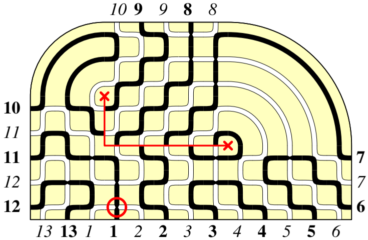
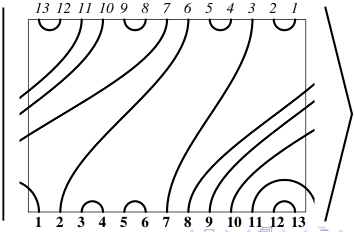
Towards a black+white Razumov-Stroganov correspondence

Andrea Sportiello, based on work with Luigi Cantini (CY Cergy Paris Université)

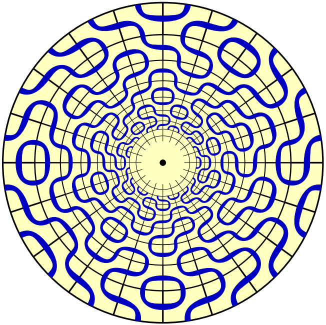




Workshop on integrable combinatorics
Chaire de la Vallée Poussin 2025

Louvain-la-Neuve, Nov 18–20, 2025

Two problems of (integrable) Random Tilings



$O(1)$ Dense Loop Model

on a semi-infinite cylinder (or strip)

$=$ XXZ Quantum Spin Chain at $\Delta = -\frac{1}{2}$

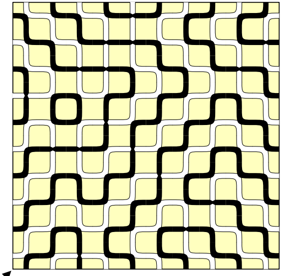
$=$ Edge-percolation (Potts Model at $Q = 1$)

Fully-Packed Loops (FPL) in a square (or some other domain mostly locally like a square lattice)

$=$ Alternating Sign Matrices (ASM)

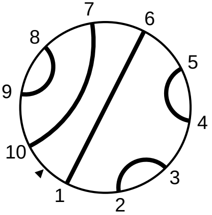
$=$ Six-Vertex Model at $\Delta = +\frac{1}{2}$ (Ice Model)

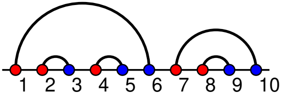
$=$ Non-Intersecting Lattice Paths



Link patterns

A link pattern $\pi \in \mathcal{LP}(2n)$ is a pairing of $\{1, 2, \ldots, 2n\}$ having no pairs (a, c), (b, d) such that $a < b < c < d$ (i.e., the drawing consists of n non-crossing arcs).





They are $C_n = \frac{1}{n+1}\binom{2n}{n}$ (the n-th *Catalan number*),

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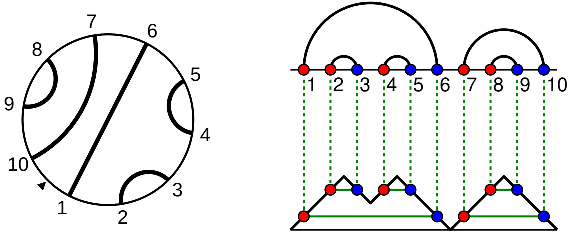

They are $C_n = \frac{1}{n+1}\binom{2n}{n}$ (the n-th *Catalan number*), are in easy bijection with Dyck Paths of length $2n$,

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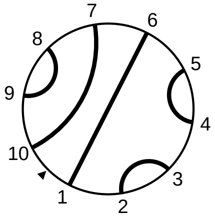
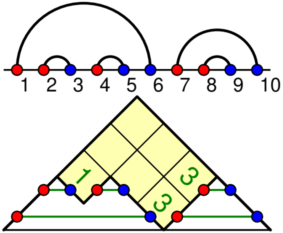
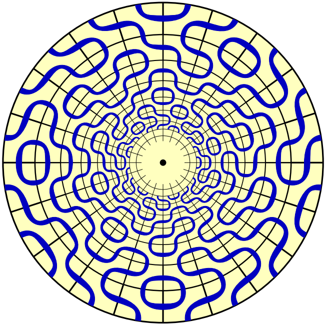




They are $C_n = \frac{1}{n+1}\binom{2n}{n}$ (the n-th *Catalan number*), are in easy bijection with Dyck Paths of length $2n$, and with integer partitions boxed in a triangle

Link patterns in the Dense Loop Model

We can associate a link pattern π to any dense-loop configuration on a semi-infinite cylinder, as the connectivity pattern among the points on the boundary.



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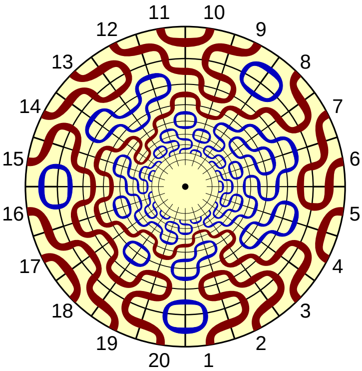

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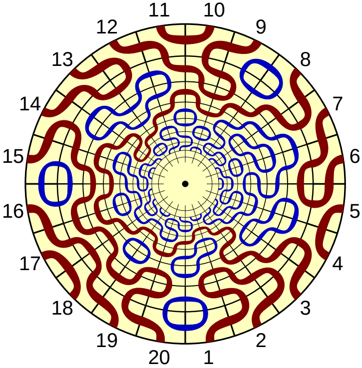
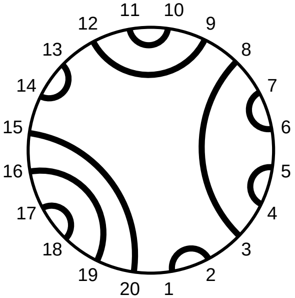
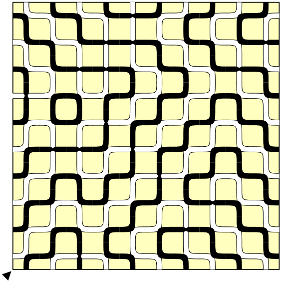


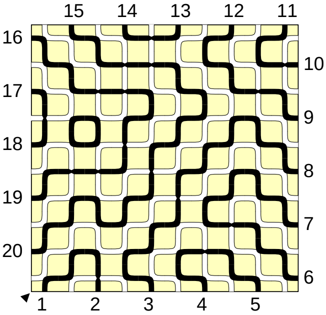


We can associate a link pattern π also to any Fully-Packed Loop configuration, as the connectivity pattern among the black terminations on the boundary.



Link patterns in Fully-Packed Loops

We can associate a link pattern π also to any Fully-Packed Loop configuration, as the connectivity pattern among the black terminations on the boundary.



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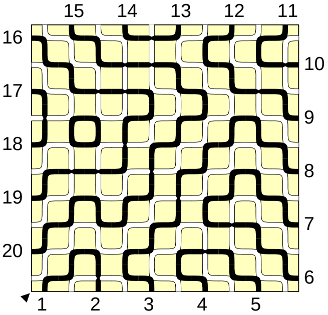
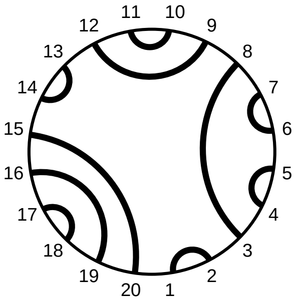
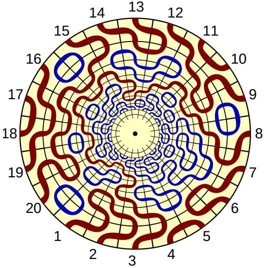
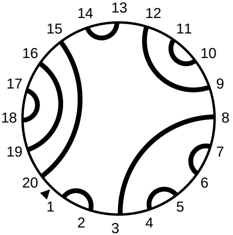
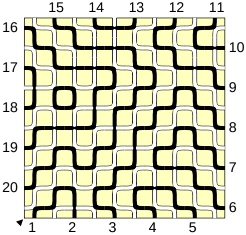




The dihedral Razumov–Stroganov correspondence



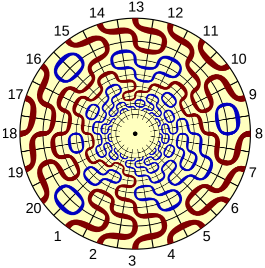


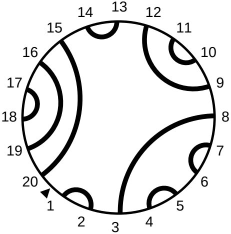


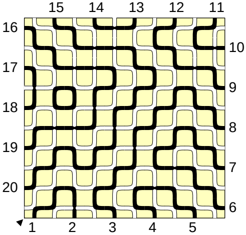
$\tilde{\Psi}_n(\pi)$: probability of π
in the $O(1)$ Dense Loop Model
in the $\{1, \ldots, 2n\} \times \mathbb{N}$ cylinder

$\Psi_n(\pi)$: probability of π
for FPL with uniform measure
in the $n \times n$ square

The dihedral Razumov–Stroganov correspondence







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Dihedral Razumov–Stroganov correspondence
(conjecture: Razumov Stroganov, 2001; proof: AS Cantini, 2010)

$$\tilde{\Psi}_n(\pi) = \Psi_n(\pi)$$

Many Razumov–Stroganov-type conjectures

In fact, there exists a whole class of Razumov–Stroganov conjectures

A.V. Razumov and Yu.G. Stroganov, *Combinatorial nature of ground state vector of $O(1)$ loop model*, Theor. Math. Phys. **138** (2004); —, *$O(1)$ loop model with different boundary conditions and symmetry classes of alternating-sign matrices*, Theor. Math. Phys. **142** (2005); J. de Gier, *Loops, matchings and alternating-sign matrices*, Discr. Math. **298** (2005); S. Mitra, B. Nienhuis, J. de Gier and M.T. Batchelor, *Exact expressions for correlations in the ground state of the dense $O(1)$ loop model*, JSTAT(2004); J. de Gier and V. Rittenberg, *Refined Razumov–Stroganov conjectures for open boundaries*, JSTAT(2004); Ph. Duchon, *On the link pattern distribution of quarter-turn symmetric FPL configurations*, FPSAC 2008

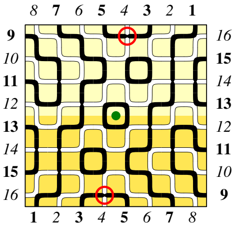
These variants are naturally arranged into two main classes:

dihedral RS: FPL domains with Wieland dihedral symmetry,
$\Leftrightarrow$ $O(1)$DLM on the cylinder (the periodic quantum spin chain)

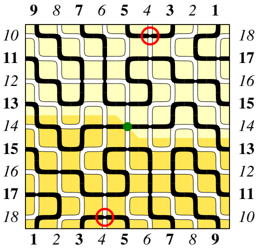
vertical RS: FPL domains with a "reflecting wall" of U-turn/O-turn $\Leftrightarrow$ versions of the $O(1)$DLM on the strip (the open or closed boundary quantum spin chain)

