

The Integrable Combinatorics of Alternating Sign Matrices

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Plan

- I. Introduce alternating sign matrices (ASMs) & state a formula for their enumeration.
- II. Introduce odd-order diagonally & antidiagonally symmetric alternating sign matrices (DASASMs) & state a formula for their enumeration.
- III. Introduce diagonally symmetric alternating sign matrices (DSASMs) & state a formula for their enumeration.
- IV. Summarize some further ASM-related results.
 - I'. Outline a proof of the ASM formula, involving the integrable six-vertex model on a square with domain-wall boundary conditions (*Kuperberg 1996*).
 - II'. Outline a proof of the odd-order DASASM formula, involving the integrable six-vertex model on an isosceles triangle with certain boundary conditions (*RB, Fischer, Konvalinka 2017*).
 - III'. Outline a proof of the DSASM formula, involving the integrable six-vertex model on an “equilateral” triangle with certain boundary conditions (*RB, Fischer, Koutschan 2023*).

I. Alternating Sign Matrices (ASMs)

ASM: square matrix for which:

- Each entry is 0, 1 or -1 .
- Each row & column contains at least one 1.
- Along each row & column, the nonzero entries alternate in sign, starting & ending with a 1.

e.g.
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

History:

- Arose during study of Dodgson condensation algorithm for determinant evaluation (*Mills, Robbins, Rumsey 1982; Robbins, Rumsey 1986*).
- Many subsequent appearances in combinatorics, algebra, mathematical physics, ...

Observations:

- First/last row/column of an ASM contains a single 1 & all other entries 0.
- Acting on an ASM with any symmetry operation of the square (reflection in 4 symmetry axes, rotation by 0° , 90° , 180° or 270°) gives another ASM.
- Any permutation matrix (exactly one 1 in each row & column, all other entries 0) is an ASM.

Elementary Bounds on Number of $n \times n$ ASMs

$$\begin{aligned}\{n \times n \text{ permutation matrices}\} &\subseteq \{n \times n \text{ ASMs}\} \\ &\subseteq \{n \times n \text{ matrices with each entry } 0, 1 \text{ or } -1\}\end{aligned}$$

implies

$$\begin{aligned}(\# \text{ of } n \times n \text{ permutation matrices}) &\leq (\# \text{ of } n \times n \text{ ASMs}) \\ &\leq (\# \text{ of } n \times n \text{ matrices with each entry } 0, 1 \text{ or } -1)\end{aligned}$$

implies

$$n! \leq (\# \text{ of } n \times n \text{ ASMs}) \leq 3^{n^2}.$$

Exact Number A_n of $n \times n$ ASMs

n=1

$$(1) \Rightarrow A_1 = 1.$$

n=2

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow A_2 = 2.$$

n=3

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow A_3 = 7.$$

$$n=4$$

- $4! = 24$ matrices without any -1 's (permutation matrices).
- 4 matrices with one -1 at position 2,2:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Similarly:

- 4 matrices with one -1 at 2,3,
- 4 matrices with one -1 at 3,2,
- 4 matrices with one -1 at 3,3.

So, 16 matrices with one -1 .

- 2 matrices with two -1 's:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

$$\Rightarrow A_4 = 24 + 16 + 2 = 42.$$

General Case

$$\# \text{ of } n \times n \text{ ASMs: } A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} = 1, 2, 7, 42, 429, 7436, \dots$$

- Recurrence: $\binom{2n}{n} A_{n+1} = \binom{3n+1}{n} A_n$.
- Conjectured: *Mills, Robbins, Rumsey 1982*.
- First proved:
 - *Zeilberger 1996* using constant term identities & known enumeration of totally symmetric self-complementary plane partitions (84 pages).
 - *Kuperberg 1996* using connections with integrable six-vertex model (12 pages).
- Several subsequent proofs – e.g. *Fischer, Konvalinka 2021* using signed bijections.
- No simple combinatorial proof (i.e. using elementary counting arguments) currently known.
- Book: D. Bressoud *Proofs & Confirmations: The Story of the ASM Conjecture* Cambridge University Press (1999), 274 pages.

