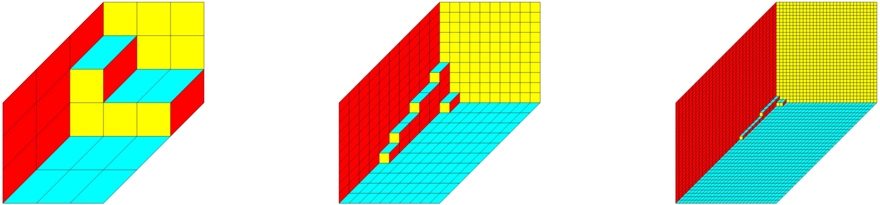
Large n and Scaling in $\mathbf{q}$

- q-Volume: $\mathfrak{w}(\varparallelogram(s,y)) = \mathbf{q}^y$

(s, y) grows linearly

$$y = nx \qquad s = nt \tag{7}$$

Large n behaviour for $\mathbf{q} \in (0,1)$ (say $\mathbf{q} = 0.5$) and $n = 3, 10, 30$



$$\mathfrak{w}(\varparallelogram(nt, nx)) = \mathbf{q}^{nx} \to 0. \tag{8}$$

Large n and Scaling in $\mathbf{q}$

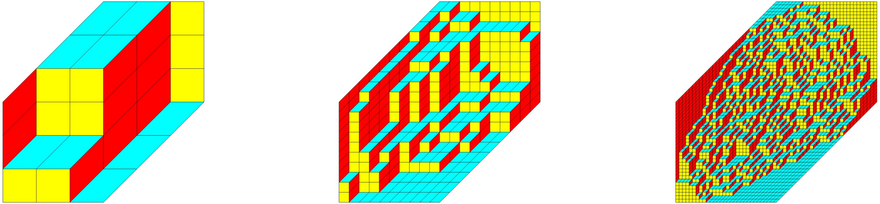
- q-Volume: $\mathfrak{w}(\lozenge(s,y)) = \mathbf{q}^y$

$$y = nx \qquad s = nt \tag{9}$$

$$\mathbf{q} = q^{\frac{1}{n}} \to 1 \tag{10}$$

$$\mathfrak{w}(\lozenge(nt,nx)) = q^x. \tag{11}$$

Large n behaviour $n = 3, 10, 30$ and $q = 0.5$



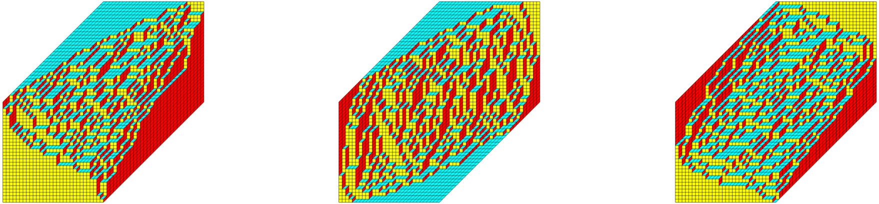
Large n and Scaling in $\mathbf{q}$

- q-Racah: $\mathfrak{w}(\lozenge(s,y)) = \kappa\mathbf{q}^{\frac{-s}{2}+y} - \left(\kappa\mathbf{q}^{\frac{-s}{2}+y}\right)^{-1}$

$$y = nx \qquad s = nt \tag{12}$$

$$\mathbf{q} = q^{\frac{1}{n}} \to 1 \tag{13}$$

Large n behaviour $n = 30$, ($q = e$ $\kappa = e$),($q = 20$ $\kappa = i$),($q = e^{i\pi/2}$ $\kappa = e^{i\pi/2}$),