Some other dihedral Razumov–Stroganov (ex-)conjectures

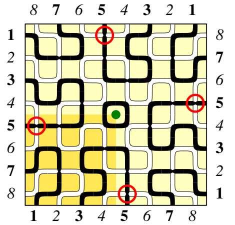
HTASM $L = 2n$ †



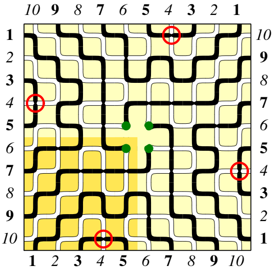
HTASM $L = 2n + 1$ †



QTASM $L = 4n$ ‡



qQTASM $L = 4n + 2$ ‡



† HTASM = Half-turn symmetric ASM's

‡ QTASM = Quarter-turn symmetric ASM's

qQTASM = quasi-Quarter-turn symmetric ASM's

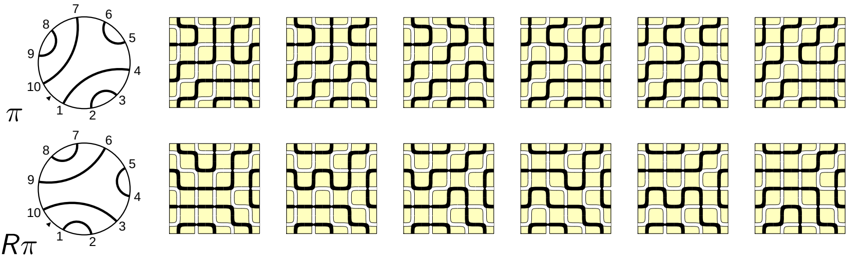
Dihedral symmetry of FPL

A corollary of the Razumov–Stroganov correspondence...
(...that was known *before* the Razumov–Stroganov conjecture)

call R the operator that rotates a link pattern by one position

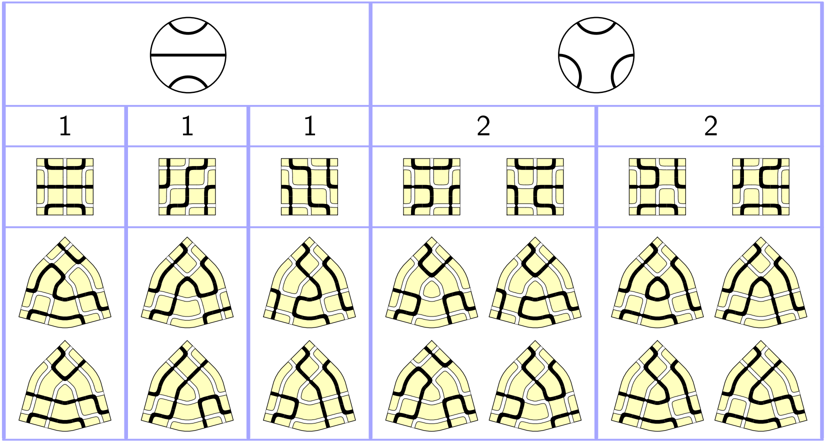
Dihedral symmetry of FPL (proof: Wieland, 2000)

$$\Psi_n(\pi) = \Psi_n(R\pi)$$



The domains where dihedral Razumov–Stroganov holds

In the case of the dihedral Razumov–Stroganov correspondence, Wieland gyration (and its generalisations) has been a crucial ingredient and led us to classify the family of domains for which RS holds



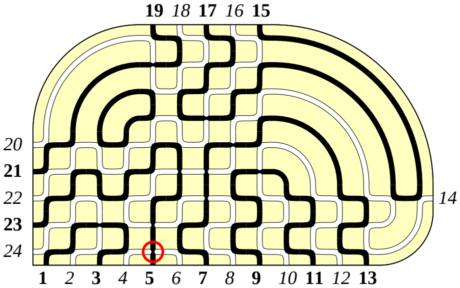
The domains where dihedral Razumov–Stroganov holds

So, in *proving* the various existing (dihedral) Razumov–Stroganov conjectures, we have been led to *generalise* them to a much larger family of domains ($\sim n^3$ different domains for $\mathcal{LP}(2n)$).

There are three subclasses, according to the type of link patterns and Temperley–Lieb algebras: ordinary, punctured even and punctured odd.

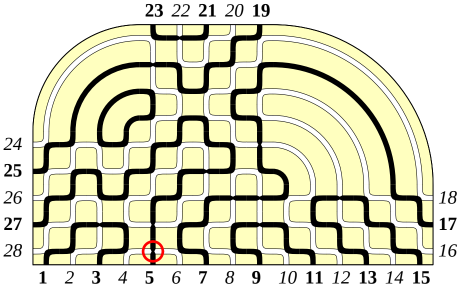
The domains where dihedral Razumov–Stroganov holds

1 corner, 3 triangles:



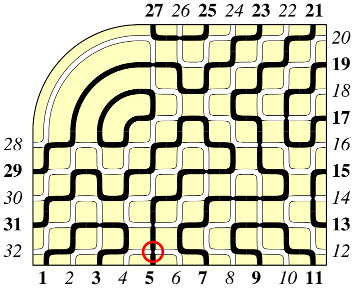
The domains where dihedral Razumov–Stroganov holds

2 corners, 2 triangles:



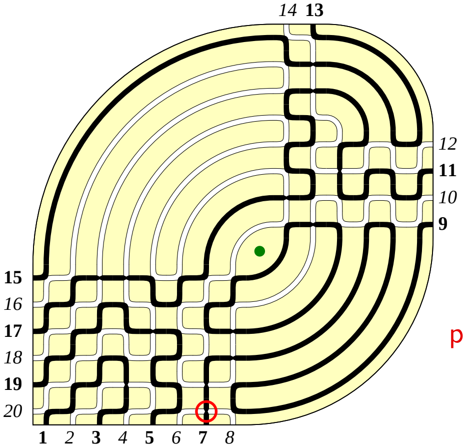
The domains where dihedral Razumov–Stroganov holds

3 corners, 1 triangle:



The domains where dihedral Razumov–Stroganov holds

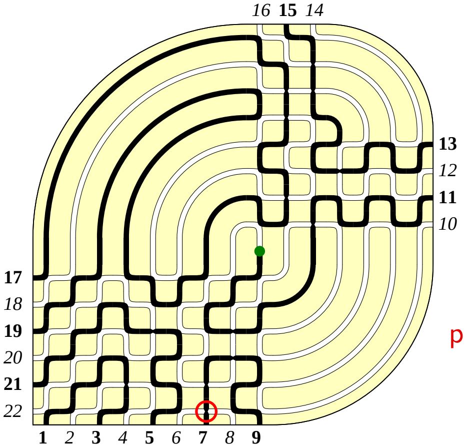
1 corner, 1 triangle, 1 face with $\ell = 2$:



(this works with punctured link patterns of even size!)

The domains where dihedral Razumov–Stroganov holds

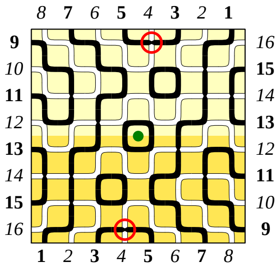
1 corner, 1 triangle, 1 vertex of degree 2:

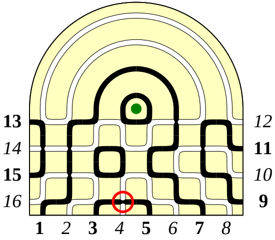


(this works with punctured link patterns of odd size!)

The domains where dihedral Razumov–Stroganov holds

2 corners, 1 face with $\ell = 2$:
(these are HTASM of even side,
half-turn symmetric ASM's,
and it works with punctured
link patterns, of even or odd size)

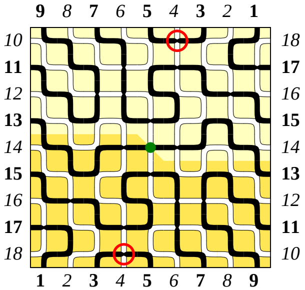


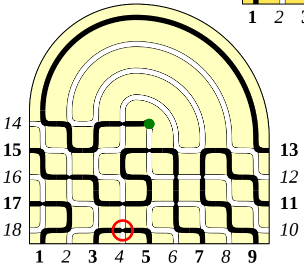


$L = 2n$

The domains where dihedral Razumov–Stroganov holds

2 corners, 1 vertex of degree 2:
(these are HTASM of odd side,
half-turn symmetric ASM's,
and it works with punctured
link patterns, of even or odd size)

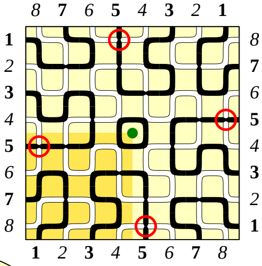


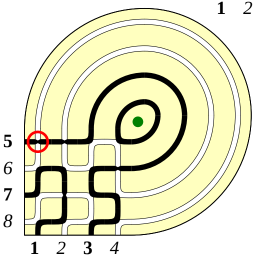


$L = 2n + 1$

The domains where dihedral Razumov–Stroganov holds

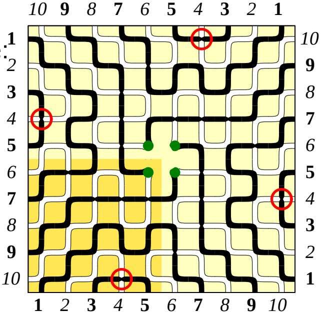
1 corner, 1 face with $\ell = 1$:
(these are QTASM,
quarter-turn symmetric ASM's)
and it works with punctured
link patterns of even size)



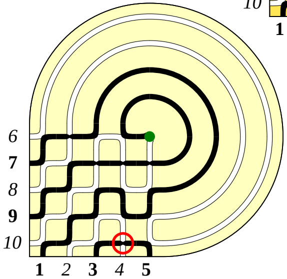


$L = 4n$

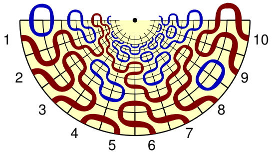
1 corner, 1 v. deg. 2 next to a triangle:
(these are *q*QTASM,
quasi–quarter-turn symmetric ASM's)
and it works with punctured
link patterns of odd size)

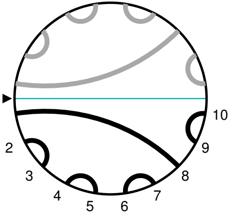


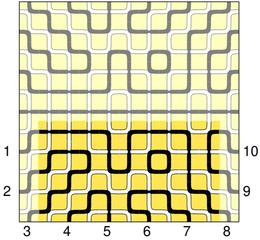
$L = 4n + 2$



A Vertical Razumov–Stroganov Conjecture



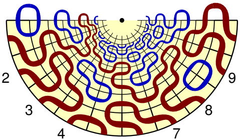


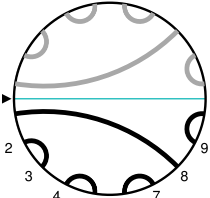


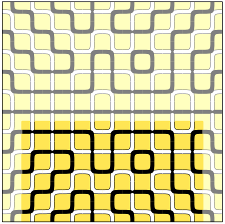
$\tilde{\Psi}_n^{\mathrm{V}}(\pi)$: probability of π
in the $O(1)$ Dense Loop Model
in the $\{1, \dots, 2n\} \times \mathbb{N}$ strip

$\Psi_n^{\mathrm{V}}(\pi)$: probability of π
for vertically-symmetric FPL
with uniform measure in the
$(2n+1) \times (2n+1)$ square

A Vertical Razumov–Stroganov Conjecture







$\tilde{\Psi}^{\mathrm{V}}_n(\pi)$: probability of π
in the $O(1)$ Dense Loop Model
in the $\{1, \dots, 2n\} \times \mathbb{N}$ strip

$\Psi^{\mathrm{V}}_n(\pi)$: probability of π
for vertically-symmetric FPL
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Vertical Razumov–Stroganov conjecture
(Razumov and Stroganov, 2001, for the square of side $2n+1$)

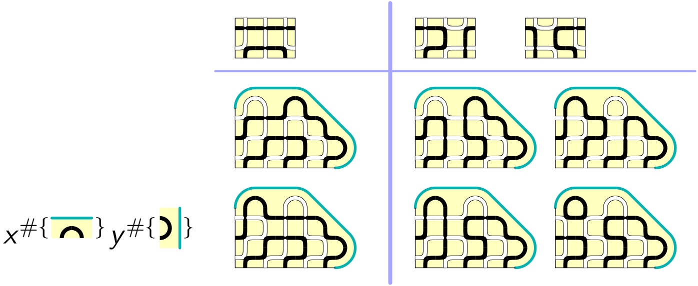
$$\tilde{\Psi}^{\mathrm{V}}_n(\pi) = \Psi^{\mathrm{V}}_n(\pi)$$

The domains where vertical Razumov–Stroganov holds

The Vertical Razumov–Stroganov conjectures are a whole second family
They involve FPL with some version of reflecting wall
and the $O(1)$ Dense Loop Model on a strip with a boundary

Our proof methods do not seem to work for any of the Vertical Razumov–Stroganov conjectures, which are all open at present

But at least we think we know the precise list of domains with Vertical RS



$3 + x + 7y + 2xy + 4y^2 + xy^2$ $\qquad$ $6 + 2x + 14y + 4xy + 8y^2 + 2xy^2$

The RS obsession: why no black+white RS?

In June 2024 there was the *"At the crossroads of physics and mathematics: the joy of integrable combinatorics — A conference in the honor of Philippe Di Francesco's 60th birthday"* in IPTh

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In June 2024 there was the *"At the crossroads of physics and mathematics: the joy of integrable combinatorics — A conference in the honor of Philippe Di Francesco's 60th birthday"* in IPTh
I was presenting my *other* recent results, on the structure constants of the canonical Grothendieck polynomials arising in the $(\pi_\bullet, \pi_\circ, \#\{\bigcirc\})$ statistics of FPL's in the VSASM's.