- Outline of Kuperberg proof (more details soon):
 - Obtain bijection between $n \times n$ ASMs & configurations of *six-vertex model* on $n \times n$ square with *domain-wall boundary conditions*.
 - Introduce spectral parameter-dependent vertex weights & consider weighted sum over all configurations of model, i.e. (inhomogeneous) *partition function*.
 - Use *Yang–Baxter equation* & other properties to obtain *Izergin–Korepin formula* for partition function as $n \times n$ determinant.
 - Evaluate determinant at certain values of parameters for which all weights are 1 (homogeneous limit).

II. Diagonally & Antidiagonally Symmetric Alternating Sign Matrices (DASASMs)

DASASM: ASM which is invariant under:

- reflection in the **diagonal** &
- reflection in the **antidiagonal**.

e.g.
$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

History:

- Arose during studies of classes of ASMs invariant under the action of subgroups of the symmetry group of the square (*Robbins 1985; Stanley 1986*).

- Observations:**
- Any DASASM is also invariant under rotation by 180° .
 - Any DASASM is uniquely determined by its entries in an isosceles triangle bounded by diagonal & antidiagonal.
 - Central entry of an *odd-order* DASASM is ± 1 .

Number D_n of $(2n+1) \times (2n+1)$ DASASMs
& numbers $D_n^\pm$ of $(2n+1) \times (2n+1)$ DASASMs with central entry ± 1

n=0

(1)

$$\Rightarrow D_0 = 1$$

$$\& \frac{D_0^-}{D_0^+} = \frac{0}{1}.$$

n=1

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow D_1 = 3$$

$$\& \frac{D_1^-}{D_1^+} = \frac{1}{2}.$$

$$n=2$$

$$\begin{aligned}
& \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \color{red}{1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \color{red}{1} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \color{red}{1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \color{red}{1} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
& \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \color{red}{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \color{red}{1} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & \color{red}{1} & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & \color{red}{1} & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \\
& \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & -1 & \color{red}{1} & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & \color{green}{-1} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & \color{green}{-1} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \color{green}{-1} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \\
& \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & \color{green}{-1} & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & \color{green}{-1} & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & \color{green}{-1} & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

$$\Rightarrow D_2 = 15$$

$$\& \frac{D_2^-}{D_2^+} = \frac{\color{green}{6}}{\color{red}{9}} = \frac{\color{green}{2}}{\color{blue}{3}}.$$

General Case

$$\# \text{ of } (2n+1) \times (2n+1) \text{ DASASMs: } D_n = \prod_{i=0}^n \frac{(3i)!}{(n+i)!} = 1, 3, 15, 126, 1782, \dots$$

- Recurrence: $\binom{2n-1}{n} D_n = \binom{3n}{n} D_{n-1}$.
- Conjectured: *Robbins 1985*.
- Proved: *RB, Fischer, Konvalinka 2017*.
- No simple combinatorial proof currently known.

$$\# \text{'s of } (2n+1) \times (2n+1) \text{ DASASMs with central entry } \pm 1: \frac{D_n^-}{D_n^+} = \frac{n}{n+1}$$

- Conjectured: *Stroganov 2008*.
- Proved: *RB, Fischer, Konvalinka 2017*.
- No simple combinatorial proof currently known.

III. Diagonally Symmetric Alternating Sign Matrices (DSASMs)

DSASM: ASM which is invariant under reflection in the main **diagonal**, i.e. under matrix transposition.

e.g.
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Observation:

- Any DSASM is uniquely determined by its upper triangular part.

Number T_n of $n \times n$ DSASMs

n=1

$$(1) \Rightarrow T_1 = 1.$$

n=2

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow T_2 = 2.$$

n=3

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow T_3 = 5.$$

General Case

of $n \times n$ DSASMs:

$$T_n = \text{Pfaffian}_{0 \text{ or } 1 \leq i, j \leq n-1} \left(\sum_{k=0}^{\min(i,j)} (3 - \delta_{k,0}) \left(\binom{i+j-2k-1}{i-k} - \binom{i+j-2k-1}{j-k} \right) \right)$$

$$= 1, 2, 5, 16, 67, 368, 2630, 24376, 293770, 4610624, \dots$$

[For an even-order skew-symmetric matrix A : $\det A = (\text{Pfaffian } A)^2$.]

- Range for i, j starts at 0 for n even, 1 for n odd.
- Proved: *RB, Fischer, Koutschan 2023*.
- T_n for $n = 1, 2, \dots, 1000$ has been computed using this formula.
- No simpler formula currently known.
- Product formula (i.e. product of ratios of factorials similar to formulae for # of ASMs or # of odd-order DASASMs) unlikely to exist, since prime factors of T_n do not seem bounded above by a polynomial in n .

