Domino tilings of the Aztec diamond in random environment

Panagiotis Zografos

Leipzig University

Based on joint work with Alexey Bufetov and Leonid Petrov

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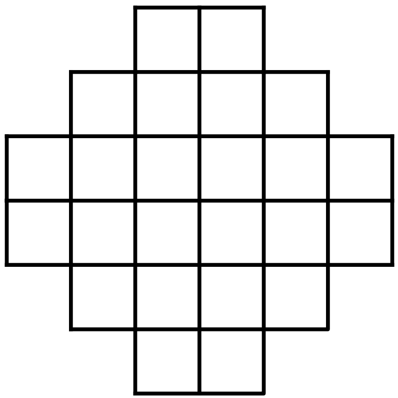
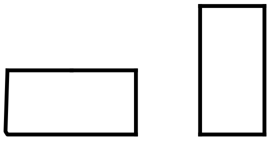
Outline

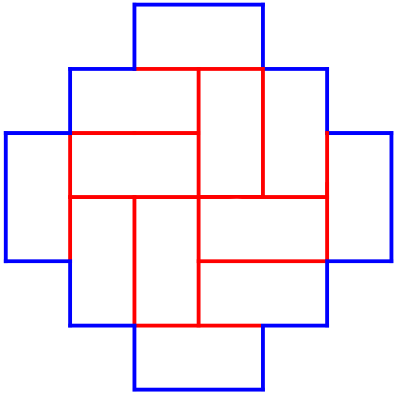
- Random domino tilings of the one-periodic Aztec diamond and the Schur process
- The one-periodic Aztec diamond in random environment: the annealed CLT
- Global fluctuations of discrete N-particle systems
- The one-periodic Aztec diamond in random environment: the quenched CLT
- Other applications

Domino tilings of the Aztec Diamond

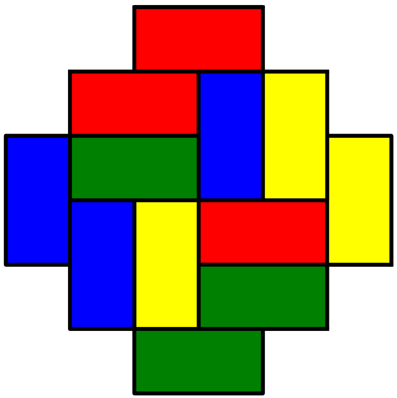
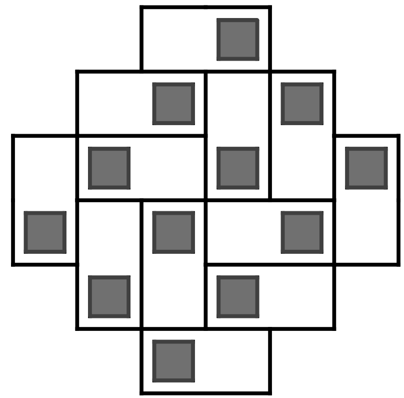
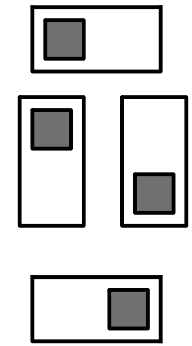
- The Aztec Diamond of size N is the set of all lattice squares which are (fully) contained in $\{(x,y):|x|+|y|\leq N+1\}$.
- Domino tilings of the Aztec diamond were introduced by Elkies-Kuperberg-Larsen-Propp'92. They proved that the number of tilings is equal to $2^{N(N+1)/2}$.

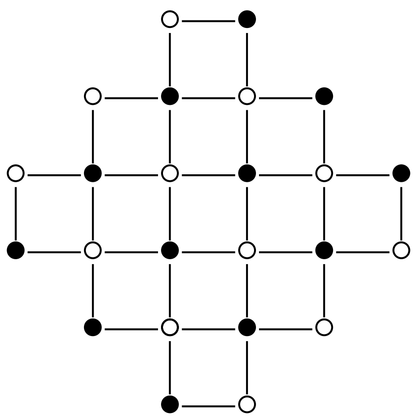
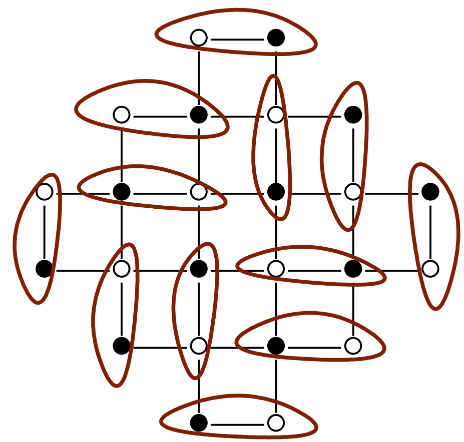


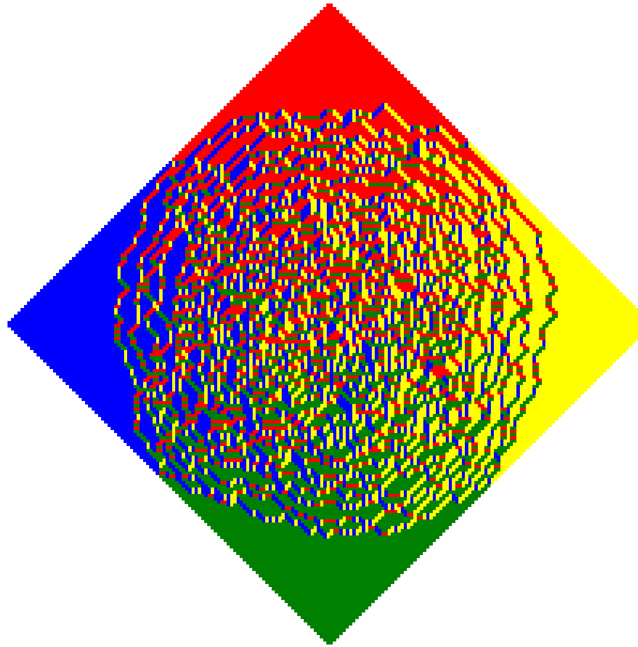




Let us consider a chessboard coloring of the Aztec diamond. It is useful to distinguish not two, but four types of dominoes



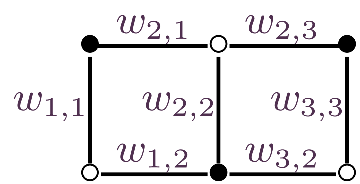
A uniformly random domino tiling of the Aztec diamond has some structure.

Theorem (Jockusch-Propp-Shor'98). Asymptotically a uniformly random tiling becomes frozen outside of a certain circle.