A point of my talk was the disappointment for the fact that we only have a (dihedral) RS correspondence for the $\pi_\bullet$ statistics, despite the fact that Wieland gyration is a statement on the triple $(\pi_\bullet, \pi_\circ, \#\{\bigcirc\})$.

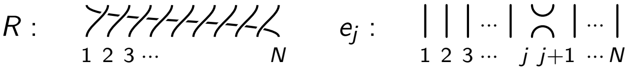
Since then, I have been thinking back to the question. And I found a sensible recipe, *and* a further generalisation…

…at this aim, we need to introduce a gauge theory for FPL's

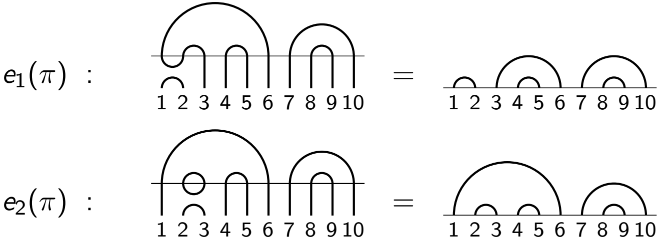
…but before doing this, we shall give a short summary of the old 2012 proof…

The Temperley-Lieb monoid $TL_N(\tau = 1)$

Consider the graphical action over link patterns $\pi \in \mathcal{LP}(N)$



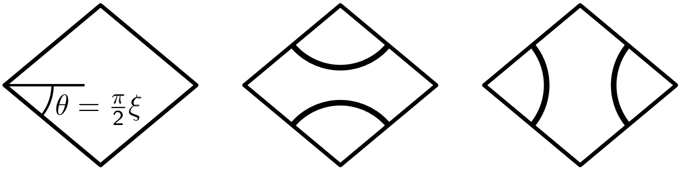
In the $TL_N(\tau)$ Algebra, $e_j^2 = \tau e_j$. When $\tau = 1$, the e_j's and $R^{\pm 1}$ act stochastically on link patterns



Consider the linear space $\mathbb{C}^{\mathcal{LP}(N)}$, linear span of basis vectors $|\pi\rangle$.
Operators e_j and $R^{\pm 1}$ are stochastic linear operators over $\mathbb{C}^{\mathcal{LP}(N)}$

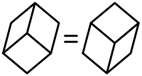
$O(1)$ Dense Loop Model: the Markov Chain over $\mathcal{LP}(N)$

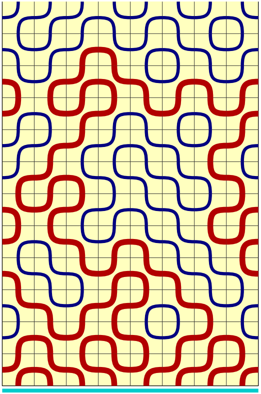
The "integrable weigths" of the $O(\tau)$ Dense Loop Model (with $\tau = 2\cos\gamma$) on isoradial graphs* are



$\xi \in [0,1]$ sin $\gamma\xi$ sin $\gamma(1-\xi)$

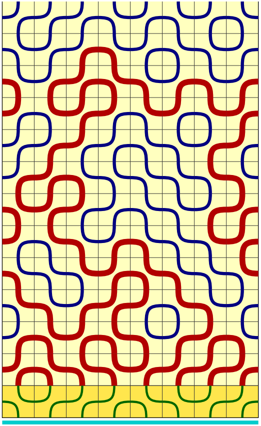
* The relation between the angle θ and the integrable weights is natural in two respects: the density of free energy is uniform, and the YBE condition corresponds to "flipping a cube"





A config with $t-1$ layers, and link pattern π

$O(1)$ Dense Loop Model: the Markov Chain over $\mathcal{LP}(N)$



A config with $t-1$ layers, and link pattern π

Add a new layer, of i.i.d. tiles, with probability $p($$)/p($$)$
$= p/(1-p) = \sin(\frac{\pi}{3}\xi)/\sin(\frac{\pi}{3}(1-\xi))$
(say, $p = 1/2$, i.e. $\xi = 1/2$)...

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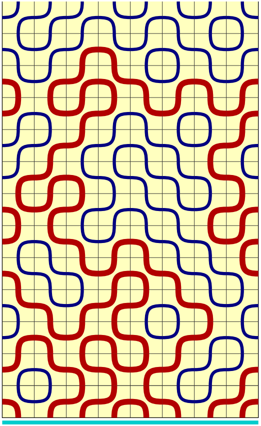

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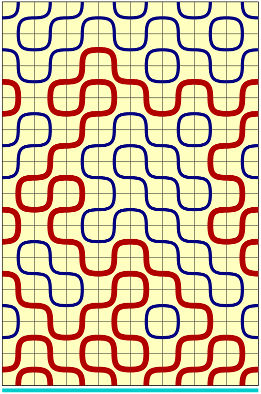
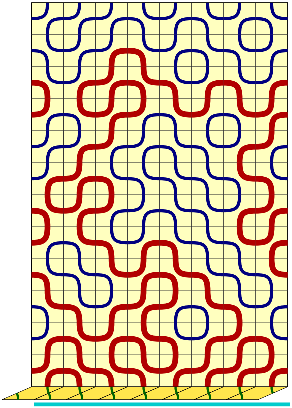
Some loops get detached from the boundary. You have a config with t layers, and a new link pattern π'.

The rates $W_p(\pi, \pi')$ are encoded by a big polynomial T_p in $R^{\pm 1}$ and the e_j's:
$T_p|\pi\rangle = \sum_{\pi'} W_p(\pi, \pi')|\pi'\rangle$

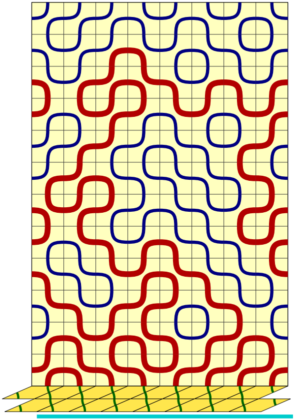
Now repeat the game,





Now repeat the game,

but add i.i.d. tiles, with prob. $p \to 0$



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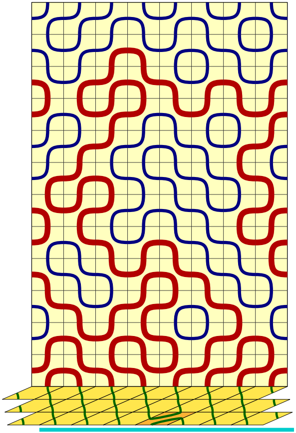
For most of the layers you just rotate



Now repeat the game,

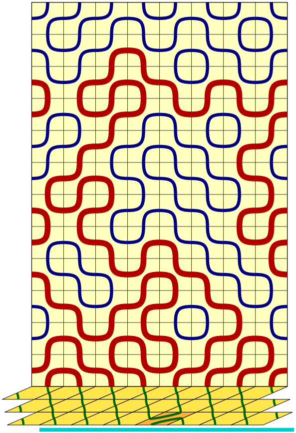
but add i.i.d. tiles, with prob. $p \to 0$

For most of the layers you just rotate From time to time, you have a single non-trivial tile. In the limit, the probability of having two non-trivial tiles in the same row vanishes.

The rates are
$W_{p\to 0}(\pi, \pi') = \delta(\pi', R\pi) + \mathcal{O}(p)$.
More precisely, the operator T_p has the form

$$T_p = R\big(I + p \textstyle\sum_j (e_j - 1) + \mathcal{O}(p^2)\big)$$

$O(1)$ Dense Loop Model: the Markov Chain over $\mathcal{LP}(N)$



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Hamiltonian H

Integrability: commutation of Transfer Matrices

The 1-parameter family of matrices for the transition rates, $W_p(\pi, \pi')$, acting on $\mathbb{C}^{\mathcal{LP}(N)}$ by filling one layer of lozenges with angle θ, form a commuting family. I.e. the family of polynomials T_p in the Temperley–Lieb Algebra form a commuting family.

Trivial: $\tilde{\Psi}_p(\pi)$, the steady state, is the unique eigenstate of $T_p(\pi, \pi')$ with all positive entries

The Yang–Baxter relation implies:
$\sum_{\pi'} W_{p_1}(\pi, \pi') W_{p_2}(\pi', \pi'') = \sum_{\pi'} W_{p_2}(\pi, \pi') W_{p_1}(\pi', \pi'')$,
or also, in TL alg., $[T_p, T_{p'}] = 0$

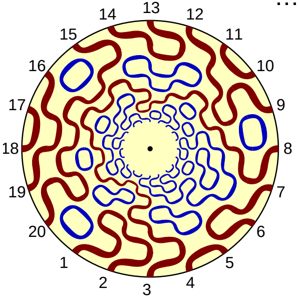
Consequence: $\tilde{\Psi}_p(\pi) \equiv \tilde{\Psi}_{p'}(\pi)$ and we can get $\tilde{\Psi}(\pi) := \tilde{\Psi}_{1/2}(\pi)$ from the study of $T_{p\to 0}$. Namely, calling $|\tilde{\Psi}_n\rangle = \sum_\pi \tilde{\Psi}(\pi)|\pi\rangle$ and $H_n = \frac{\partial}{\partial p}(R^{-1} T_p)|_{p=0} = \sum_{i=1}^{2n}(e_i - 1)$, we have

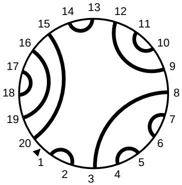
$$H_n|\tilde{\Psi}_n\rangle = 0$$

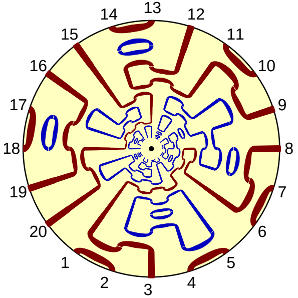
linear-algebra characterization of $\tilde{\Psi}(\pi)$

Integrability: commutation of Transfer Matrices

...said with a picture...







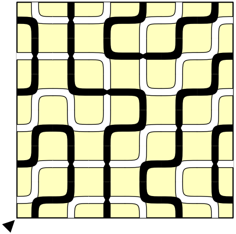
$$|\tilde{\Psi}_n\rangle := \sum_{\pi \in \mathcal{LP}(2n)} \tilde{\Psi}_n(\pi)|\pi\rangle \qquad |\tilde{\Psi}_n\rangle := \sum_{\pi \in \mathcal{LP}(2n)} \tilde{\Psi}_n(\pi)|\pi\rangle$$

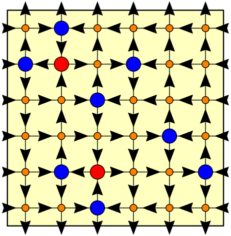
$$(T_n - 1)|\tilde{\Psi}_n\rangle = 0 \qquad H_n|\tilde{\Psi}_n\rangle = 0$$

the two linear equations for $|\tilde{\Psi}_n\rangle$ are equivalent!

Refinement position in Fully-Packed Loops

Fully-Packed Loops have a unique straight tile on any external line (and Alternating Sign Matrices have a unique $+1$ on any external line)





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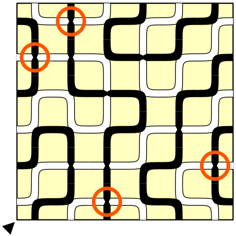

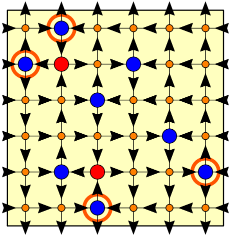

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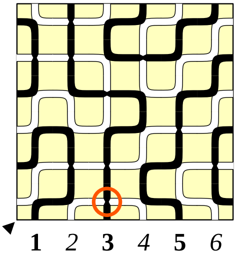
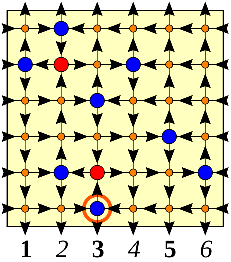
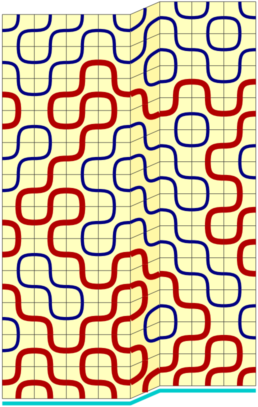


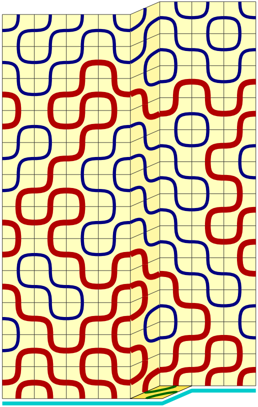


Concentrate on the bottom row, and call refinement position the corresponding column index.