IV. Further ASM-Related Results

- Product formulae conjectured for # of ASMs in most symmetry classes: *Robbins 1985*.
- All these formulae now proved:
 - Odd-order vertically symmetric ASMs: *Kuperberg 2002*.
 - Odd-order vertically & horizontally symmetric ASMs: *Okada 2006*.
 - Even-order half-turn symmetric ASMs: *Kuperberg 2002*.
 - Odd-order half-turn symmetric ASMs: *Razumov, Stroganov 2006*.
 - Order 0 mod 4 quarter-turn symmetric ASMs: *Kuperberg 2002*.
 - Odd-order quarter-turn symmetric ASMs: *Razumov, Stroganov 2006*.
 - Odd-order diagonally & antidiagonally symmetric ASMs (DASASMs):
RB, Fischer, Konvalinka 2017.
- # of ASMs in remaining nonempty symmetry classes:
 - Diagonally symmetric ASMs (DSASMs): Pfaffian formula obtained
RB, Fischer, Koutschan 2023.
 - Odd-order totally symmetric ASMs: multiple contour-integral formula obtained
Liénardy, Walmsley Hagendorf 2025.
 - Even-order diagonally & antidiagonally symmetric ASMs (DASASMs): no formula currently known.

- Various formulae also known for # of all ASMs & # of symmetric ASMs with fixed values of certain statistics (e.g. positions of 1's in first/last row/column, # of -1 's, # of 'inversions').
- e.g. # of all ASMs with fixed positions of 1's in first/last row/column:
 - SINGLE 1: *Zeilberger 1996*.
 - TWO 1's on opposite or adjacent boundaries: *Stroganov 2006*
 - THREE or FOUR 1's: *Ayyer, Romik 2013; RB 2013*.
- No simple combinatorial proofs of any of the previous ASM enumeration formulae currently known.

ASMs & Other Combinatorial Objects

Razumov–Stroganov correspondence

- Involves associating ‘fully-packed loop configurations’ & ‘link patterns’ with ASMs & characterizing $\#$ of ASMs with fixed link pattern
Cantini, Sportiello 2011; Cantini, Sportiello 2014.
- Proved using elegant combinatorial arguments: a rare case of such a proof for a nontrivial ASM-related result.

ASM – descending plane partition correspondence

- Known (non-bijectively) that:
 $\#$ of $n \times n$ ASMs with fixed values of four statistics
= $\#$ of order- n ‘descending plane partitions’ with fixed values of four statistics
RB, Di Francesco, Zinn-Justin 2012; RB, Di Francesco, Zinn-Justin 2013
[see also *Fischer, Schreier-Aigner 2023*].
- Gives:
 $\#$ of arrow-preserving orientations of certain square lattice with domain-wall boundaries
= $\#$ of rotationally-invariant tilings of certain triangular lattice with central hole.
- Square $\leftrightarrow$ triangle duality holds:
 $\#$ arrow-preserving orientations of certain triangular lattice with domain-wall boundaries
= $\#$ of rotationally-invariant tilings of certain square lattice with central hole
Di Francesco, Gitter 2020; Di Francesco 2021.

ASM – totally symmetric self-complementary partition correspondence

- Known (non-bijectively) that:

of $n \times n$ ASMs with fixed values of two statistics

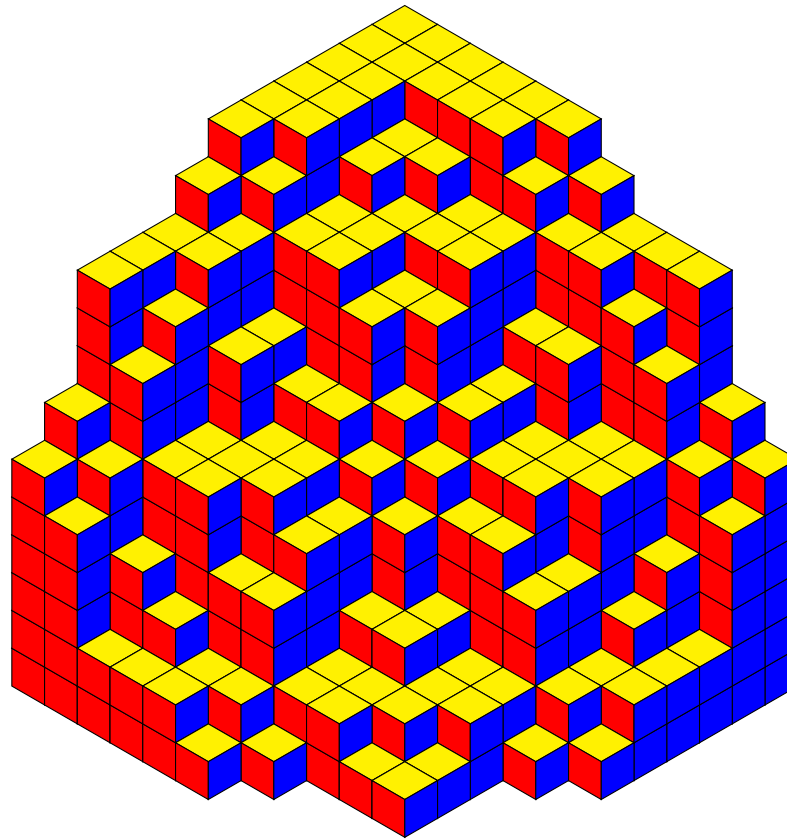
= # of 'totally symmetric self-complementary partitions' in a $2n \times 2n \times 2n$ box
with fixed values of two statistics *Fonseca, Zinn-Justin 2008.*

Totally Symmetric Self-Complementary Plane Partitions (TSSCPPs)

TSSCPP: aligned stack of unit cubes in a box, which is invariant under reflections, rotations & box-complementation

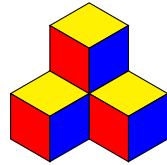
- Introduced: *Stanley 1986*

e.g. TSSCPP in $12 \times 12 \times 12$ box:



Number P_n of TSSCPPs in a $2n \times 2n \times 2n$ box

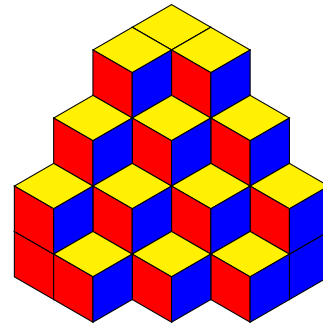
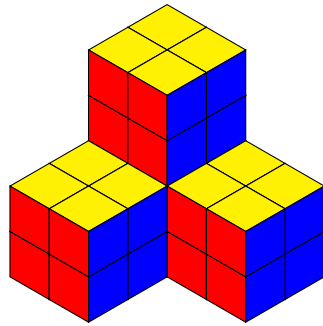
$n=1$