Weighted dimers

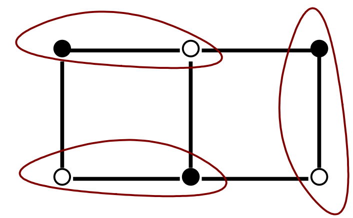
Weight function on edges:

$$w\colon E - \mathbb{R}_{>0}$$



Associated weight of a dimer covering:

$$w(D) = \prod_{e \in D} w(e)$$



$$w(D) = w_{1,2} \cdot w_{2,1} \cdot w_{3,3}$$

Partition function:

$$Z = \sum_D w(D)$$

$$Z = w_{1,2} \cdot w_{2,1} \cdot w_{3,3} + w_{2,3} \cdot w_{3,2} \cdot w_{1,1} + w_{1,1} \cdot w_{2,2} \cdot w_{3,3}$$

Probability measure on dimer coverings

$$\mathbb{P}(D) = \frac{w(D)}{Z}$$

There are many methods how to study this object:

- Analysis of a sampling (shuffling) algorithm: Jockusch-Propp-Shor'98
- Kasteleyn matrix: Kenyon'00, Cohn-Kenyon-Propp'02, Kenyon-Okounkov-Sheffield'06, Kenyon-Okounkov'07, ...

Johansson'03, Chhita-Johansson-Young'12, Chhita-Johansson'14, Johansson-Mason'23, ...

Duits-Kuijlaars'17, Berggren-Borodin'23, Boutillier-de Tilière'24, Bobenko-Bobenko'24, Berggren-Nicoletti'25

- Schur generating functions: Bufetov-Gorin'13,'16: can be efficiently used at least for the global behavior: limit shape and fluctuations.

Gorin-Panova'13, Panova'14, Bufetov-Knizel'16, Boutillier-Li'17, Gorin-Sun'18,

Huang'18, Benaych-Georges-Cuenca-Gorin'21, Keating-Li-Prause'23, Bufetov-Z'24,

Cuenca-Dolega'25, ...

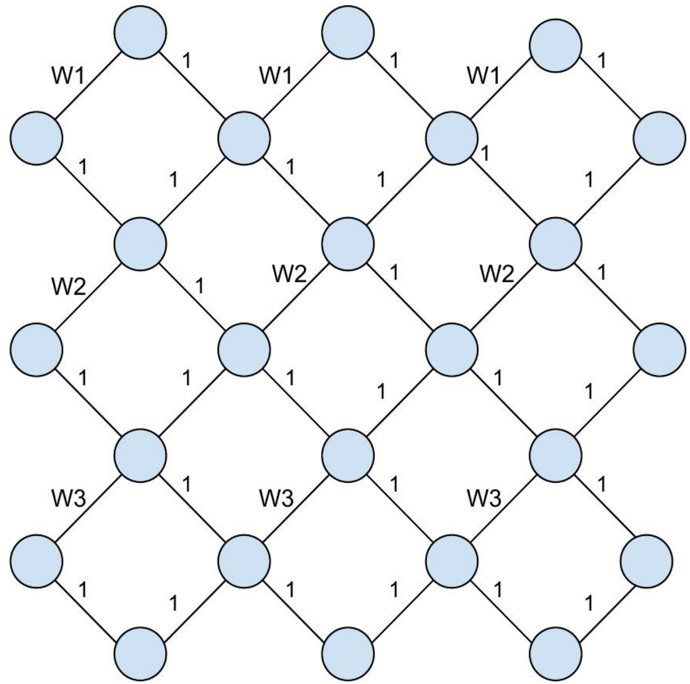
In this talk we will consider the Aztec diamond with random edge weights.

Physics literature: There are studies for the Aztec diamond with all edge weights being chosen i.i.d. Perret-Ristivojevic-Le Doussal-Schehr-Wiese'12. This model is conjectured to display a super-rough region.

Mathematics literature: Dimer models with layered disorder—the weights are periodic in one direction Bufetov-Petrov-Z.'25, Moulard-Toninelli'25, Z.'25. Despite the disorder integrable structure is preserved.

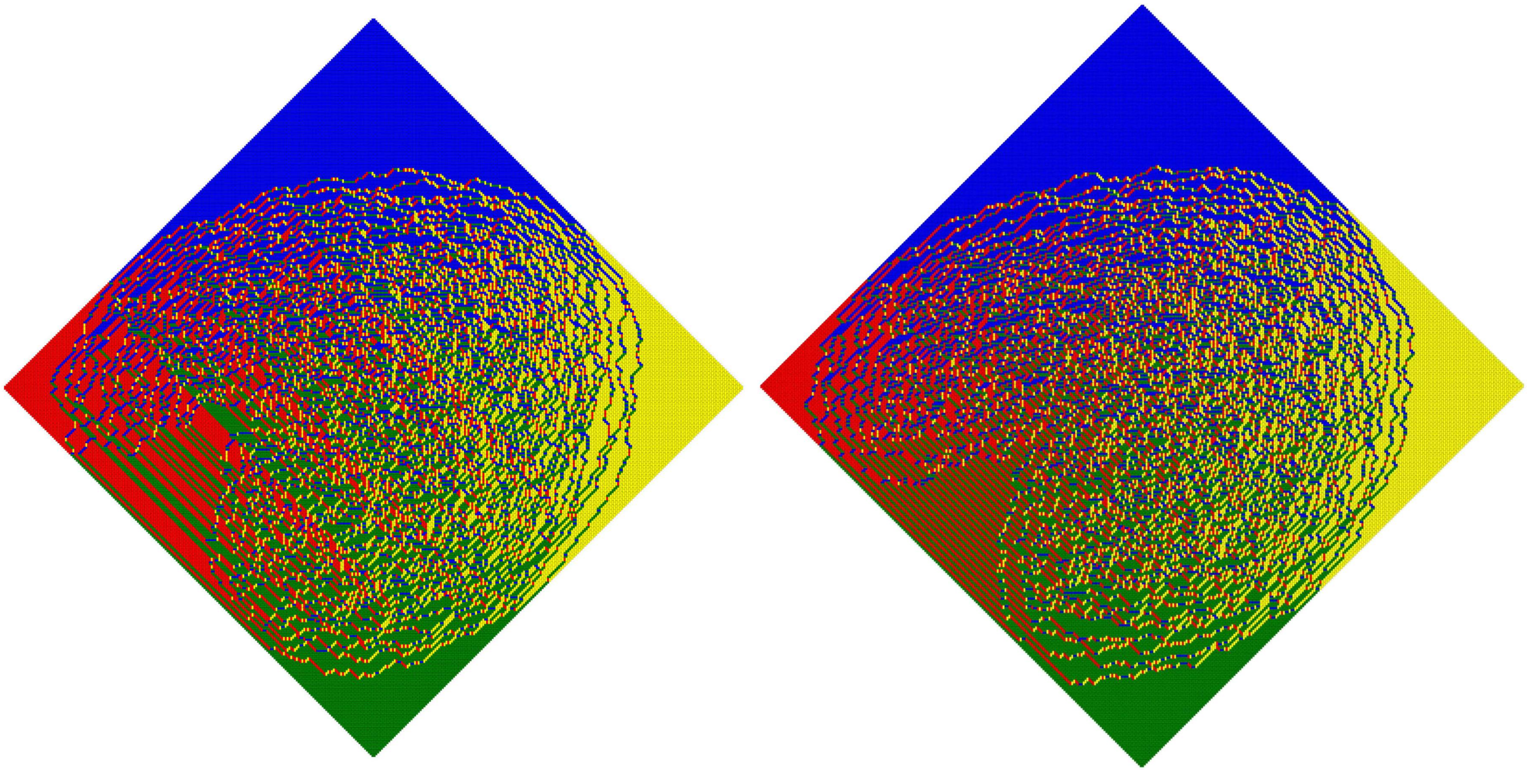
In the context of random environment it is common to distinguish between two types of expectations:

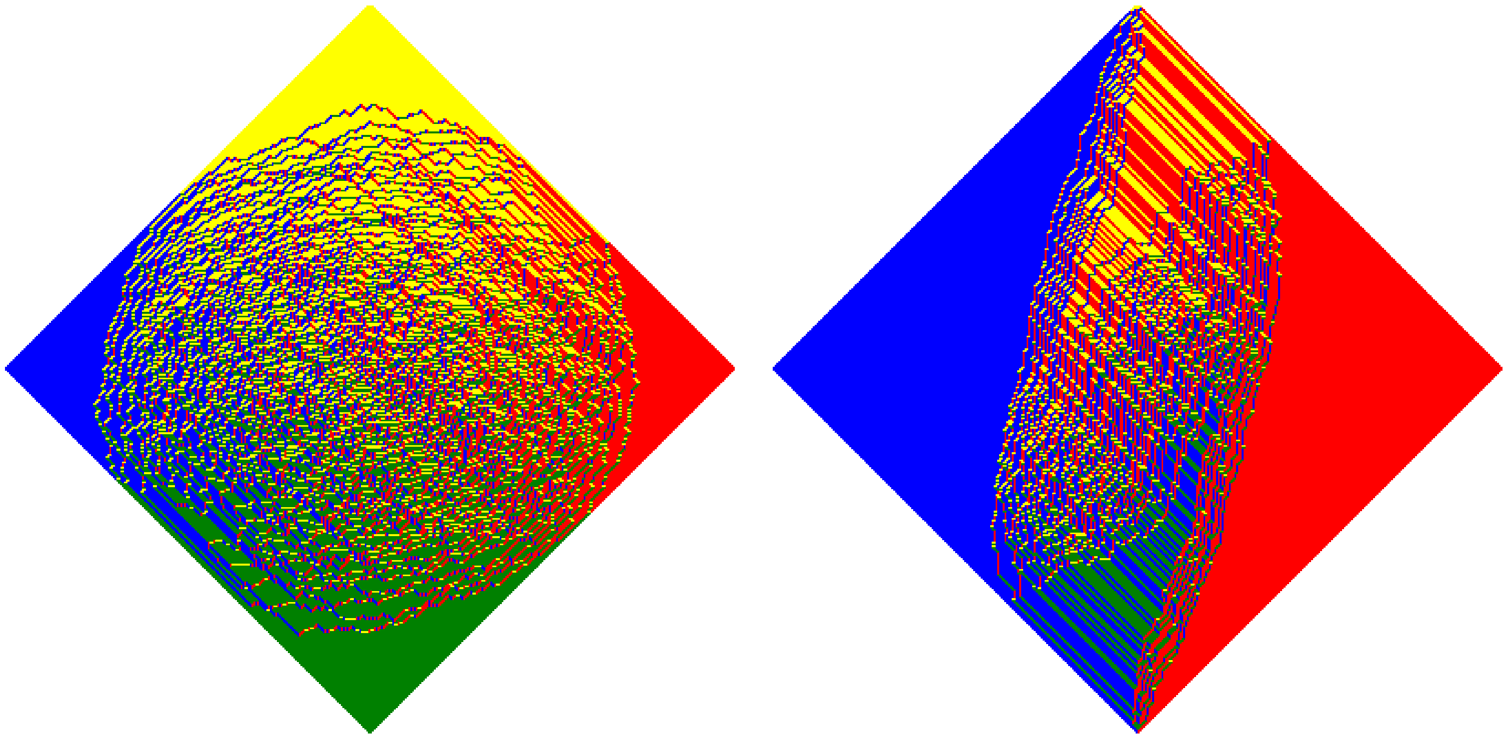
- Quenched: We first fix the random environment and then compute expectations.
- Annealed: We average in both over the randomness in the tilings and the randomness in the environment.