The Izergin–Korepin determinant gives us the total number of FPL configurations, possibly refined according to these 4 statistics, but not the numbers refined according also to the link patterns. . .

Repeat the game once more...



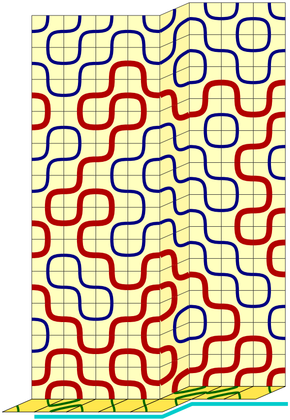


Repeat the game once more...

...but this time keep all tiles frozen, except for the one in column $i+1$

$$RX_i(t) = R(t + (1-t)e_i)$$

These simple operators seem to have nothing to do with the original problem.



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Nonetheless, calling $S_i(t) = (RX_i(t))^N$ the Scattering Matrix on column i, we have $S_i(1-t) = 1 + t\,H + \mathcal{O}(t^2)$

So, if we understand the Frobenius vector $\Psi(t)$ of $S_i(1-t)$, (i.e. the Frobenius vector of $RX_i(1-t)$), we also understand the RS vector Ψ.

Dihedral covariance of the eigenvectors $|\tilde{\Psi}_n^{(i)}(t)\rangle$

In the original formulation of the Razumov–Stroganov conjecture we have $|\tilde{\Psi}_n\rangle = \sum_\pi \tilde{\Psi}(\pi)|\pi\rangle$, satisfying $H_n|\tilde{\Psi}_n\rangle = 0$

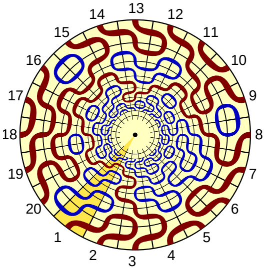
The operators $RX_i(t)$, and the scattering matrices $S_i(t)$, induce the deformation
$|\tilde{\Psi}_n^{(i)}(t)\rangle = \sum_\pi \tilde{\Psi}^{(i)}(t;\pi)|\pi\rangle$, satisfying $(RX_i(t) - 1)|\tilde{\Psi}_n^{(i)}(t)\rangle = 0$.

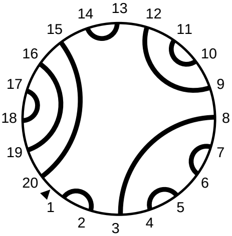
Because of a dihedral covariance of these equations, (and unicity of the Frobenius vector) it suffices to study $RX_1(t)$ and $|\tilde{\Psi}_n^{(1)}(t)\rangle$

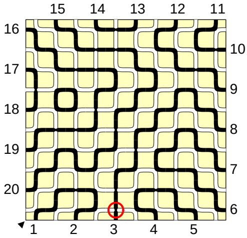
i.e., $0 = (X_i(t) - R^{-1})|\tilde{\Psi}_n^{(i)}(t)\rangle = R(X_{i+1}(t) - R^{-1})R^{-1}|\tilde{\Psi}_n^{(i)}(t)\rangle$
implies $|\tilde{\Psi}_n^{(i+1)}(t)\rangle \propto R^{-1}|\tilde{\Psi}_n^{(i)}(t)\rangle$

Call $\mathrm{Sym} = N^{-1}\sum_{i=0}^{N-1} R^i$, the operator that projects on the rotationally-invariant subspace of $\mathbb{C}^{\mathcal{LP}(N)}$.

The refined Razumov–Stroganov correspondence







$\tilde{\Psi}_n(t;\pi)$: probability of π in the $O(1)$ Dense Loop Model with dynamics given by $RX_1(t)$

$\Psi_n(t;\pi)$: count FPL's ϕ having link pattern π give $t^{h(\phi)-1}$ weight

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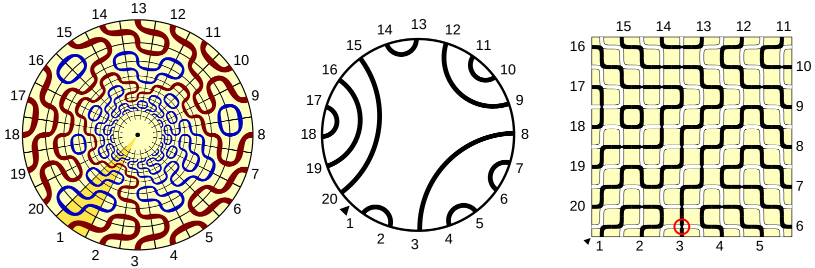

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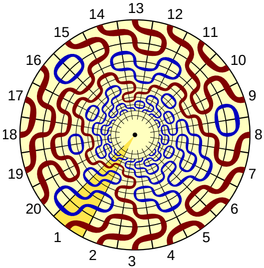
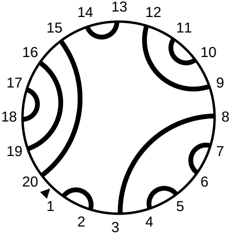
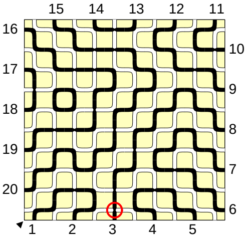
(conjecture: Di Francesco, 2004; proof: AS Cantini, 2012)

$$\tilde{\Psi}_n(t;\pi) \neq \Psi_n(t;\pi)$$







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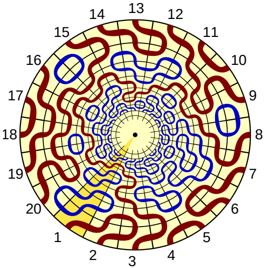

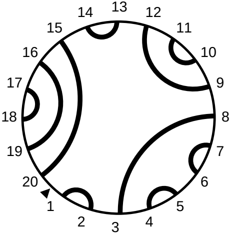

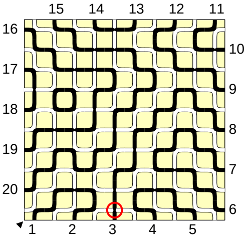

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$$\mathrm{Sym}\,|\tilde{\Psi}_n(t)\rangle = \mathrm{Sym}\,|\Psi_n(t)\rangle$$

A quest for a new strategy

In 2010, Cantini and myself gave a first proof of the (unrefined) Razumov–Stroganov conjecture. Later on, in 2012 we gave a proof for the refined Di Francesco conjecture, which also provides a *(more illuminating?)* proof of the original RS

2010:
- Realize that $H|\tilde{\Psi}\rangle = 0$ fixes $|\tilde{\Psi}\rangle$ univocally;
- Prove combinatorially that also $|\Psi\rangle$ satisfies $H|\Psi\rangle = 0$...

...But the $|\tilde{\Psi}^{(i)}\rangle$'s differ (they are only dihedrally covariant), and satisfy different linear equations (with $RX_i(t)$)...

...and $\mathrm{Sym}\,|\tilde{\Psi}^{(i)}\rangle$ does not satisfy any simple linear equation that fixes it univocally!

2012:
- Find a new way $\pi'(\phi)$ of associating link patterns to FPL;
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Bonus: The new enumeration is interesting by itself

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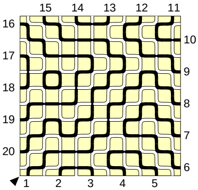
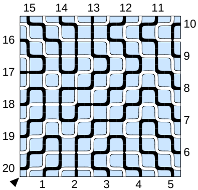
The heretical enumeration



The role of black and white is symmetrical...

The heretical enumeration

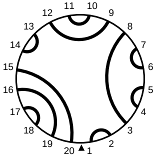


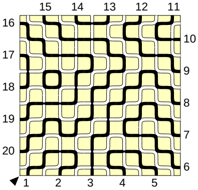
...who's who is a matter of convention.

Swapping coloration in all FPL's leads to an equivalent conjecture

The heretical enumeration

$\pi(\phi)$

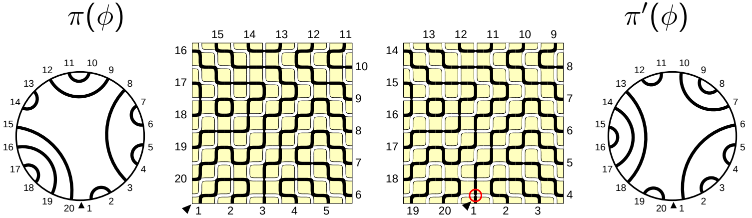




$\pi'(\phi)$

Here's the new rule: if the refinement position is odd...

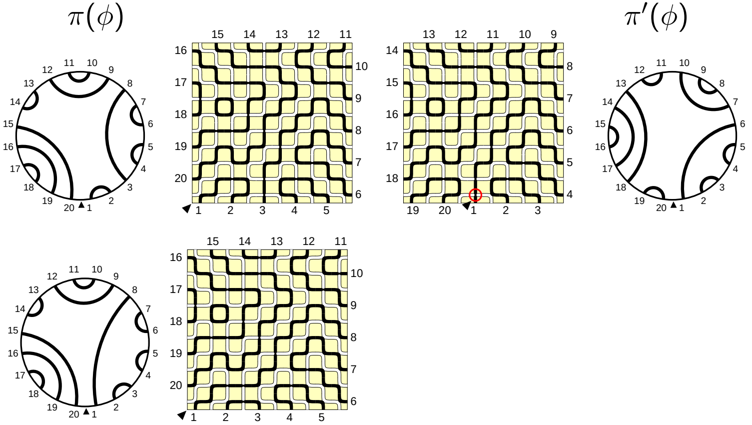
The heretical enumeration



Here's the new rule: if the refinement position is odd...

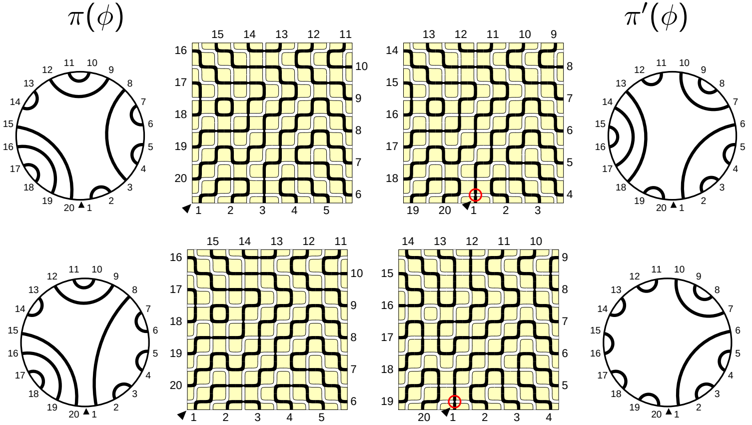
...you just rotate the starting point to the refinement position

The heretical enumeration



if the refinement position is even...

The heretical enumeration



if the refinement position is even…

…you swap black and white, and rotate the starting point

Use projectors to get 2 simple equations (instead of 1 difficult eq.)