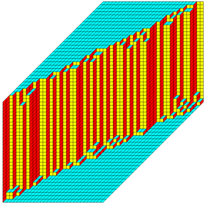
Large n and No Scaling in $\mathbf{q}$

- q-Racah: $\mathfrak{w}(\text{▱}(s,y)) = \kappa\mathbf{q}^{\frac{-s}{2}+y} - \left(\kappa\mathbf{q}^{\frac{-s}{2}+y}\right)^{-1}$

$$y = nx \qquad s = nt \tag{14}$$

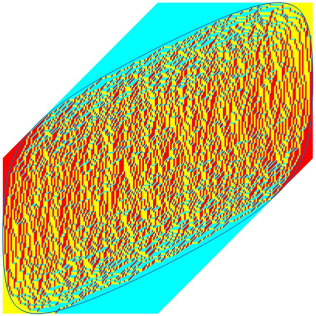
$$\mathbf{q} \in (0,1) \tag{15}$$

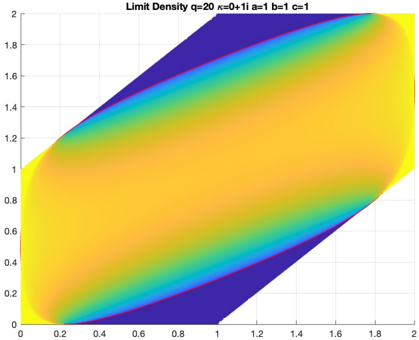
Large n behaviour $n = 30$, ($\mathbf{q} = 0.5$ $\kappa = i$)



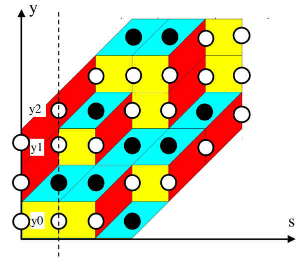
First look of global behaviour of large n

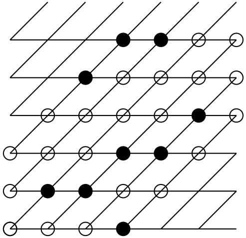
As $n \to \infty$ almost surely

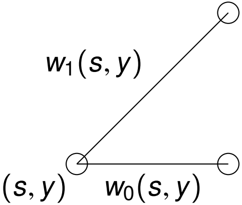




Tiling and q-Racah OPE







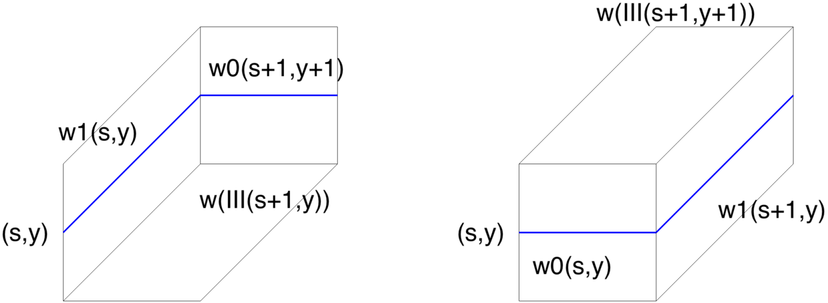
- Tiling = non-intersecting paths on a lattice
- Transition Matrix:

$$T_s(y,z) := w_1(s,z)\delta_{z,y+1} + w_0(s,y)\delta_{z,y}. \tag{16}$$

- LGV Theorem: the white dots $(y_j(s))$ form a determinantal point process e.g.,

$$\mathbb{P}\left((y_j(s))_{j,s=0,1}^{2,5}\right) \propto \prod_{s=0}^{5} \det\left(T_s(y_i(t), y_j(s+1))_{i,j=0}^{2}\right) \tag{17}$$

Connection between Paths and Lozenges



$$\frac{w_1(s,y)w_0(s+1,y+1)}{w_0(s,y)w_1(s+1,y)} = \frac{\mathfrak{w}(\lozenge(s+1,y))}{\mathfrak{w}(\lozenge(s+1,y+1))} \tag{18}$$

Extended q-Racah OPE

Theorem (Borodin, Gorin, and Rains (2010))

- *For q-Racah weights, for any fixed time s, the position of the white dots* $\{y_j\}_{j=0}^{na-1}$ *forms a q-Racah OPE*

$$\sim \prod_{0\leq i<j\leq na-1} (\mu_n(y_i) - \mu_n(y_j))^2\, w_n(y_1)\dots w_n(y_n), \tag{19}$$