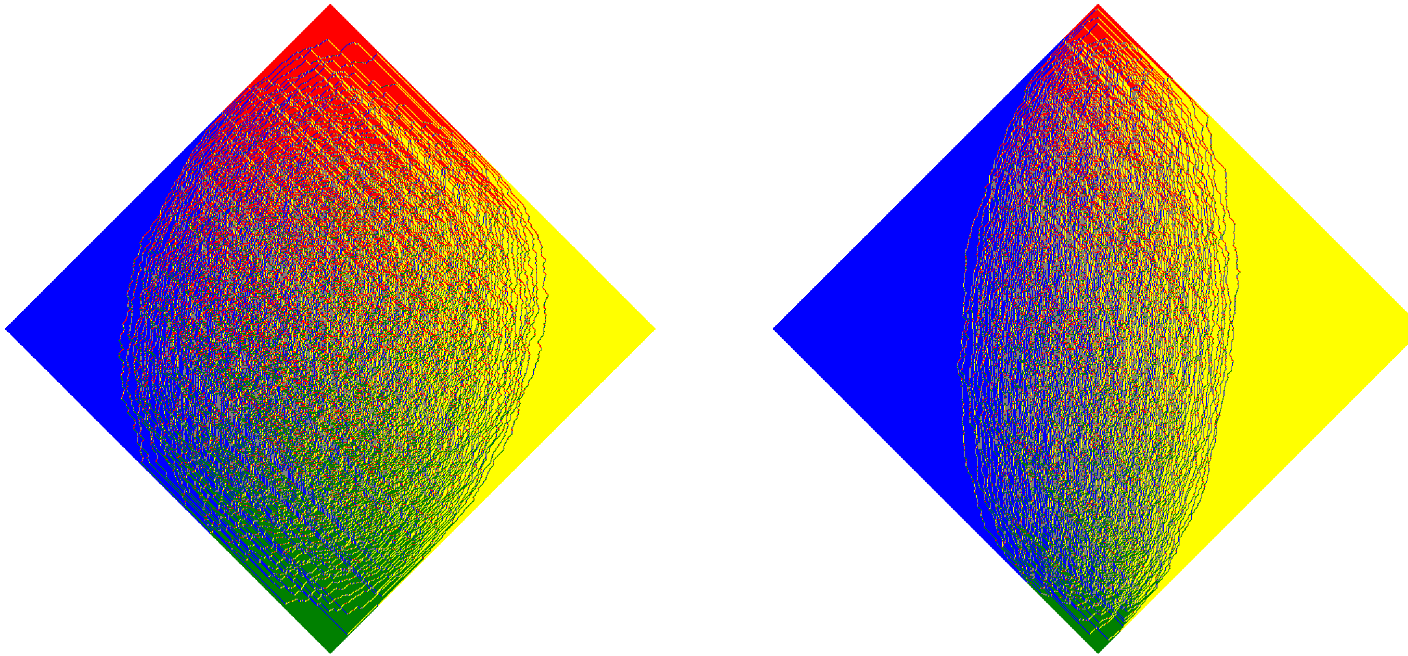


One-periodic model of weights-given by Schur measure/process introduced by Okounkov and Okounkov-Reshetikhin.

Let us take the parameters $w_1, \ldots, w_M$ random: independent and identically distributed.







Signatures and Schur functions

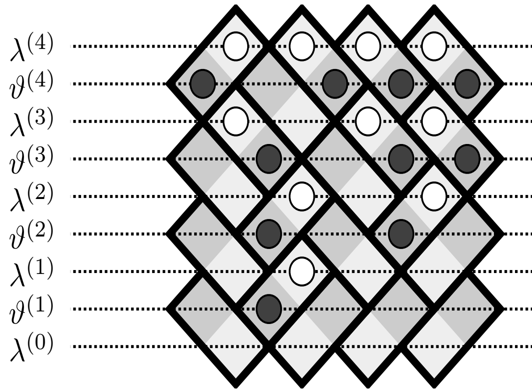
- A signature of length N is an N-tuple of integers $\lambda=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_N)$. We will use the notation $l_i:=\lambda_i+N-i$.
- Sign_N–the set of all signatures of length N.

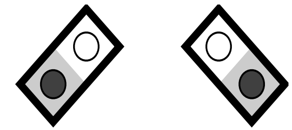
- The Schur function is defined by

$$s_\lambda(x_1,\ldots,x_N):=\frac{\det(x_i^{l_j})_{i,j=1,\ldots,N}}{\prod_{1\leq i<j\leq N}(x_i-x_j)},$$

where λ is a signature of length N. The Schur function is a Laurent polynomial in $x_1,\ldots,x_N$.

- Signatures $\lambda\in\mathrm{Sign}_N$ and $\mu\in\mathrm{Sign}_{N-1}$ interlace ($\lambda\succ\mu$), if $\lambda_i\geq\mu_i\geq\lambda_{i+1}$, for all $i=1,\ldots,N-1$.
- Signatures $\lambda,\theta\in\mathrm{Sign}_N$ interlace vertically ($\theta\succ_v\lambda$), if $\theta_i-\lambda_i\in\{0,1\}$, for all $i=1,\ldots,N$.





Domino tilings of the Aztec diamond of size M are in bijection with sequences of signatures $\{\lambda^{(i)},\theta^{(i)}\}_{i=1}^{M}$, of the form

$$0 \prec \theta^{(1)} \succ_v \lambda^{(1)} \prec \theta^{(2)} \succ_v \lambda^{(2)} \prec \cdots \prec \theta^{(M)} \succ_v \lambda^{(M)} = 0,$$

where $\lambda^{(i)},\theta^{(i)} \in \mathrm{Sing}_i$.

The one-periodic probability measure with parameters $w_1,\ldots,w_M$ make these signatures random. Our main application is to describe the behavior of such random signatures.

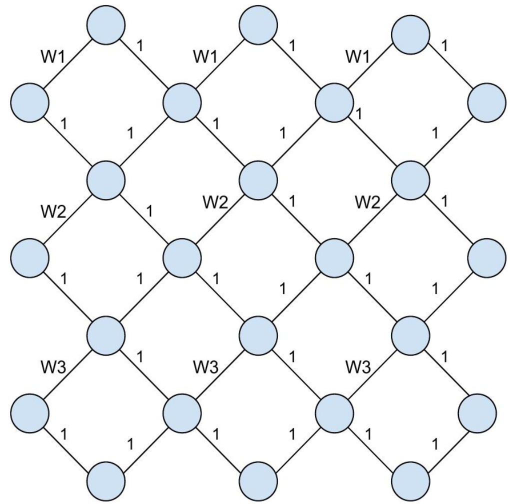
$$0 \prec \theta^{(1)} \succ_v \lambda^{(1)} \prec \theta^{(2)} \succ_v \lambda^{(2)} \prec \cdots \prec \theta^{(M)} \succ_v \lambda^{(M)} = 0,$$

One-periodic probability measure:

$$\mathbb{P}_{\beta_1,\ldots,\beta_M}(\{\lambda^{(i)},\theta^{(i)}\}_{i=1}^{M}) = \prod_{i=1}^{M} \frac{1}{(w_i+1)^i} w_i^{|\theta^{(i)}|-|\lambda^{(i)}|}.$$

For $N < M$, the marginal distribution of $\lambda^{(N)}$ is given by the Schur measure:

$$\mathbb{P}_{\beta_1,\ldots,\beta_M}(\lambda^{(N)} = \lambda) = \prod_{i=N+1}^{M} \frac{1}{(w_i+1)^N} \cdot s_\lambda(1,\ldots,1) s_{\lambda'}(w_M,\ldots,w_{N+1}).$$



We assume that $w_i = \beta_i/(1-\beta_i)$, $\beta_i \in (0,1)$, and $\beta_1, \dots, \beta_M$ are independent and identically distributed, with distribution β.