We wanted to prove Di Francesco 2004 conjecture:

$$\mathrm{Sym}\ |\tilde{\Psi}(t)\rangle = \mathrm{Sym}\ |\Psi(t)\rangle$$

with $|\tilde{\Psi}(t)\rangle$ solving $(X_1(t) - R^{-1})|\tilde{\Psi}(t)\rangle = 0$
and $|\Psi(t)\rangle = \sum_\phi t^{h(\phi)-1}|\pi(\phi)\rangle$

We have been led to split this in two parts:

$|\tilde{\Psi}(t)\rangle = |\Psi'(t)\rangle$ and $\mathrm{Sym}\ |\Psi'(t)\rangle = \mathrm{Sym}\ |\Psi(t)\rangle$
with $|\Psi'(t)\rangle = \sum_\phi t^{h(\phi)-1}|\pi'(\phi)\rangle$

The first relation is proven if you show that

$$(X_1(t) - R^{-1})|\Psi'(t)\rangle \equiv (t - R^{-1} - (t-1)e_1)|\Psi'(t)\rangle = 0$$

recalling that $e_1^2 = e_1$, and $(1-e_1)^2 = (1-e_1)$:

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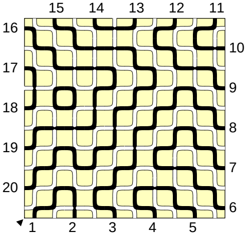
FPL config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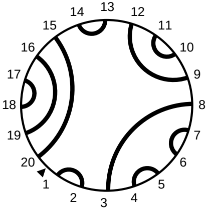
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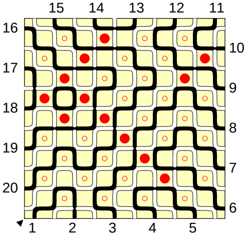
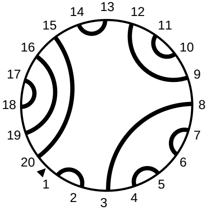
Wieland gyration: how it works

Mark faces  and ,
of given parity





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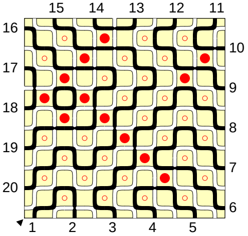
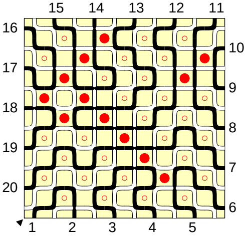
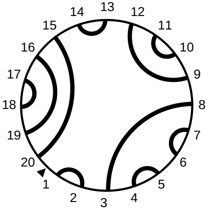


Exchange ⇔





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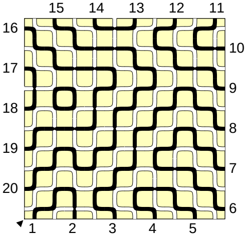

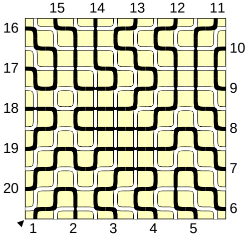

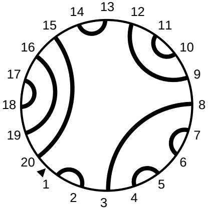

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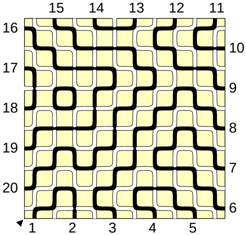

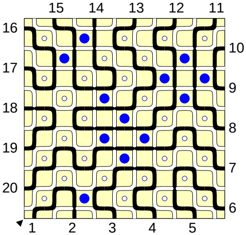

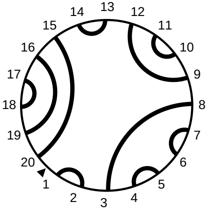

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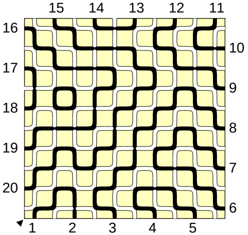

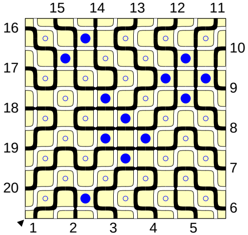

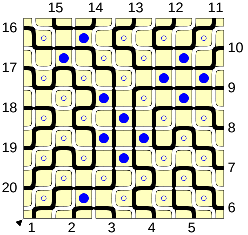

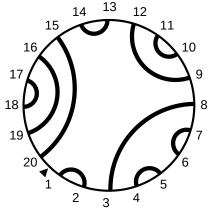

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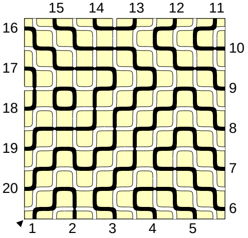
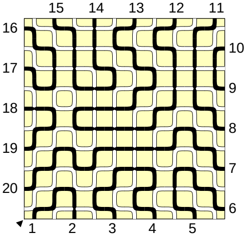
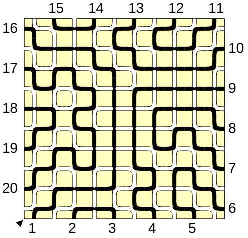
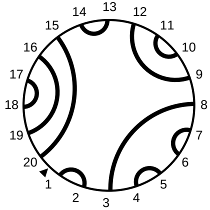


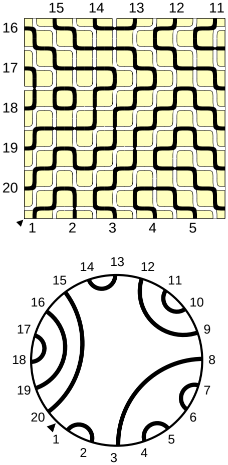


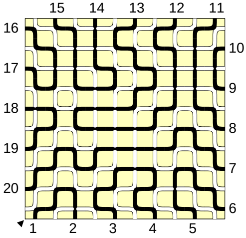




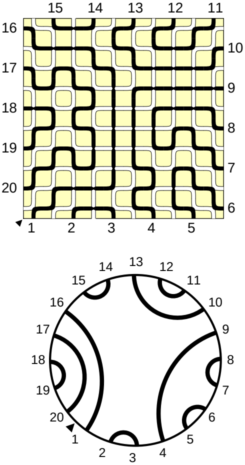
Wieland gyration: how it works

Link pattern π...



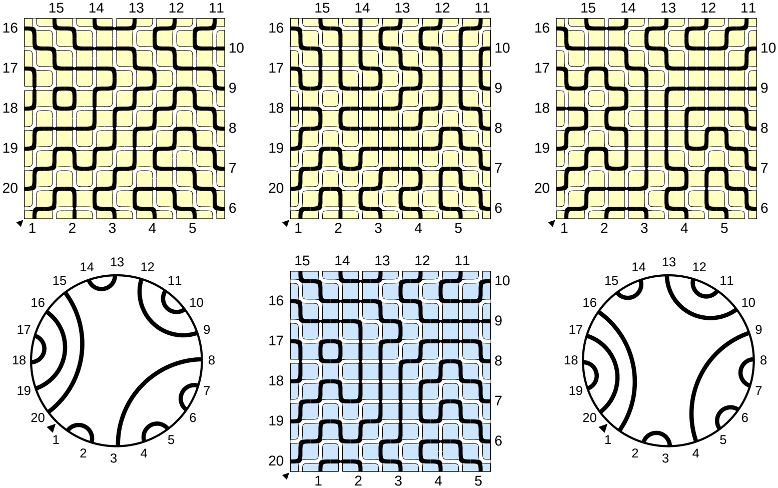


...and $R\,\pi$...



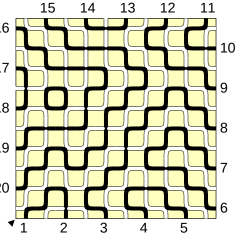
Wieland gyration: how it works

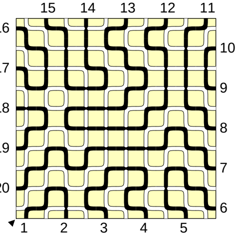
Link pattern π... | ...and, on the conjugate of the intermediate step... | ...and $R\,\pi$...

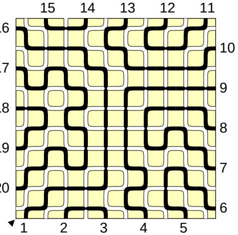


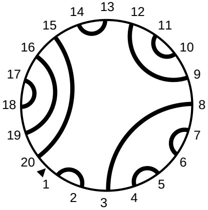
Wieland gyration: how it works

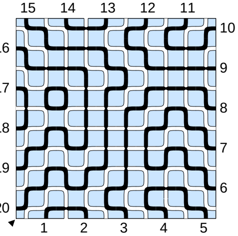
Link pattern π... ...$R^{\frac{1}{2}}\,\pi$... ...and $R\,\pi$...

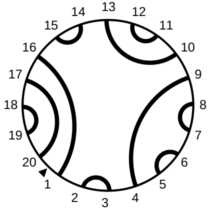






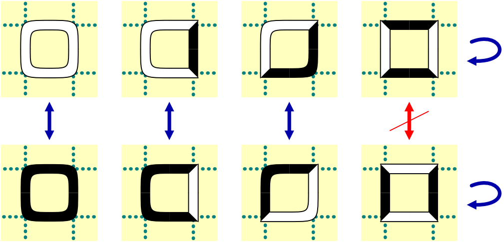






Wieland gyration: why it works

Easier to visualize the ⇔ exchange on the few , faces...
...but better use the conjugate config at intermediate step,
and think that , are the only faces fixed in the transformation



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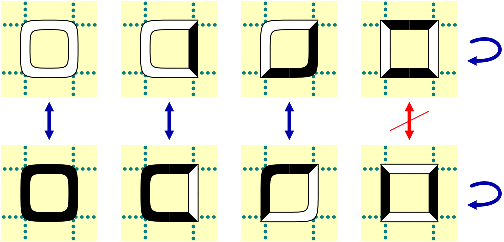
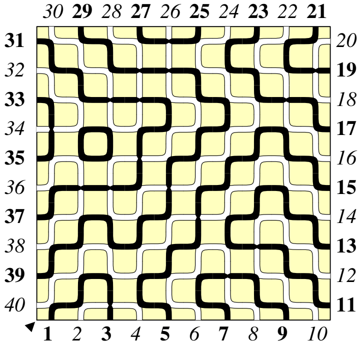


This rule inverts $\deg_{\text{black}}(v) \leftrightarrow \deg_{\text{white}}(v)$,
and preserves connectivity of open-path endpoints

Wieland gyration: where it works

...in the original square domain for FPL we have "external legs" (i.e., vertices of degree 1)... if we pair them, to produce triangles, we solve this annoyance...



A configuration on (Λ, τ_+)
(i.e., first leg is black)

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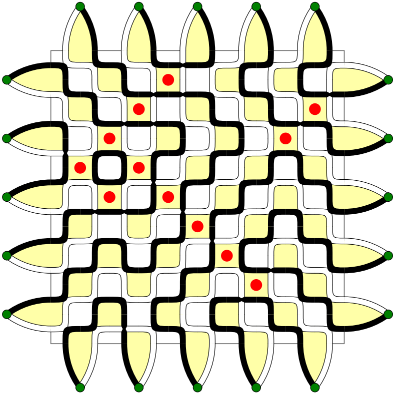

The construction of $\mathcal{G}_+$, pairing $(2j-1, 2j)$ legs (plaquettes are in yellow)

mark in red  and 

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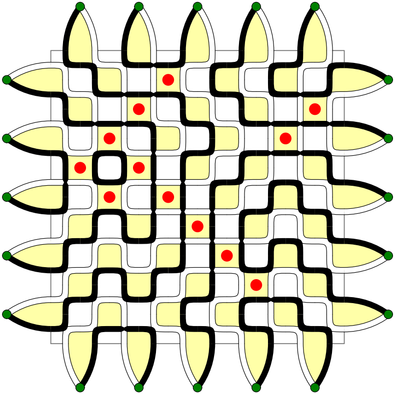

The result of map H_+

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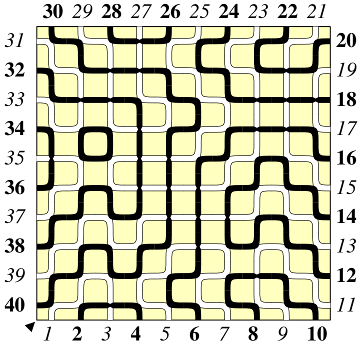

Split auxiliary vertices to recover the (Λ, τ_-) geometry
(i.e., first leg is white)

Wieland gyration: where it works

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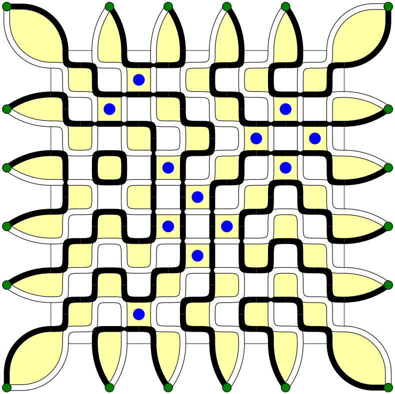

The construction of $\mathcal{G}_-$, pairing $(2j, 2j+1)$ legs

mark in blue  and 

Wieland gyration: where it works

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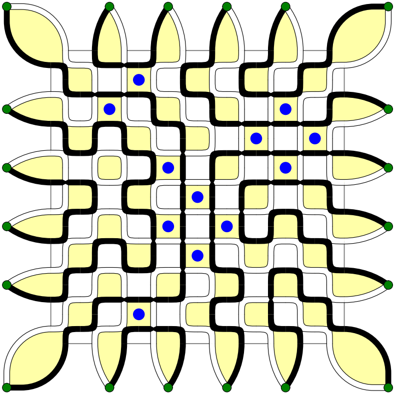

The result of map H_-

Wieland gyration: where it works

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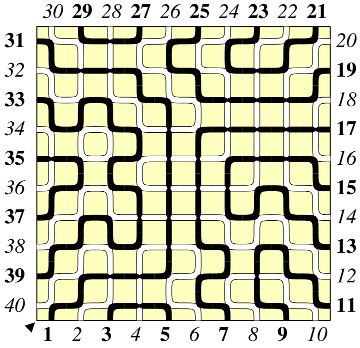

Split auxiliary vertices to recover the (Λ, τ_+) original geometry (with a rotated link pattern)...