$\Rightarrow$

$$P_1 = 1$$

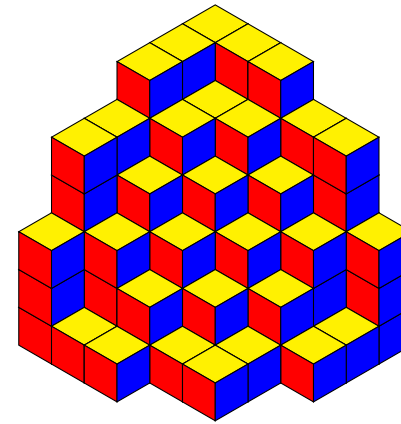
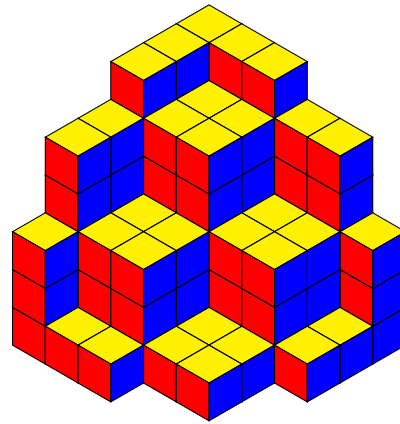
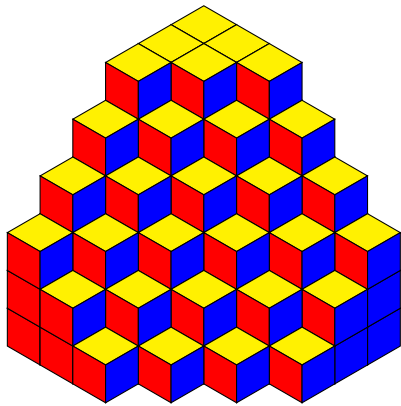
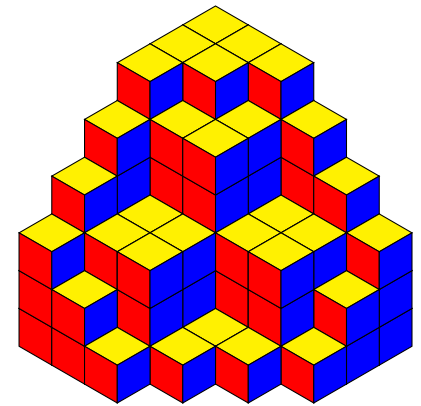
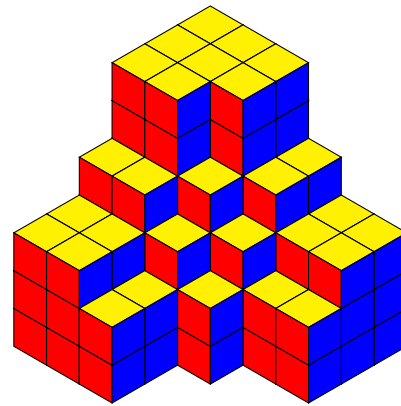
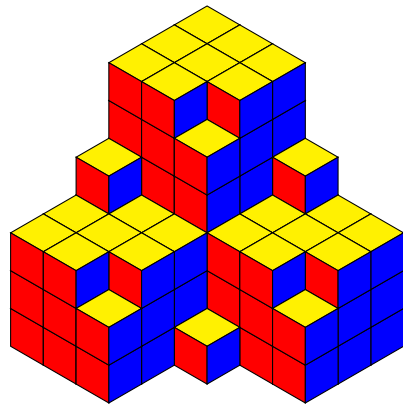
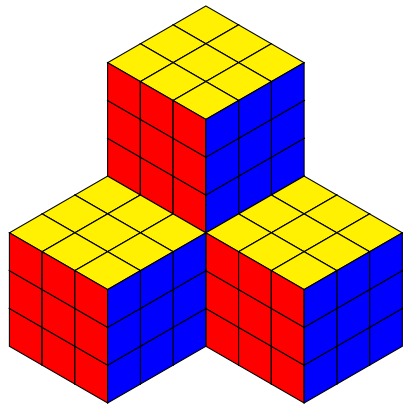
$n=2$



$\Rightarrow$

$$P_2 = 2$$

$n=3$



$\Rightarrow P_3 = 7$

General Case

$$(\# \text{ of TSSCPPs in } 2n \times 2n \times 2n \text{ box}): P_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} = 1, 2, 7, 42, 429, 7436, \dots$$

- Conjectured: *Mills, Robbins, Rumsey 1986.*
- Proved: *Andrews 1994.*
- Proof: Use sequences of nonintersecting lattice paths, Lindström–Gessel–Viennot theorem & determinant evaluation.
- Therefore

$$(\# \text{ of TSSCPPs in } 2n \times 2n \times 2n \text{ box}) = (\# \text{ of } n \times n \text{ ASMs})$$

- No explicit bijection currently known between $\{\text{TSSCPPs in } 2n \times 2n \times 2n \text{ box}\}$ & $\{n \times n \text{ ASMs}\}$ for arbitrary n .
- “*This is one of the most intriguing open problems in the area of bijective proofs.*”
(*R. Stanley 2009*)
- “*The greatest, still unsolved, mystery concerns the question of what plane partitions have to do with alternating sign matrices.*” (*C. Krattenthaler 2016*)

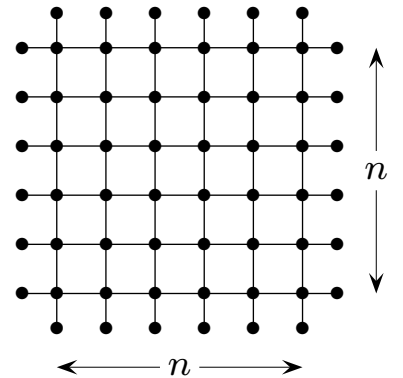
I'. Outline of Proof of ASM Formula

$$(\# \text{ of } n \times n \text{ ASMs}) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

- (1) Obtain bijection between $n \times n$ ASMs & configurations of *six-vertex model* on $n \times n$ square with *domain-wall boundary conditions*.
- (2) Introduce spectral parameter-dependent vertex weights & consider weighted sum over all configurations of model, i.e. (inhomogeneous) *partition function*.
- (3) Use *Yang–Baxter equation* & other properties to obtain *Izergin–Korepin formula* for partition function as $n \times n$ determinant.
- (4) Evaluate determinant at certain values of parameters for which all weights are 1 (homogeneous limit).