We study two cases:

First, we assume that the variance of β decays like M^{-1},

$$\lim_{M\to\infty} \mathbb{E}[\beta] = b, \quad \lim_{M\to\infty} M \operatorname{Var}(\beta) = \sigma^2.$$

Second, we study the case of fixed distribution β.

$$0 \prec \theta^{(1)} \succ_v \lambda^{(1)} \prec \theta^{(2)} \succ_v \lambda^{(2)} \prec \cdots \prec \theta^{(M)} \succ_v \lambda^{(M)} = 0$$

For $N < M$, let

$$m[\lambda^{(N)}] = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda_i^{(N)} + N - i), \qquad p_k^{(N)} = \frac{1}{N} \sum_{i=1}^{N} (\lambda_i^{(N)} + N - i)^k.$$

- Every domino tiling is uniquely determined by the height function,

$$H(y,\eta) := |\{1 \le i \le \lfloor \eta N \rfloor : \lambda_i^{(\lfloor \eta N \rfloor)} + \eta N - i \ge Ny\}|, \qquad y \in \mathbb{Z}_{>0},\ \eta \in (0,1).$$

Theorem Bufetov-Petrov-Z.'25

In the case of decreasing variance, $\left\{\frac{1}{M^k}(p_k^{(\lfloor \eta M \rfloor)} - \mathbb{E}[p_k^{(\lfloor \eta M \rfloor)}])\right\}_{k \in \mathbb{Z}_{>0}, \eta \in (0,1)}$ converge to jointly Gaussian distribution, whose covariance can be computed explicitly.

Covariance structure: For $\lim_{M\to\infty}\mathbb{E}[\beta]=b$ and, $\lim_{M\to\infty}M\,\mathrm{Var}(\beta)=\sigma^2$, we have:

$$\lim_{M\to\infty}\frac{1}{M^{k_1+k_2}}\mathrm{Cov}(p_{k_1}^{(\lfloor\eta_1 M\rfloor)},p_{k_2}^{(\lfloor\eta_2 M\rfloor)})$$

$$=\frac{1}{(2\pi \mathrm{i})^2}\oint_{|w|=\varepsilon}\oint_{|z|=2\varepsilon}\left(\frac{\eta_1}{z}+\eta_1+\frac{(1+z)(1-\eta_1)b}{1+bz}\right)^{k_1}\left(\frac{\eta_2}{w}+\eta_2+\frac{(1+w)(1-\eta_2)b}{1+bw}\right)^{k_2}$$

$$\times\left(\frac{\min\{1-\eta_1,1-\eta_2\}\sigma^2}{(1+bz)^2(1+bw)^2}+\frac{1}{(z-w)^2}\right)dzdw.$$

- $\frac{1}{(z-w)^2}$ is a Gaussian Free Field term.
- $\frac{\min\{1-\eta_1,1-\eta_2\}\sigma^2}{(1+bz)^2(1+bw)^2}$ is a new term, it is generated by a one-dimensional Brownian motion.

This case actually fits the setup of general theorems from Bufetov-Gorin'16.

Gaussian Free Field convergence for uniform tilings of Aztec diamond: Chhita-Johansson-Young'12, Bufetov-Gorin'16.

Theorem Bufetov-Petrov-Z.'25

In the case of fixed distribution β, the vector

$$\left\{\frac{p_k^{(N)}}{N^{k+1}}\right\}_{k\in\mathbb{Z}>0}, \quad \text{where} \quad p_k^{(N)}=\sum_{i=1}^{N} (\lambda_i^{(N)}+N-i)^k,$$

converges in probability to

$$\lim_{\substack{N,M\to\infty \\ N/M\to\eta}}\frac{\mathbb{E}[p_k^{(N)}]}{N^{k+1}}=\frac{1}{2\pi\mathrm{i}(k+1)}\oint_{|z|=1}\frac{1}{z+1}\left(\frac{1}{z}+1+\frac{1-\eta}{\eta}F(z)\right)^{k+1}dz,$$

where

$$F(z)=\mathbb{E}_\beta\left[\frac{\beta+\beta z}{1+\beta z}\right].$$

Theorem Bufetov-Petrov-Z.'25

In the case of fixed distribution β, the vector

$$\left\{\frac{p_k^{(\lfloor\eta M\rfloor)}-\mathbb{E}[p_k^{(\lfloor\eta M\rfloor)}]}{M^{k+\frac{1}{2}}}\right\}_{k\in\mathbb{Z}_{>0},\eta\in(0,1)}, \quad \text{where} \quad p_k^{(\lfloor\eta M\rfloor)}=\sum_{i=1}^{\lfloor\eta M\rfloor}(\lambda_i^{(\lfloor\eta M\rfloor)}+\lfloor\eta M\rfloor-i)^k,$$

converges to a jointly Gaussian distribution with covariance given by

$$\lim_{M\to\infty}\frac{1}{M^{k_1+k_2+1}}\mathrm{Cov}(p_{k_1}^{(\lfloor\eta_1 M\rfloor)},p_{k_2}^{(\lfloor\eta_2 M\rfloor)})=\frac{1-\eta_2}{(2\pi\mathrm{i})^2}\oint_{|z|=2\varepsilon}\oint_{|w|=\varepsilon}\left((1-\eta_1)F(z)+\eta_1+\frac{\eta_1}{z}\right)^{k_1}$$

$$\times\left((1-\eta_2)F(w)+\eta_2+\frac{\eta_2}{w}\right)^{k_2}G(z,w)dzdw,$$

for $\eta_1<\eta_2$, where

$$F(z)=\mathbb{E}_\beta\left[\frac{\beta+\beta z}{1+\beta z}\right], \qquad G(z,w)=\mathrm{Cov}_\beta\left(\frac{\beta}{1+\beta z},\frac{\beta}{1+\beta w}\right).$$

Abstract setting

- Discrete particle system: $l_1 > \cdots > l_N$, $l_i \in \mathbb{Z}$
- We consider a probability measure on $\mathbb{R}$:

$$m[l] := \frac{1}{N} \sum_{i=1}^{N} \delta\left(\frac{l_i}{N}\right)$$

- If l_i are random, the $m[l]$ is a random probability measure on $\mathbb{R}$.
- We are interested in asymptotic behavior of $m[l]$, as $N \to \infty$. Concentration? Gaussian fluctuations?
- Characteristic functions??? N-tuples of integers are a dual object to the unitary group of dimension N.
- Goal: Describe the convergence in terms of such characteristic functions.

Sign_N—the set of all signatures of length N. Let $\mathrm{Sign}_N \ni \lambda \mapsto \rho(\lambda)$ be a probability measure on Sign_N.