- *The multi-time case: extended q-Racah OPE.*
- *Uniform weights: Johansson (2005) shows it gives the (extended) Hahn poylnomial ensemble.*

Connection Between Three Tiling Models: Askey Scheme

$$
\begin{tikzcd}
 & q\text{-Racah} \arrow[dl, "\kappa \to 0"'] \arrow[dr, "\kappa \to \infty"] & \\
q^{-Volume} \arrow[dr, "q \to 1"'] & & q^{Volume} \arrow[dl, "q \to 1"] \\
 & \text{Uniform} &
\end{tikzcd}
$$

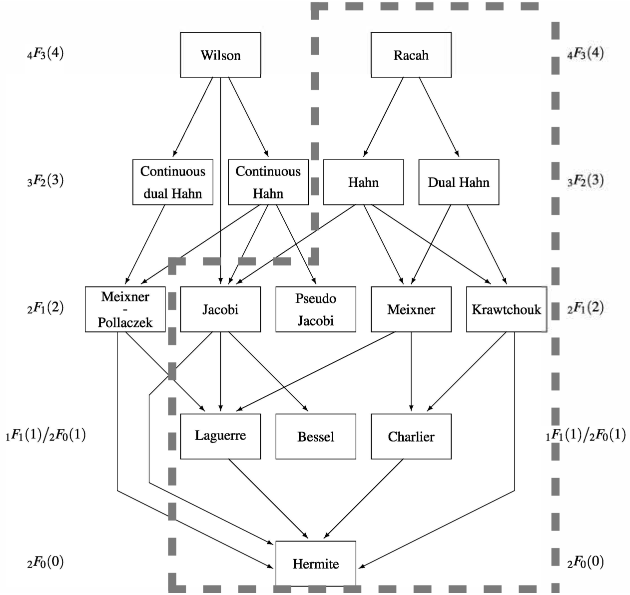
$$
\begin{tikzcd}
q\text{-Racah Polynomial} \arrow[d, "\kappa \to \infty \text{ or } 0"] \\
q\text{-Hahn Polynomial} \arrow[d, "q \to 1"] \\
\text{Hahn Polynomial}
\end{tikzcd}
$$

Connection Between Three Tiling Models: Asky Scheme

q-Racah Polynomial

$$\Big\downarrow\; q \to 1$$

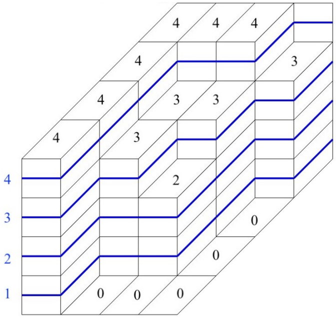
Racah Polynomial



Height Function

Height Function: counting paths below a point (t, x)

$$h_n(t, x) = \sum_{j=0}^{n-1} \chi_{\{x_j(t) \leq x\}} \tag{20}$$



The asymptotics of the height function as $n \to \infty$,

$$\frac{1}{n} h_n(t, x) \to \int_{\text{lower bound}}^{x} \text{Limit Shape}, \quad \text{weakly almost surely} \tag{21}$$

$$\sum_{j=1}^{m} h_n(t_j, x) - \mathbb{E}[h_n(t_j, x)] \to \text{Gaussian Field}, \quad \text{weakly in distribution} \tag{22}$$

Known Results

- **Uniform:**
  1. Cohn, Larsen, and Propp (1998) (calculus of variations) mean convergence
  2. Bufetov and Gorin (2018) (Schur generating functions)
  3. Petrov (2015) (steepest descent) GFF
  4. Duits (2018) (OPE) GFF
  5. and more on local scale... (with inverse Kasteleyn)
- ***q*-Volume:**
  1. Ahn (2020) (Macdonald process) LLN (in prob) and GFF
  2. Barhoumi and Duits (2025) (q-Jacobi OP) local fluctuation along arctic curve
- ***q*-Racah:**
  1. Dimitrov and Knizel (2019) (Discrete Loop Equation) LLN (in prob) 1D Log Gaussian
  2. Gorin and Huang (2024) (Dynamical Loop Equation) LLN (in prob) GFF
  3. Duits, Duse, and L' (2024) (q-Racah OP) LLN (a.s.) GFF
  4. Knizel and Petrov (2025) (q-Racah OP) unscaled q-Racah waterfall LLN