Wieland gyration: where it works

So, the trick is:

- invert $\deg_{\mathrm{black}}(v) \leftrightarrow \deg_{\mathrm{white}}(v)$
- preserve connectivity of open paths

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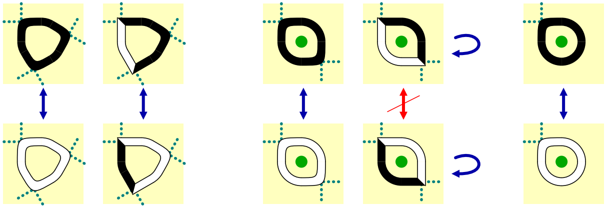

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- At boundaries, pair external legs to produce triangles

Wieland gyration: where it works

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- invert $\deg_{\mathrm{black}}(v) \leftrightarrow \deg_{\mathrm{white}}(v)$
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- Works with the Wieland recipe, on faces $\ell = 4$
- Works even more easily on faces $\ell = 1, 2, 3$
- Can't work on faces $\ell \geq 5$
- Stay tuned for the forgotten plaquette! (will come out later on...)
- At boundaries, pair external legs to produce triangles

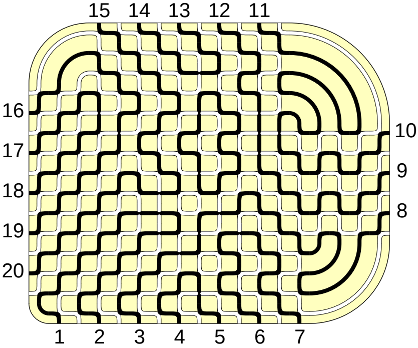
A single move exists on plenty of graphs...

then, rotation comes from two moves

...many more domains than just $n \times n$ squares have this property!

Wieland gyration: where it works

We can trade corners for points of curvature (i.e., faces with less than 4 sides). But we need at least one corner, because closed spectral lines have a trivial behaviour $(1 + R)$



(bottom line: an elementary generalization of Wieland strategy gives rotational symmetry for FPL enumerations above)

Yet one word on gyration... the boundary conditions

We constructed a whole family of domains where Wieland gyration works, and thus the enumerations must be rotationally symmetric. But, so far, we only used alternating boundary conditions

What does it happen if we generalise on boundary conditions, adding defects?

Pairing consecutive legs of the same colour produces arcs, and "loses link-pattern information": gyration implies a relation on linear combinations of $\Psi(\pi)$'s

These linear combinations, induced by arcs, are well-described by Temperley-Lieb operators.

We will not need this in full generality...
the study of a single defect is sufficient at our purposes.

Yet one word on gyration... the boundary conditions

I now like to think of this as a "Ward Identity": if G is a group of invariance for both the measure and the Hamiltonian, that is $\mathrm{d}\mu(\phi) = \mathrm{d}\mu(g \circ \phi)$ and $H(\phi) = H(g \circ \phi)$ for all $g \in G$, then

$$Z = \sum_{\phi} e^{-\beta H(\phi)} = \sum_{\phi} e^{-\beta H(g \circ \phi)} \qquad \forall g \in G$$

and also, more importantly,

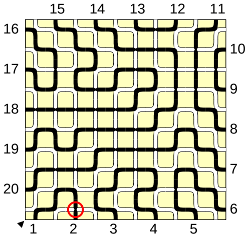
$$Z \langle A(\phi) \rangle = \sum_{\phi} A(\phi) e^{-\beta H(\phi)} = \sum_{\phi} A(g \circ \phi) e^{-\beta H(g \circ \phi)} = Z \langle A(g \circ \phi) \rangle$$

that is $\langle A(\phi) - A(g \circ \phi) \rangle = 0$. It is the fact that $Ae^{-\beta H}$ is "almost invariant but not exactly" that produces interesting relations...

Alternating boundary conditions, with one defect

Example: the state $|\Psi^{[j]}\rangle = \sum_{\phi\,:\,h(\phi)=j} |\pi'(\phi)\rangle$ satisfies

$$(R\, e_{j-1} - e_j)|\Psi^{[j]}\rangle = 0$$



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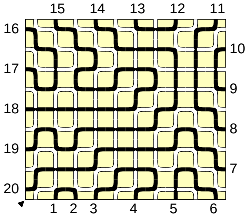

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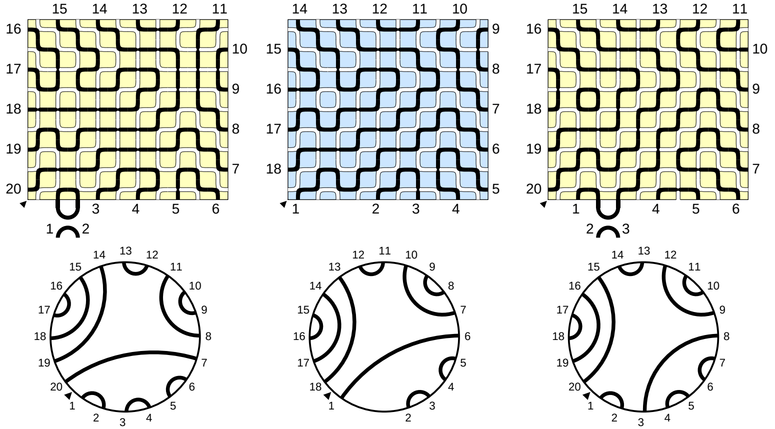

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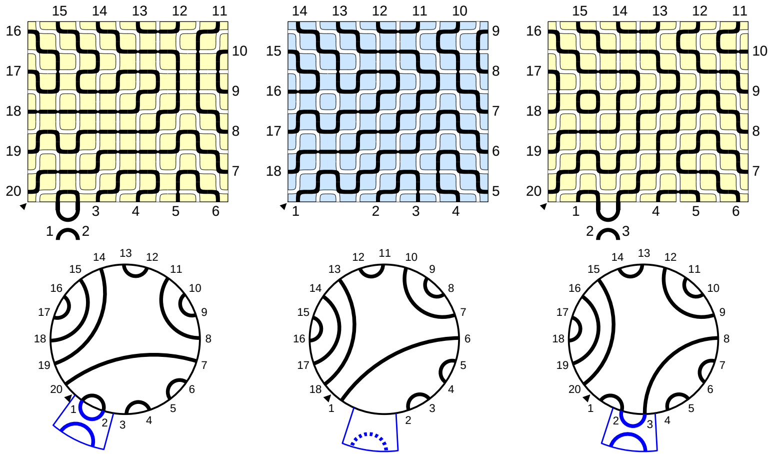

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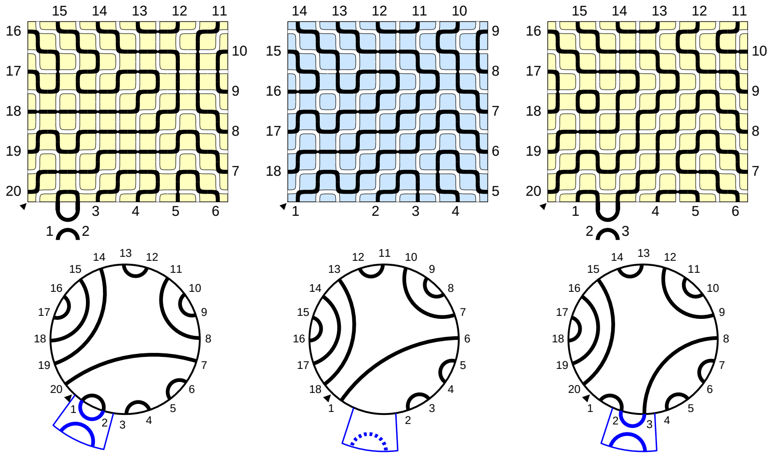
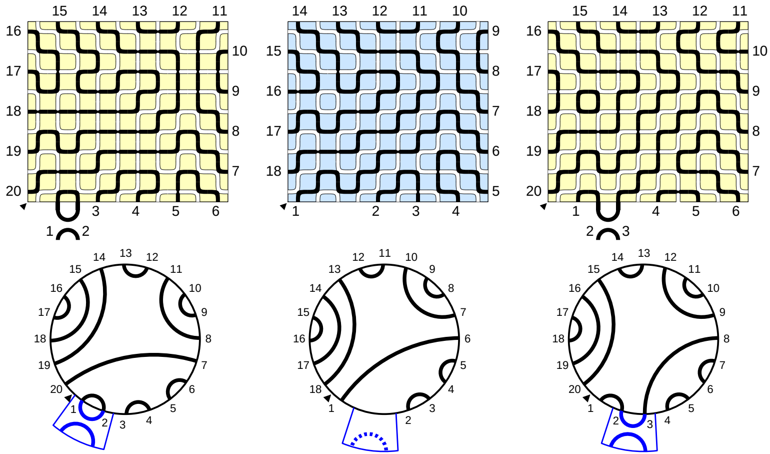


Alternating boundary conditions, with one defect

Example: the state $|\Psi^{[j]}\rangle = \sum_{\phi\,:\,h(\phi)=j} |\pi'(\phi)\rangle$ satisfies

$$R e_j (1 - R^{-1}) |\Psi^{[j]}\rangle = 0$$



A first consequence

Recall our checklist of identities:

(1) : $e_1\,(1 - R^{-1})|\Psi'(t)\rangle = 0$

(2) : $(1 - e_1)\,(t - R^{-1})|\Psi'(t)\rangle = 0$

(3) : $\mathrm{Sym}\,|\Psi'(t)\rangle = \mathrm{Sym}\,|\Psi(t)\rangle$

(2) is equivalent to ask that $t\Psi(t;\pi) = \Psi(t;R^{-1}\pi)$,
for all π such that $1 \not\sim 2$...

but this is easily seen: $1 \not\sim 2$ forces a small region,
that in turns implies a simple behaviour of the refinement position under Wieland gyration

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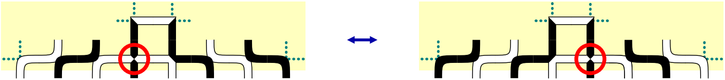

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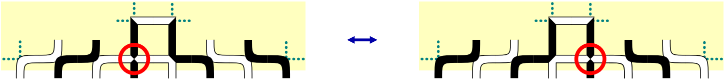

The final orbit lemma

Consider the orbits under Wieland half-gyration
As FPL in the same orbit have the same link pattern up to rotation, $\mathrm{Sym}\ |\Psi'(t)\rangle = \mathrm{Sym}\ |\Psi(t)\rangle$ follows if, for every j, and every orbit, there are as many contributions t^{j-1} to $|\Psi'(t)\rangle$ as to $|\Psi(t)\rangle$.

Study the behavior of the trajectory $h(x)$ of the refinement position:

- $h(x+1) - h(x) \in \{0, \pm 1\}$
- In a periodic function, any height value is attained alternately on ascending and descending portions (or maxima/minima)
- All maxima/minima plateaux have length 2, the slope is ± 1 elsewhere
- Ascending/descending parts of the trajectory have respectively black and white refinement position

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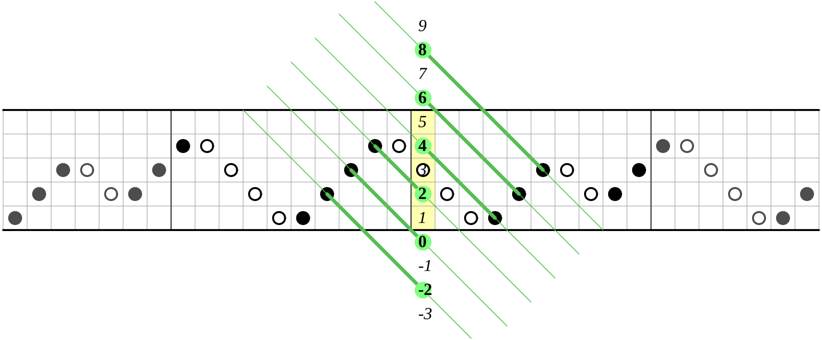
The final orbit lemma

As a consequence, in any orbit $\mathcal{O}$, and for any value j, the numbers of $\phi \in \mathcal{O}$ such that $h(\phi) = j$, and

- are in even (resp. odd) position in the orbit;
- or have a black (resp. white) refinement position;

are all equal.