Definition (Bufetov-Gorin) The Schur generating function of ρ is defined by

$$S_\rho(x_1,\dots,x_N) := \sum_{\lambda \in \mathrm{Sign}_N} \rho(\lambda) \frac{s_\lambda(x_1,\dots,x_N)}{s_\lambda(1,\dots,1)}.$$

Claim: A Schur generating function is a good analog of a characteristic function for asymptotic questions.

$$0 \prec \theta^{(1)} \succ_v \lambda^{(1)} \prec \theta^{(2)} \succ_v \lambda^{(2)} \prec \cdots \prec \theta^{(M)} \succ_v \lambda^{(M)} = 0$$

- For $N < M$, the (quenched) Schur generating function of the distribution of $\lambda^{(N)}$, is

$$S_{\mathrm{Law}(\lambda^{(N)})}(x_1,\dots,x_N) = \prod_{i=1}^{N} \prod_{j=N+1}^{M} (1 - \beta_j + \beta_j x_i).$$

$$m[\rho_N]:=\frac{1}{N}\sum_{i=1}^{N}\delta\left(\frac{\lambda_i+N-i}{N}\right),\quad \text{where}\quad (\lambda_1\geq\cdots\geq\lambda_N)\text{ is }\rho_N\text{-distributed.}$$

Theorem Bufetov-Petrov-Z.'25

Assume that ρ_N is a sequence of probability measure on Sign_N, such that for any fixed $k\geq 1$,

$$\lim_{N\to\infty}\frac{1}{N}\log S_{\rho_N}(u_1,\dots,u_k,1,\dots,1)=F_k(u_1,\dots,u_k).$$

Then, $m[\rho_N]$ converges in probability to a deterministic probability measure on $\mathbb{R}$, with moments given by

$$\mu_k=\sum_{m=0}^{k}\binom{k}{m}\frac{1}{(m+1)!}\sum_{n=1}^{m+1}\binom{m+1}{n}(-1)^{n+1}\sum_{l_1+\cdots+l_n=m}$$

$$\times\binom{m}{l_1,\dots,l_n}\partial_{u_n}^{l_n}\dots\partial_{u_1}^{l_1}[u_1^k(\partial_{u_1}F_k(u_1,\dots,u_k))^{k-m}]\Big|_{u_i=1}.$$

$$\lim_{N\to\infty}\frac{1}{N}\log S_{\rho_N}(u_1,\ldots,u_k,1,\ldots,1)=F_k(u_1,\ldots,u_k)$$

- The case where $F_k(u_1,\ldots,u_k)=\Psi(u_1)+\cdots+\Psi(u_k)$, was considered by Bufetov-Gorin'13.

$$0\prec\theta^{(1)}\succ_v\lambda^{(1)}\prec\theta^{(2)}\succ_v\lambda^{(2)}\prec\cdots\prec\theta^{(M)}\succ_v\lambda^{(M)}=0, \quad S_{\mathrm{Law}(\lambda^{(N)})}(x_1,\ldots,x_N)=\prod_{i=1}^{N}\prod_{j=N+1}^{M}(1-\beta_j+\beta_j x_i).$$

- For the uniform probability measure on domino tilings of the Aztec diamond, we have for $N<M$,

$$\frac{1}{N}\log S_{\mathrm{Law}(\lambda^{(N)})}(x_1,\ldots,x_k,1,\ldots,1)=\frac{M-N}{N}\sum_{i=1}^{k}\log\left(\frac{1+x_i}{2}\right).$$

- For i.i.d. edge weights, for the annealed Schur generating function we have

$$\frac{1}{N}\log S_{\mathrm{Law}(\lambda^{(N)})}(x_1,\ldots,x_k,1,\ldots,1)=\frac{M-N}{N}\log\left(\mathbb{E}_\beta\left[\prod_{i=1}^{k}(1-\beta+x_i\beta)\right]\right).$$

$$m[\rho_N]:=\frac{1}{N}\sum_{i=1}^{N}\delta\left(\frac{\lambda_i+N-i}{N}\right), \quad \text{where} \quad (\lambda_1\geq\cdots\geq\lambda_N) \text{ is } \rho_N\text{-distributed.}$$

Theorem Bufetov-Petrov-Z.'25

Assume that ρ_N is a sequence of probability measure on Sign_N, such that for any fixed $k\geq 1$,

$$\lim_{N\to\infty}\frac{1}{N}\log S_{\rho_N}(u_1,\ldots,u_k,1,\ldots,1)=F_k(u_1,\ldots,u_k).$$

Then, the moments of $m[\rho_N]$ have Gaussian fluctuations of order $\sqrt{N}$.

The formula for the covariance in the general case is explicit, but quite complicated.

These complicated formulas in the abstract setting vastly simplify in the case of Aztec diamond with i.i.d. edge weights.

- In the annealed case the GFF fluctuations become invisible, washed away by the much larger ones of the random environment.

Question. Does the global behavior of the one-periodic Aztec diamond change, when we condition on the environment?

$$0 \prec \theta^{(1)} \succ_v \lambda^{(1)} \prec \theta^{(2)} \succ_v \lambda^{(2)} \prec \cdots \prec \theta^{(M)} \succ_v \lambda^{(M)} = 0, \qquad p_k^{(N)} = \sum_{i=1}^{N} (\lambda_i^{(N)} + N - i)^k.$$

Theorem Z.'25 Assume that the distribution of β is fixed. For almost every realization of the environment, the vector $\left\{\frac{1}{M^k}(p_k^{(\lfloor \eta M \rfloor)} - \mathbb{E}_\lambda[p_k^{(\lfloor \eta M \rfloor)}])\right\}_{k \in \mathbb{Z}_{>0}, \eta \in (0,1)}$ converges to a jointly Gaussian distribution, and

$$\lim_{M\to\infty} \frac{1}{M^{k_1+k_2}} \mathrm{Cov}_\lambda(p_{k_1}^{(\lfloor \eta_1 M \rfloor)}, p_{k_2}^{(\lfloor \eta_2 M \rfloor)}) = \frac{1}{(2\pi \mathrm{i})^2} \oint_{|z|=2\varepsilon} \oint_{|w|=\varepsilon} \left((1-\eta_1)\mathbb{E}\left[\frac{\beta+\beta z}{1+\beta z}\right] + \eta_1 + \frac{\eta_1}{z} \right)^{k_1}$$

$$\times \left((1-\eta_2)\mathbb{E}\left[\frac{\beta+\beta w}{1+\beta w}\right] + \eta_2 + \frac{\eta_2}{w} \right)^{k_2} \frac{1}{(z-w)^2} dz dw.$$

Randomness of the environment in Quenched and Annealed

$$0 \prec \theta^{(1)} \succ_v \lambda^{(1)} \prec \theta^{(2)} \succ_v \lambda^{(2)} \prec \cdots \prec \theta^{(M)} \succ_v \lambda^{(M)} = 0, \qquad p_k^{(N)} = \sum_{i=1}^{N} (\lambda_i^{(N)} + N - i)^k.$$

$$p_k^{(N)} - \mathbb{E}_{\beta,\lambda}[p_k^{(N)}] = p_k^{(N)} - \mathbb{E}_{\lambda}[p_k^{(N)}] + \mathbb{E}_{\lambda}[p_k^{(N)}] - \mathbb{E}_{\beta,\lambda}[p_k^{(N)}].$$

- The one-dimensional Brownian motions of the annealed CLTs arise due to the fluctuations of the quenched mean $\mathbb{E}_{\lambda}[p_k^{(N)}]$.