Limit Shape Density

Theorem (Duits, Duse, and L' (2024))

$$\partial_x h(t,x) = \begin{cases} 1 & \xi_-(x) < \xi_+(x) \leq 1 \\ \frac{1}{\pi} \arccos\left(\frac{2y(1)-y(\xi_-(x))-y(\xi_+(x))}{|y(\xi_-(x))-y(\xi_+(x))|}\right) & \xi_+(x) > 1 > \xi_-(x) \\ 0 & \xi_+ > \xi_-(x) \geq 1. \end{cases} \tag{23}$$

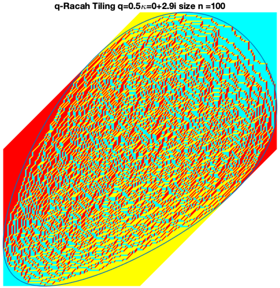
where

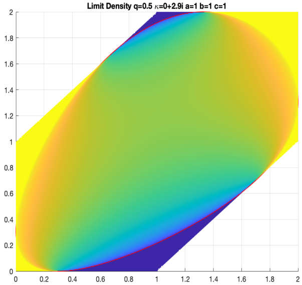
- $y(\xi) := q^{-\xi} + q^{\xi-b-c-2a}$,
- $\xi_-(x) \leq \xi_+(x)$ are two real solutions to

$$4a(\xi)^2 - (\mu - b(\xi))^2 = 0. \tag{24}$$

where a and b are limits of recurrence coefficients of q-Racah polynomials.

Limit Shape Density Example: $q = 0.5$, $\kappa = 29i$



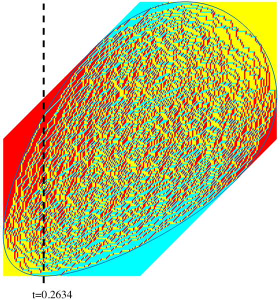


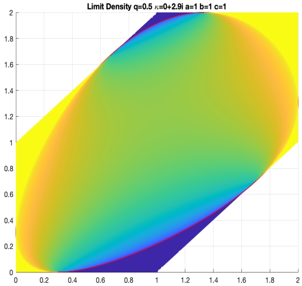
1. Frozen regions $\partial_x h(t,x) \in \{0,1\}$, liquid region $\partial_x h(t,x) \in (0,1)$
2. Arctic curve is algebraic of degree two in $\mu(t,x)$ and q^{-t}:

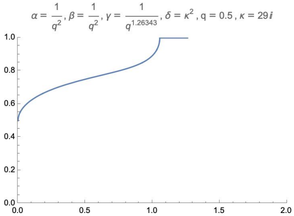
$$4a(1)^2 - (\mu - b(1))^2 = 0$$

3. Six turning points: $\xi_-(x_0) = \xi_+(x_0) = 1$.

Limit Shape Density Example: $q = 0.5$, $\kappa = 29i$







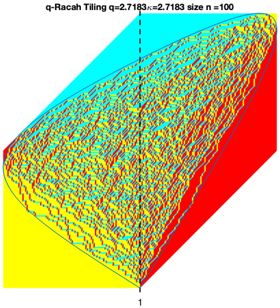
Turning point : $x_0 = 0$ and $t \approx 0.26$

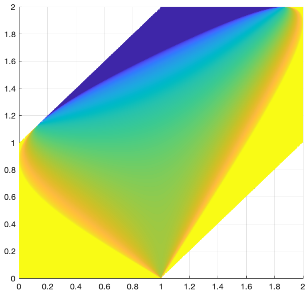
$\xi_+(x_0) = \xi_-(x_0) = 1$ and $\mu'(x_0) \neq 0$.