This completes the proof *...and we can finally pass to new things...*

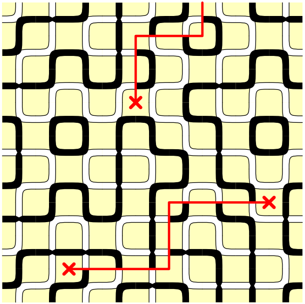
Colour-changing cuts in FPL's

Let G be a 4-valent planar graph, and consider FPL on G

Let $\mathcal{C}$ be a subset of edges of the dual G^* (the *cuts*)

Our FPL paths change colour when they cross $\mathcal{C}$

The set of FPL's is defined up to a gauge transformation of $\mathcal{C}$, that is, the lattice version of local deformation of cuts, while keeping the endpoints fixed.
In particular, the enumeration according to the link pattern is invariant under cut deformations.



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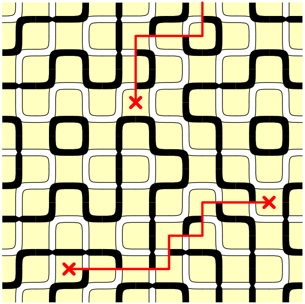

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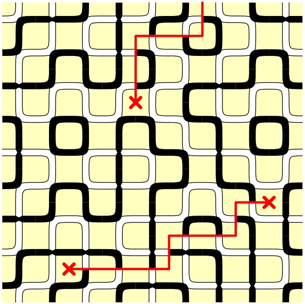

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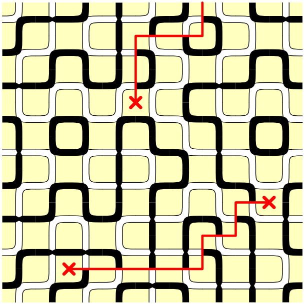

The forgotten dihedral domains

Above, we classified the domains allowing for the (dihedral, $\pi_\bullet$) Razumov–Stroganov correspondence. None of them involved cuts (as on such graphs we find paths that "start black and finish white", and we wouldn't know what to do with them...)

If we had an idea on how to construct a bicoloured $(\pi_\bullet, \pi_\circ)$ Razumov–Stroganov correspondence, nothing would prevent in principle to consider also domains with cuts.

but, again, where exactly does Wieland gyration work?

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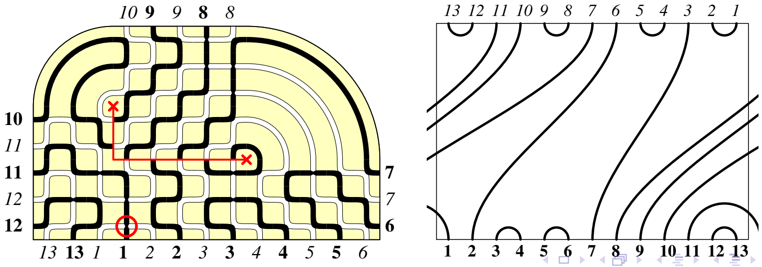
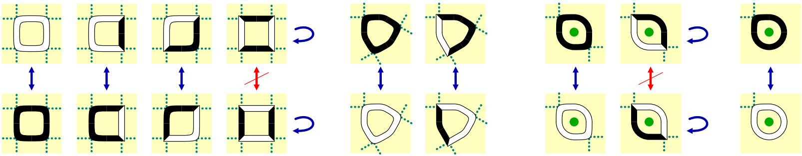
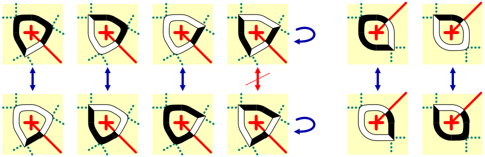


The forgotten plaquettes



We said that: ❶ squares barely work (must not swap if B/W/B/W) ❷ triangles and lower work easily ❸ 2-gons and 1-gons work so well that you can even put a puncture in them.



Now, with cut endpoints: ❶ triangles barely work (must not swap if B/W/B/W) ❷ 2-gons and lower work easily ❸ no room for a puncture anymore.

The forgotten plaquettes

Old domains: up to three triangles (or one triangle and one 2-gon, or one 1-gon, or one triangle and one vertex of degree 2, these cases allow for a puncture)

New domains: as each cut needs two triangles (or 2-gons, or 1-gons), we can have only up to one cut, connecting two triangles (and possibly a third triangle is left normal), or a triangle and a 2-gon

no room for a puncture anymore, and no room for two cuts!

Reverse-engineering RS from FPL towards a loop model

We want to "invent" a black+white Razumov–Stroganov correspondence

But are disappointed to find no known good candidate on the DLM side (the "rotor model" would have been promising, if it weren't for certain incompatibility issues on known enumerations)

So, we try to build on what we already know, from the FPL side, and hope to interpret *a posteriori* what we find as a loop model.

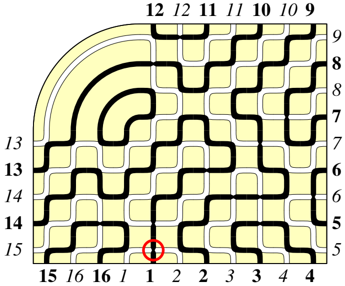
Our best starting point is the Di Francesco 2004 (ex-)conjecture, i.e. our proof that the heretical-enumeration-vector for FPL $|\Psi'(t)\rangle$ satisfies

$$e_1^{\bullet}\,(1-R_{\bullet}^{-1})|\Psi'(t)\rangle = 0$$
$$(1-e_1^{\bullet})\,(t-R_{\bullet}^{-1})|\Psi'(t)\rangle = 0$$

The black+white equations

So, we consider the black+white heretical enumeration $|\Psi_{\rm bw}(t)\rangle$, e.g. with the convention that:

- the refinement is always on a black leg
- the refinement leg has black index ❶
- the black indices grow CCW
- the white leg left of ❶ has white index ①
- the white indices grow CCW



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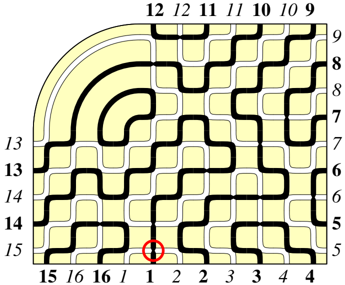

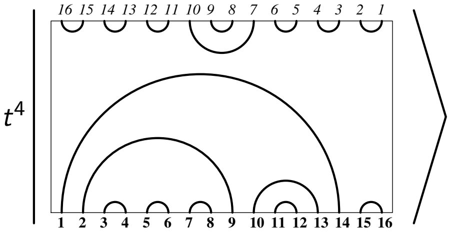

The black+white equations

We can work out lemmas as in our previous proof, but now keeping track of both black and white legs. It is more complicated but similar in spirit. We find equations depending only on $e_1^\bullet$, e_1°, $R_\bullet$ and $R_\circ$, namely: (note: these equations do not have a unique solution!)

(we conjecture that there are $C_{\lceil \frac{N}{2} \rceil - 1}$ lin. indep. solutions)

$$(1 - e_1^\circ)|\Psi_{\rm bw}(t)\rangle = 0$$

$$(1 - e_1^\bullet)\,(t - R_\bullet^{-1})|\Psi_{\rm bw}(t)\rangle = 0$$

$$e_1^\bullet e_1^\circ \left(1 - \frac{1}{1 - tR_\circ(1 - e_1^\circ)} R_\circ R_\bullet^{-1}\right) |\Psi_{\rm bw}(t)\rangle = 0$$

These are consistent with our previous equations. Indeed, the first one states that the refinement leg is black, and the other two, under the identification $R_\circ \to 1$, $e_1^\circ \to 1$, become

$$e_1^\bullet\,(1 - R_\bullet^{-1})|\Psi'(t)\rangle = 0$$

$$(1 - e_1^\bullet)\,(t - R_\bullet^{-1})|\Psi'(t)\rangle = 0$$

One big problem ahead: the solution is not unique!

In the ordinary dihedral Razumov–Stroganov correspondence the RS vector is the same for all the dihedral domains, up to a multiplicative constant. On the DLM side, it is the Frobenius vector of the corresponding Markov chain.

But this cannot work here, because the black+white enumerations for different domains are not proportional

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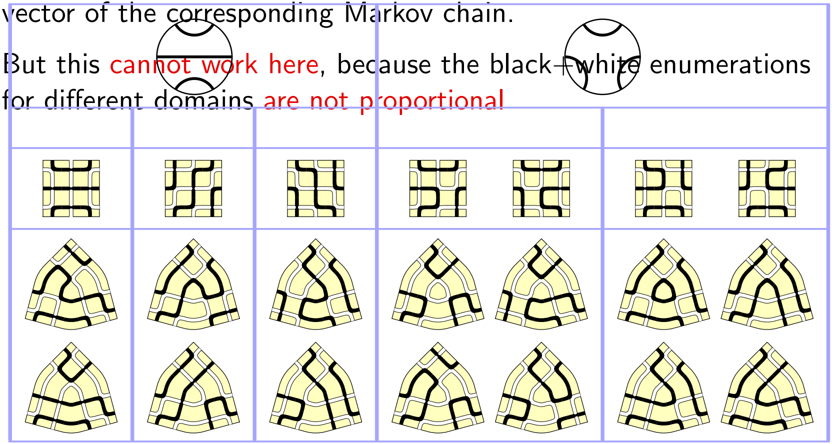

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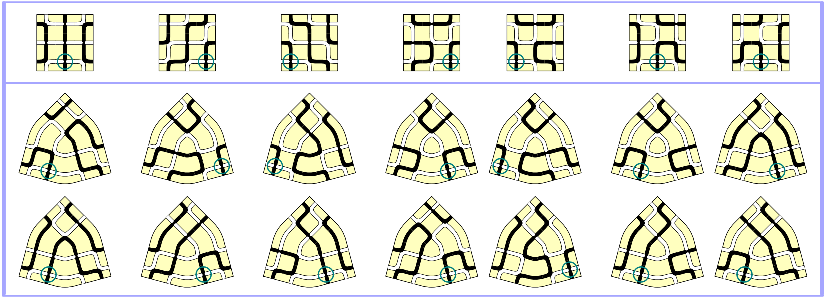

First nice surprise: a hidden symmetry

The heretical enumeration breaks the symmetry between black and white.

Define the combination

$$|\Phi(t)\rangle = \frac{R_\bullet^{-1}}{1 - t(1 - e_1^\circ)R_\circ} |\Psi_{\mathrm{bw}}(t)\rangle$$

Our equations above read

$$\begin{aligned}
(1 - e_1^\circ)\,(1 - tR_\circ)|\Phi(t)\rangle &= 0 \\
(1 - e_1^\bullet)\,(1 - tR_\bullet)|\Phi(t)\rangle &= 0 \\
e_1^\bullet e_1^\circ\,(R_\circ - R_\bullet)\,|\Phi(t)\rangle &= 0
\end{aligned}$$

which are both very compact, and symmetric in black and white.