Question. Where is the randomness of the environment in the quenched CLT?

- Analogously, the higher-order cumulants of $\{N^{-k} p_k^{(N)}\}_{k \geq 1}$ (with respect to $\mathbb{E}_{\lambda}$) concentrate to those of the Gaussian Free Field, and they have Gaussian fluctuations.

It is also possible to consider non i.i.d. edge weights.

$$0\prec\theta^{(1)}\succ_v\lambda^{(1)}\prec\theta^{(2)}\succ_v\lambda^{(2)}\prec\cdots\prec\theta^{(M)}\succ_v\lambda^{(M)}=0, \quad m[\lambda^{(N)}]=\frac{1}{N}\sum_{i=1}^{N}\delta\left(\frac{\lambda_i^{(N)}+N-i}{N}\right).$$

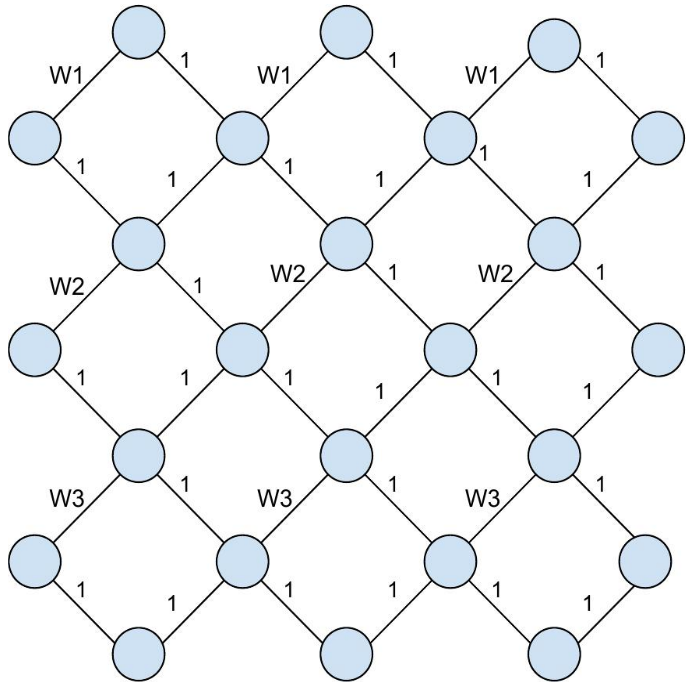
Global conditions asymptotic for the random measure

$$m_N[\beta]=\frac{1}{N}\sum_{i=1}^{N}\delta(\beta_i), \quad \text{where} \quad w_i=\frac{\beta_i}{1-\beta_i}.$$

are sufficient to extract analogous global asymptotic results for $m[\lambda^{(N)}]$.

- If $m_N[\beta]$ has a limit shape, then $m[\lambda^{(N)}]$ also has a limit shape.
- If $m_N[\beta]$ has Gaussian fluctuations, then $m[\lambda^{(N)}]$ also has Gaussian fluctuations.

In the remaining time, we consider a different setup, where the edge weights are deterministic.



Let us take $w_1 = W$, and all other parameters to be equal to 1.

$$0 \prec \theta^{(1)} \succ_v \lambda^{(1)} \prec \theta^{(2)} \succ_v \lambda^{(2)} \prec \cdots \prec \theta^{(M)} \succ_v \lambda^{(M)} = 0,$$

$$\log S_{\mathrm{Law}(\lambda^{(N)})}(x_1,\ldots,x_k,1,\ldots,1) = (M-N-1)\sum_{i=1}^{k} \log\left(\frac{1+x_i}{2}\right) + \sum_{i=1}^{r} \log\left(\frac{1+Wx_i}{1+W}\right).$$

Theorem Bufetov-Z.'24

Assume that ρ_N is a sequence of probability measure on Sign_N, such that for any fixed $k \geq 1$,

$$\lim_{N\to\infty}\left(\log S_{\rho_N}(x_1,\ldots,x_k,1,\ldots,1) - N\sum_{i=1}^{k} \Psi(x_i)\right) = \sum_{i=1}^{k} \Phi(x_i),$$

where Ψ, Φ are analytic functions in a complex neighborhood of 1, and the above convergence is uniform in a complex neighborhood of 1^k. Then

$$\lim_{N\to\infty}\left(\sum_{\lambda\in\mathrm{Sign}_N} \rho_N(\lambda)\cdot\sum_{i=1}^{N} \left(\frac{\lambda_i+N-i}{N}\right)^k - N\mu_k\right) = \mu_k',$$

where μ_k, μ_k' can be computed explicitly.

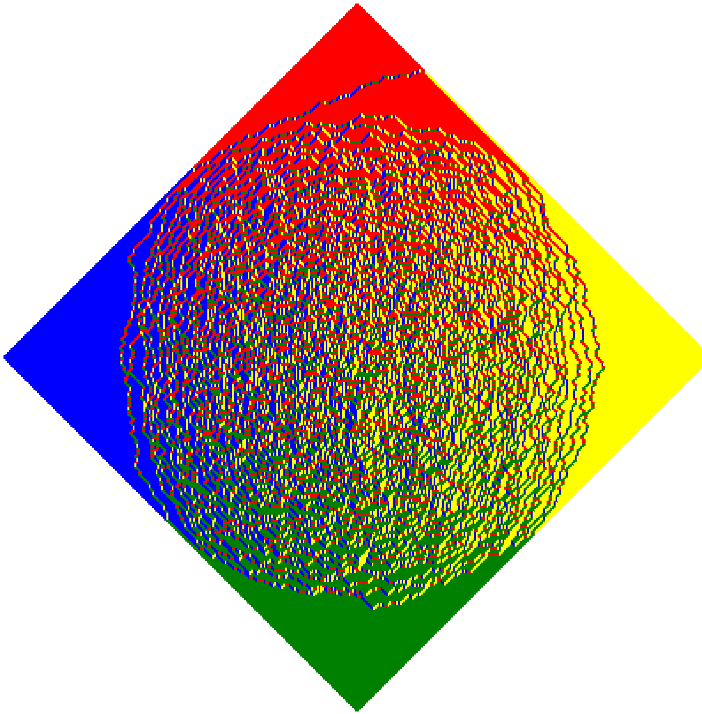
$$\lim_{N\to\infty}\left(\log S_{\rho_N}(x_1,\ldots,x_k,1,\ldots,1)-N\sum_{i=1}^{k}\Psi(x_i)\right)=\sum_{i=1}^{k}\Phi(x_i)$$