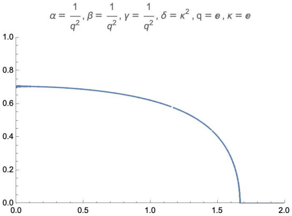
$$\partial_x h(t,x) = \frac{1}{\pi} \arccos\left(c_1 \sqrt{x} + c_2 x^{\frac{3}{2}} + O(x^{\frac{5}{2}})\right), \qquad (25)$$

$c_1 < 0$.

Limit Shape Density Singular Example: $q = e, \kappa = e$





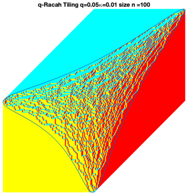


$x_0 = 0$ Singular point $\xi_+(x_0) = \xi_-(x_0) = 1$ and $\mu'(x_0) = 0$.

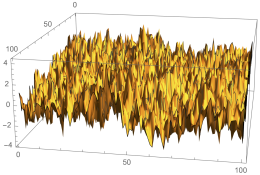
$$\partial_x h(t, x) = \frac{1}{\pi} \arccos\left(c_3 + c_4 x^2 + O(x^3)\right). \tag{26}$$

Height Function and Gaussian Free Field

Find a diffeomorphism: $F : \{(t,x) : \text{Liquid Region}\} \to \mathbb{R} \times (0,\pi)$



$$h_n \circ F^{-1} \to GFF$$



Theorem (Duits, Duse, and L', 2024; Gorin and Huang, 2024)

For $\phi \in C_c^2(\mathbb{R} \times (0,\pi))$ and $\Delta\phi$ being piecewise constant in first coordinate,

$$(h_n \circ F^{-1}, -\Delta\phi) - \mathbb{E}[(h_n \circ F^{-1}, -\Delta\phi)] \to^d \mathcal{N}(0, \|\phi\|_\nabla^2) \tag{27}$$

where $\nabla = (-\Delta)^{\frac{1}{2}}$.

Variational Problem

Connection between limit shape and GFF.

Variational Problem

Surface tension: $\sigma(t,x)$ such that $\det D^2\sigma = 1$ in a convex polygon. (Lobachevsky function)

$$min_{u \in \mathscr{A}_N(P,u_0)} \int\int_{\text{Hexagon}} \sigma(\nabla u(t,x)) + u(t,x)w(t,x)dtdx \tag{28}$$

$$w(t,x) := \frac{\partial}{\partial x} \log\left|\partial_x \mu(t,x)\right|, \qquad \mu(t,x) = q^{-x} + \kappa^2 q^{-t-c+x}$$

$\mathscr{A}_N(P,u_0)$ the space of all possible height functions in the Hexagon with boundary condition.

1. Uniform tiling ($w(t,x) = 0$) is studied by Kenyon and Okounkov (2007). Astala, Duse, Prause, and Zhong (2020) classify the regularity of minimizers.
2. q-Racah tiling: proposed by Borodin, Gorin, and Rains (2010)

Variational Problem: inhomogeneous term

Eular-Lagrangian: in the liquid region

$$\mathrm{div} \nabla \sigma(\nabla u^*) = w \tag{29}$$

Morales, Pak, and Tassy (2022): $h_{tile}(t, x) := x - h(t, x)$ is a solution, denoted by u^*.

1. ***q*-Racah:**

$$w(t, x) = -\log(q) \frac{q^{c+t} + \kappa^2 q^{2x}}{q^{c+t} - \kappa^2 q^{2x}}$$

2. ***q*-Volume ($\kappa = 0$):**

$$w(t, x) = -\log(q)$$

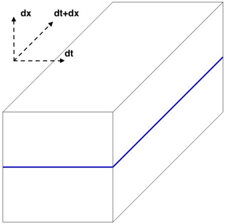
3. **Uniform ($\kappa = 0$, $q \to 1$):**

$$w(t, x) = 0$$

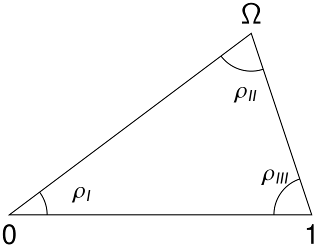
Height Function II

$h_{tile}(t,x) := x - h(t,x)$ is a solution.
Density of the paths: $\partial_x h(t,x)$.
Density of lozenges:

$$\rho_I/\pi = -\partial_t h, \qquad \rho_{II}/\pi = \partial_t h + \partial_x h, \qquad \rho_{III}/\pi = 1 - \partial_x h \tag{30}$$