Configurations of Six-Vertex Model on a Square with Domain-Wall Boundary Conditions

- Consider the $n \times n$ grid graph $\mathcal{G}_n :=$



- The set of config'ns of the six-vertex model on $\mathcal{G}_n$ with domain-wall boundary conditions is

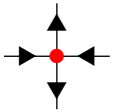
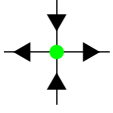
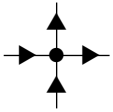
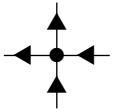
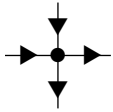
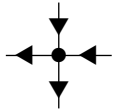
$$6V(n) := \left\{ \begin{array}{l} \text{edge orientations} \\ \text{of } \mathcal{G}_n \end{array} \left| \begin{array}{l} \bullet \text{ 2 in \& 2 out arrows at each degree-4 vertex} \\ \quad (\Rightarrow 6 \text{ options: } \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array}, \begin{array}{c} \uparrow \downarrow \\ \rightarrow \leftarrow \end{array}, \begin{array}{c} \leftarrow \rightarrow \\ \uparrow \downarrow \end{array}, \begin{array}{c} \leftarrow \rightarrow \\ \downarrow \uparrow \end{array}, \begin{array}{c} \rightarrow \leftarrow \\ \uparrow \downarrow \end{array}, \begin{array}{c} \rightarrow \leftarrow \\ \downarrow \uparrow \end{array}) \\ \bullet \text{ all arrows out of } \mathcal{G}_n \text{ at top \& bottom boundaries} \\ \bullet \text{ all arrows into } \mathcal{G}_n \text{ at left \& right boundaries} \end{array} \right\}.$$

- e.g. $6V(1) = \{ \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} \}, \quad 6V(2) = \left\{ \begin{array}{cc} \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} & \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} \\ \begin{array}{c} \leftarrow \rightarrow \\ \uparrow \downarrow \end{array} & \begin{array}{c} \leftarrow \rightarrow \\ \uparrow \downarrow \end{array} \end{array} \right\},$

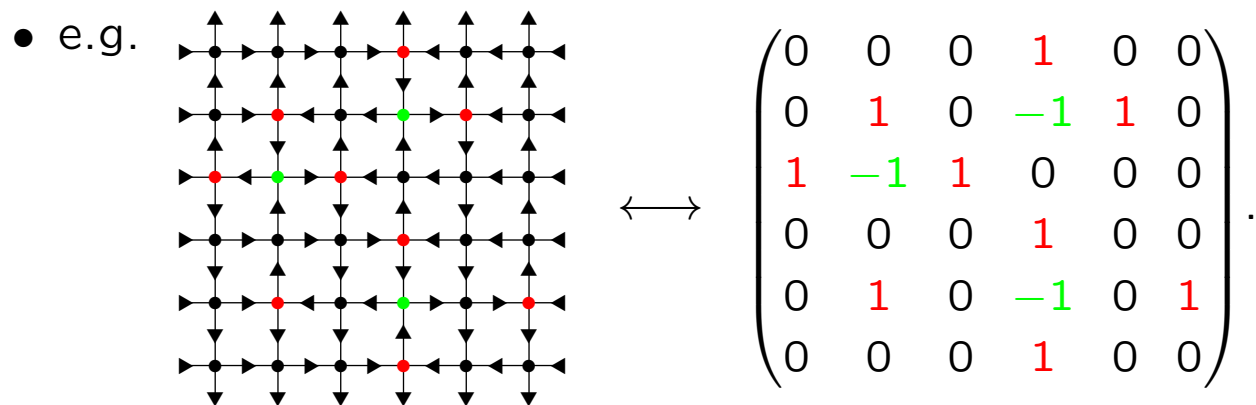
$$6V(3) = \left\{ \begin{array}{ccccccc} \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} & \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} & \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} & \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} & \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} & \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} & \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} \\ \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} & \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} & \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} & \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} & \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} & \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} & \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} \\ \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} & \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} & \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} & \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} & \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} & \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} & \begin{array}{c} \uparrow \downarrow \\ \leftarrow \rightarrow \end{array} \end{array} \right\}.$$