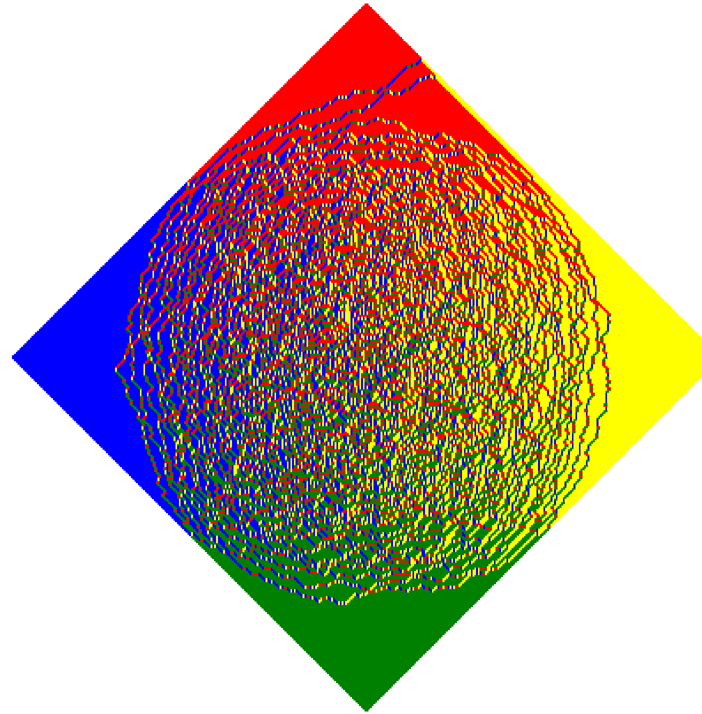
The moments μ_k, μ_k' are given by

$$\mu_k=\sum_{m=0}^{k}\binom{k}{m}\frac{1}{(m+1)!}\frac{\mathrm{d}^m}{\mathrm{d}x^m}(x^k\Psi'(x))^{k-m}\bigg|_{x=1},$$

$$\mu_k'=\sum_{m=0}^{k-1}\binom{k}{m+1}\frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(x^k\left(\Phi'(x)-\frac{1}{2x}\right)(\Psi'(x))^{k-m-1}\right)\bigg|_{x=1}.$$

- The $1/N$ correction has been studied extensively in the context of random matrices Shlyakhtenko'15. In certain cases, explicit formulas for the signed measure μ', lead to Baik-Ben Arous-Peche phase transitions.
- It also fits into the abstract framework of Infinitesimal Free Probability.





We again study this picture via moments

$$\sum_{\lambda\in\mathrm{Sign}_N}\rho_N(\lambda)\cdot\sum_{i=1}^{N}\left(\frac{\lambda_i^{(N)}+N-i}{N}\right)^k=N\cdot\mu_{k;W}+\mu'_{k;W}+o(1),\qquad \frac{N}{M}\to\eta,$$

$$\mu_{k;W}=\int_{\mathbb{R}}t^k\mu_W(dt),\qquad \mu'_{k;W}=\int_{\mathbb{R}}t^k\mu'_W(dt).$$

We have

$$\mu'_W(dt)=-\mathbf{1}_{\eta>\frac{(W+1)^2}{2(W^2+1)}}\cdot\delta\left(\frac{-1+2\eta W-W}{\eta(W^2-1)}\right)-\frac{1}{2}\mathbf{1}_{\eta=\frac{(W+1)^2}{2(W^2+1)}}\cdot\delta\left(\frac{-1+2\eta W-W}{\eta(W^2-1)}\right)$$

$$+\left(\eta+\frac{1-2\eta}{4t}-\frac{\eta(2\eta-1)}{4(\eta t-1)}+\frac{\eta(2W^2\eta-W^2-2W+2\eta-1)}{2(-\eta t+W^2\eta t-2W\eta+1+W)}\right)\frac{\mathbf{1}_{1-(2\eta-1)^2-(2\eta t-1)^2>0}}{\pi\sqrt{1-(2\eta-1)^2-(2\eta t-1)^2}}dt,$$

$$\mu_W(dt)=\mathbf{1}_{(1-2\eta t)^2\le 1-(1-2\eta)^2}\frac{1}{\pi}\arccos\left(\frac{1-2\eta}{\sqrt{1-(1-2\eta t)^2}}\right)dt+\mathbf{1}_{1\ge(1-2\eta t)^2>1-(1-2\eta)^2}dt.$$

Method of proof

Consider the differential operators

$$\mathcal{D}_k(\cdot) := \prod_{i<j} \frac{1}{x_i - x_j} \sum_{i=1}^{N} \left(x_i \frac{\partial}{\partial x_i} \right)^k \left(\prod_{i<j} (x_i - x_j) \cdot \right).$$

They act nicely on Schur generating functions.

$$S_{\rho_N}(x_1, \dots, x_N) = \sum_{\lambda \in \mathrm{Sign}_N} \rho_N(\lambda) \frac{s_\lambda(x_1, \dots, x_N)}{s_\lambda(1, \dots, 1)}.$$

The Schur polynomials are eigenfunctions of $\mathcal{D}_k$:

$$\mathcal{D}_k s_\lambda(x_1, \dots, x_N) = \sum_{i=1}^{N} (\lambda_i + N - i)^k \cdot s_\lambda(x_1, \dots, x_N).$$

Method of proof

$$S_{\rho_N}(x_1,\ldots,x_N) = \sum_{\lambda\in \mathrm{Sign}_N} \rho_N(\lambda) \frac{s_\lambda(x_1,\ldots,x_N)}{s_\lambda(1,\ldots,1)}.$$

$$\mathcal{D}_{k_1}\ldots\mathcal{D}_{k_\nu} S_{\rho_N}(x_1,\ldots,x_N)\Big|_{x_i=1} =$$

$$\sum_{\lambda\in \mathrm{Sign}_N} \rho_N(\lambda) \left(\sum_{i=1}^{N} (\lambda_i+N-i)^{k_1}\right)\ldots\left(\sum_{i=1}^{N} (\lambda_i+N-i)^{k_\nu}\right).$$

General conditions on S_{ρ_N} allow to compute the left hand side. The right hand side divided by $N^{k_1+\cdots+k_\nu+\nu}$ approximates expectations of the moments of (a priori random) limit measure

$$\mathbb{E}\left[\prod_{i=1}^{\nu} \int_{\mathbb{R}} x^{k_i} \mu(dx)\right].$$