Define Complex Slope: Ω



Variational Problem $t + ix = z \in \mathbb{C}$

Theorem (due to Astala and Duse)

Define

$$f(z) := \overline{(I + \nabla\sigma)(\nabla u^*(z))}. \tag{31}$$

Then f solves the inhomogeneous -quasilinear Beltrami equation:

$$f_{\bar{z}}(z) = \mathcal{H}'_\sigma(f(z))f_z(z) + w(z) \tag{32}$$

where $\mathcal{H}_\sigma(z)$ is the Cayley transform of $\nabla\sigma$.
In the case $w = 0$, we have

$$\mathcal{H}'_\sigma(f) = \frac{\partial_{\bar{z}} f}{\partial_z f}. \tag{33}$$

$$\mathcal{H}'_\sigma(f) = -\frac{1 + (i - 1)\Omega}{-1 + (1 + i)\Omega}. \tag{34}$$

- In the case of uniform tiling, f is the diffeomorphism for GFF!

Complex Structure and Complex Slope

In the case $w \neq 0$, ansatz,

$$f = \eta \circ F$$

with

$$\mathcal{H}'_{\sigma}(f) = \frac{\partial_{\bar{z}} F}{\partial_z F} \tag{35}$$

and

$$\mathcal{H}'_{\sigma}(f) = -\frac{1 + (i-1)\Omega}{-1 + (1+i)\Omega}. \tag{36}$$

Find η, F?

Conjecture

For any suitable w, F given by (36) will always be a diffeomorphism that gives a Gaussian Free Field out of the liquid region.

Complex Burgers Equation

Recall

$$(h_n \circ F^{-1}, -\Delta\phi) - \mathbb{E}[(h_n \circ F^{-1}, -\Delta\phi)] \to^d \mathcal{N}(0, \|\phi\|_\nabla^2) \tag{37}$$

Diffeomorphism F : liquid region $\to \mathbb{R} \times (0, \pi)$ is given by

$$F(t,x) = \frac{1}{2}\log\left(\frac{q^t - 1}{1 - q^{-b-c+t}}\right) + i \arccos \frac{\mu(t,x) - b(1;t)}{2\sqrt{a(1;t)}}$$

Surprisingly, F from our computation is the F in the variational problem theory.

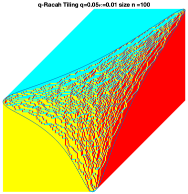
Theorem Duits, Duse, and L' (2024)

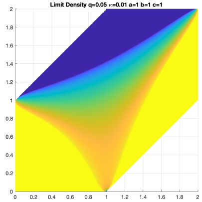
In the q-Racah tiling model we have in the liquid region

$$\frac{\partial_{\bar{z}} F}{\partial_z F} = -\frac{1 + (i-1)\Omega}{-1 + (1+i)\Omega} \tag{38}$$

$$\frac{\Omega_t}{\Omega} - \frac{\Omega_x}{1 - \Omega} = w(t,x) \tag{39}$$

The Story of tilings with q-Racah weights



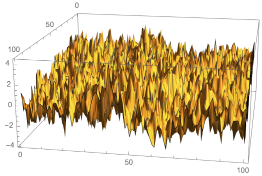


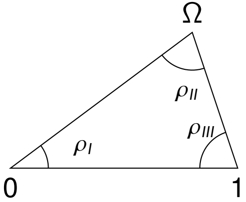
$$\frac{1}{n}h_n \to h$$

$$F : \text{liquid region} \to \text{strip}$$

$$\nabla h = \begin{pmatrix} -\frac{\rho_I}{\pi} \\ 1 - \frac{\rho_{III}}{\pi} \end{pmatrix} \Longleftrightarrow \Omega$$

$$\frac{\partial_{\bar{z}} F}{\partial_z F} = -\frac{1+(i-1)\Omega}{-1+(1+i)\Omega}$$





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