$$\Rightarrow |6V(1)| = 1, \quad |6V(2)| = 2, \quad |6V(3)| = 7.$$

- Simple bijection between $6V(n)$ & $\{n \times n \text{ ASMs}\}$ obtained by associating six-vertex model local configurations at degree-4 vertices with ASM entries according to:

<u>six-vertex model</u>		<u>ASM</u>
	$\longleftrightarrow$	1
	$\longleftrightarrow$	-1
 ,  ,  , 	$\longleftrightarrow$	0

(Elkies, Kuperberg, Larsen, Propp 1992).



Vertex Weights

- Associate weights with six-vertex model local configurations at degree-4 vertices:

$$W(\begin{array}{c} \uparrow \\ \leftarrow \bullet \rightarrow \\ \downarrow \end{array}, u) = W(\begin{array}{c} \downarrow \\ \leftarrow \bullet \rightarrow \\ \uparrow \end{array}, u) = \frac{\sigma(qu)}{\sigma(q^2)},$$

$$W(\begin{array}{c} \uparrow \\ \rightarrow \bullet \leftarrow \\ \downarrow \end{array}, u) = W(\begin{array}{c} \downarrow \\ \rightarrow \bullet \leftarrow \\ \uparrow \end{array}, u) = \frac{\sigma(qu^{-1})}{\sigma(q^2)},$$

$$W(\begin{array}{c} \uparrow \\ \rightarrow \bullet \rightarrow \\ \downarrow \end{array}, u) = W(\begin{array}{c} \downarrow \\ \rightarrow \bullet \rightarrow \\ \uparrow \end{array}, u) = 1,$$

where $\sigma(x) := x - x^{-1}$, u = 'spectral parameter', q = global parameter.

- Simple properties of weights:

$$- \text{ At } u = 1 \text{ \& } q = e^{i\pi/3}: \quad W\left(\begin{array}{c} d \\ a \rightarrow \bullet \leftarrow c \\ b \end{array}, 1\right) \Big|_{q=e^{i\pi/3}} = 1.$$

$$- \text{ At } u = q^{\pm 1}: \quad W\left(\begin{array}{c} d \\ a \rightarrow \bullet \leftarrow c \\ b \end{array}, q^{-1}\right) = \delta_{ab'} \delta_{cd'} \quad \& \quad W\left(\begin{array}{c} d \\ a \rightarrow \bullet \leftarrow c \\ b \end{array}, q\right) = \delta_{ad'} \delta_{bc'},$$

where a' denotes reversal of orientation a (i.e. in $\leftrightarrow$ out).

- Weights satisfy the Yang–Baxter Equation (YBE):

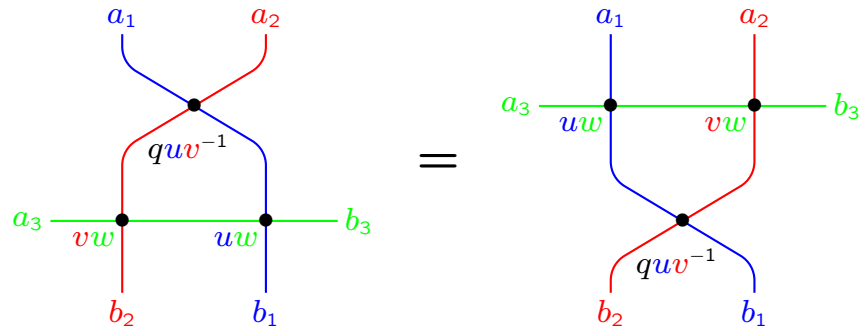
For all $a_1, b_1, a_2, b_2, a_3, b_3 \in \{\text{in}, \text{out}\}$,