Second nice surprise: separation of $TL_\bullet$ and $TL_\circ$

In terms of the vector $|\Psi'_{\rm bw}(t)\rangle = R_\bullet^{-1}|\Psi_{\rm bw}(t)\rangle$,
we can write our equations as

$$\begin{aligned}
(1-e_1^\circ)|\Psi'_{\rm bw}(t)\rangle &= 0 \\
(1-e_1^\bullet)\,(1-tR_\bullet)|\Psi'_{\rm bw}(t)\rangle &= 0 \\
\left(e_1^\circ \frac{1}{1-tR_\circ} - \frac{1}{1-tR_\bullet} e_1^\bullet\right)|\Psi'_{\rm bw}(t)\rangle &= 0
\end{aligned}$$

These equations can be interpreted as if $TL_\bullet$ and $TL_\circ$ act on the bottom and top sides of a cylinder, with the stochastic operator:

- apply e_1
- apply R^ℓ, with probability $p(\ell) = (1-t)t^\ell$

and its adjoint (that is, the two steps are performed in reverse order)

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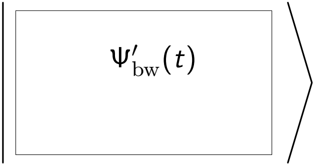
$$\left(e_1^\circ \frac{1}{1 - tR_\circ} - \frac{1}{1 - tR_\bullet} e_1^\bullet \right) |\Psi'_{\rm bw}(t)\rangle = 0$$



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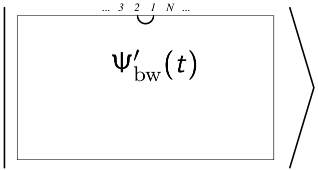

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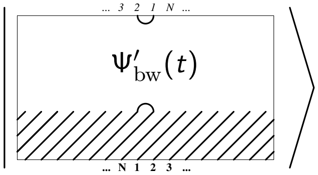

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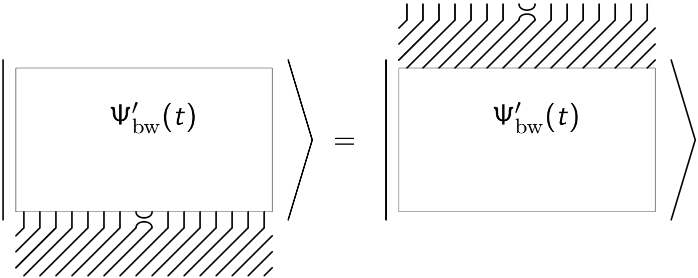

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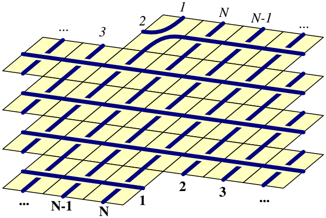
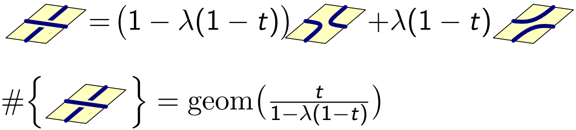
Corollary 1: A one-parameter family of solutions is given by

$$\Psi(t;\lambda) = (1-\lambda)\frac{e_1 \frac{1-t}{1-tR}}{1-\lambda e_1 \frac{1-t}{1-tR}} = e_1 \frac{\frac{(1-\lambda)(1-t)}{1-\lambda(1-t)}}{1-\frac{t}{1-\lambda(1-t)} R\, X_1(1-\lambda(1-t))}$$

(for $\lambda \in [0,1]$ this is a prob. measure), where elements in the Temperley–Lieb Algebra are interpreted as "cylindric link patterns", with white at the top and black at the bottom.





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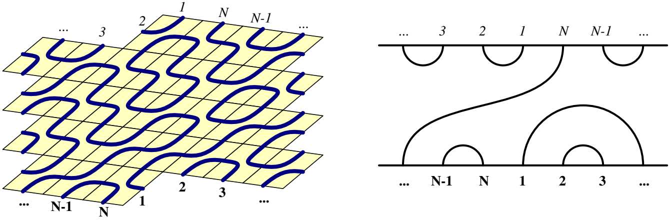

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Corollary 2: the solutions to our equations form a $\mathbb{C}[t]$–non-unital ring within TL_N. Indeed, Ψ is a solution if

$$(1-e_1)\Psi = 0 \qquad \Psi(1-tR)(1-e_1) = 0 \qquad \left[\Psi, e_1 \frac{1}{1-tR}\right] = 0$$

so, if Ψ and Φ are both solutions, we have

$$\begin{aligned}
&(1-e_1)\Psi\Phi = 0\\
&\Psi\Phi(1-tR)(1-e_1) = 0\\
&\left[\Psi\Phi, e_1 \tfrac{1}{1-tR}\right] = \Psi\left[\Phi, e_1 \tfrac{1}{1-tR}\right] + \left[\Psi, e_1 \tfrac{1}{1-tR}\right]\Phi = 0
\end{aligned}$$

The two nice cases at $n = 3$ tell us that it must be a non-commutative algebra, but at least $\Psi = e_1 \frac{1}{1-tR}$ is in the center

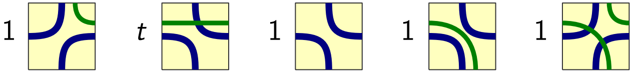
Can we construct a black+white Razumov–Stroganov?

So, we still do not have a DLM counterpart as in a proper Razumov–Stroganov correspondence!

Let us try to invent a loop model whose TM is $e_1 \frac{1}{1-tR}$
It must contain loop diagrams, for the Temperley–Lieb action
It must also contain a "mark" from where to start counting
But, as $1/(1-tR) = 1 + tR + t^2R^2 + t^3R^3 + \cdots$ has arbitrarily many terms, the mark may move arbitrarily far in a single row

We may interpret the mark as a particle like in a 6VM in NILP representation, in the sector in which there is a single particle

A set of tiles with these properties is a 5VM mixed to a $O(1)$DLM



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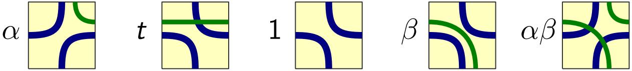
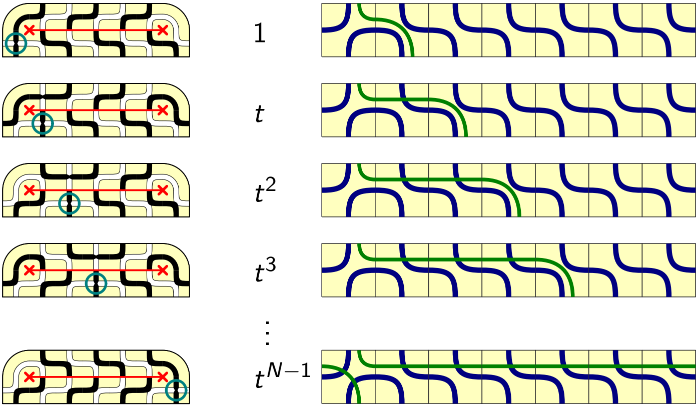


However, we are not able to "Baxterise" these weights. . .

Third nice surprise: FPL domains with a cut as $T(t)$

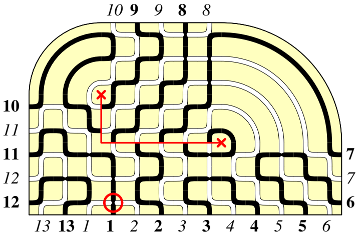
…but we get a nice surprise: the Transfer Matrix of this model is the partition function for FPL in the simplest cut domain (two rows)

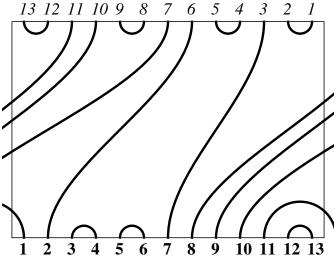


Third nice surprise: FPL domains with a cut as $T(t)$

Why is it conceivable that FPL domains with a cut act as transfer matrices? Because, in disguise, they correspond to domains with a cylindic topology (symmetric under reflection plus complementation)

The symmetry is important:
if we did not have it,
we would have two hexagonal faces,
and Wieland gyration wouldn't work.





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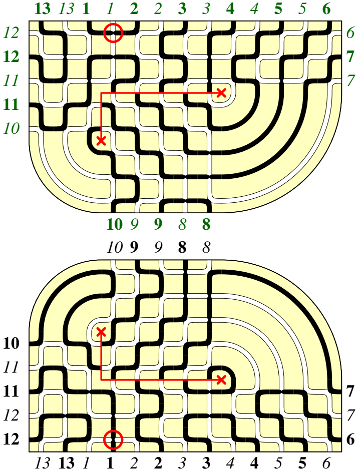

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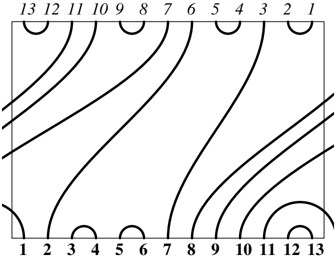

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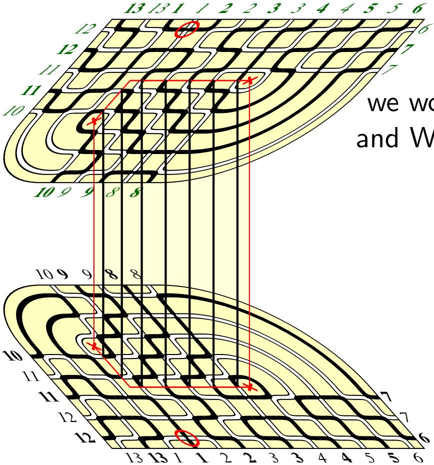

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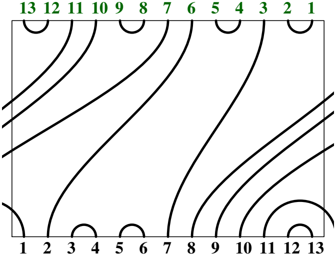

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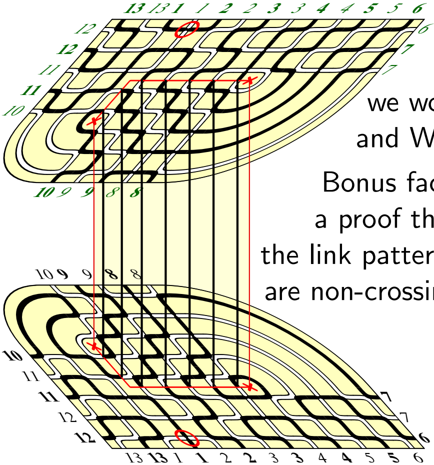

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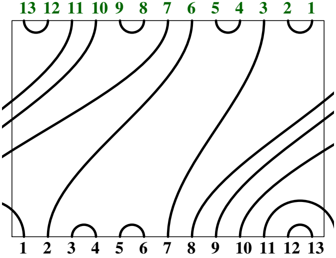
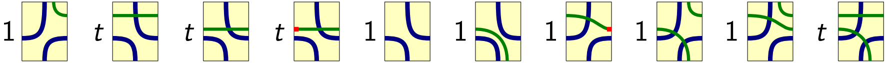
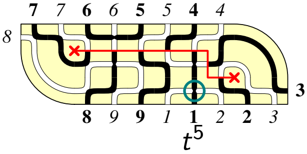
Bonus fact:
a proof that
the link patterns
are non-crossing

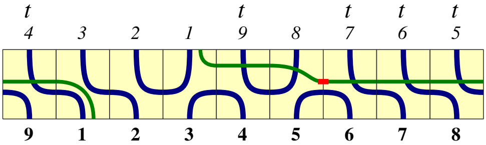


More transfer matrices

More general cut domains can also be interpreted as Transfer Matrices, but with more complicated (horizontal) auxiliary space, and more complicated tiles (as for a “higher spin” line in the 6VM)







This family of elements of the ring is interesting. It has “low degree” in the e_i’s (degree $k-1$ if there are k horizontal lines), and degree $N-1$ in t (so that they are not trivially related). Their limit for $t \to 1$, once symmetrised, gives the Hamiltonians of ordinary TL, $H_k = \frac{\partial^k}{\partial \lambda^k} T(\lambda)|_{\lambda=0}$.

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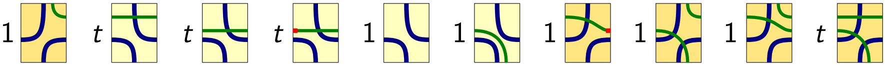

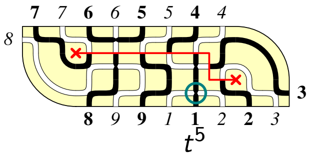

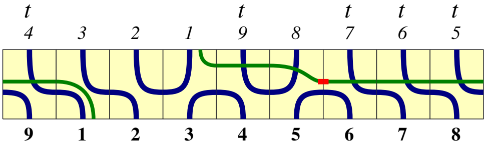

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Conclusions and open questions

What is done:

- We have identified a system of equations in $\mathrm{TL}_\bullet(N) \times \mathrm{TL}_\circ(N)$ for the black+white heretical enumeration $A_D(\pi)$ of FPL on dihedral domains $D \in \mathcal{D}^\star_N$ (with a cut or a puncture or none)
- The eqns do not fix the vector univocally. This is compulsory, as different domains have non-proportional enumerations.

What is not done:

- We did not study the algebra of solutions. We believe that it has dim. $C_{\lceil \frac{N}{2} \rceil - 1}$ but do not have a proof. We don't know its center, and its structure constants. We don't know if it is generated by the $\{A_D(\pi)\}_{D \in \mathcal{D}^\star_N}$.
- We do not have a Razumov–Stroganov correspondence, i.e. an integrable DLM that, for any given FPL domain D, produces the enumerations A_D when realised in some geometry dependent on D.