$$\sum_{c_1, c_2, c_3} W\left(\begin{array}{c} a_1 \\ \text{---} \bullet \text{---} \\ c_2 \text{---} \bullet \text{---} \\ c_1 \end{array}, q u v^{-1}\right) W\left(\begin{array}{c} c_2 \\ \text{---} \bullet \text{---} \\ a_3 \text{---} \bullet \text{---} \\ b_2 \end{array}, v w\right) W\left(\begin{array}{c} c_1 \\ \text{---} \bullet \text{---} \\ c_3 \text{---} \bullet \text{---} \\ b_1 \end{array}, u w\right)$$

$$= \sum_{c_1, c_2, c_3} W\left(\begin{array}{c} a_1 \\ \text{---} \bullet \text{---} \\ a_3 \text{---} \bullet \text{---} \\ c_1 \end{array}, u w\right) W\left(\begin{array}{c} a_2 \\ \text{---} \bullet \text{---} \\ c_3 \text{---} \bullet \text{---} \\ b_3 \end{array}, v w\right) W\left(\begin{array}{c} c_1 \\ \text{---} \bullet \text{---} \\ b_2 \text{---} \bullet \text{---} \\ c_2 \end{array}, q u v^{-1}\right),$$

where sums are over all $c_1, c_2, c_3 \in \{\text{in}, \text{out}\}$ with 2 in & 2 out arrows at each vertex.

- YBE depicted as:



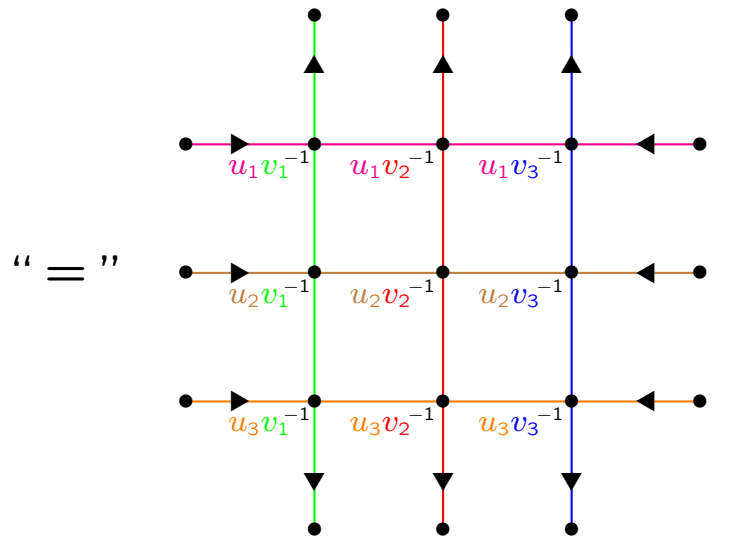
Partition Function

- Define the *partition function* as

$$Z(u_1, \dots, u_n, v_1, \dots, v_n) := \sum_{C \in 6V(n)} \prod_{i,j=1}^n W(C_{ij}, u_i v_j^{-1}),$$

where C_{ij} = local configuration at vertex in row i & column j of grid $\mathcal{G}_n$.

- e.g. $Z(u_1, u_2, u_3, v_1, v_2, v_3) =$ sum of 7 terms, each consisting of a product of 9 weights



- Since $W(C_{ij}, 1)|_{q=e^{i\pi/3}} = 1$ for all i, j & $6V(n)$ is in bijection with $\{n \times n \text{ ASMs}\}$:

$$Z(\underbrace{1, \dots, 1}_{2n})|_{q=e^{i\pi/3}} = |6V(n)| = (\# \text{ of } n \times n \text{ ASMs}).$$

Izergin–Korepin Formula

- The *Izergin–Korepin Formula* is

$$Z(u_1, \dots, u_n, v_1, \dots, v_n) = \frac{\prod_{i,j=1}^n \sigma(q u_i v_j^{-1}) \sigma(q u_i^{-1} v_j)}{\sigma(q^2)^{n(n-1)} \prod_{1 \leq i < j \leq n} \sigma(u_i u_j^{-1}) \sigma(v_i^{-1} v_j)} \det_{1 \leq i, j \leq n} \left(\frac{1}{\sigma(q u_i v_j^{-1}) \sigma(q u_i^{-1} v_j)} \right) \quad (\text{Izergin 1987})$$

Proof outline:

(a) Show that a function $X(u_1, \dots, u_n, v_1, \dots, v_n)$ which satisfies the following conditions is uniquely determined:

- (i) $X(u_1, v_1) = 1$ (*Initial condition*).
- (ii) $X(u_1, \dots, u_n, v_1, \dots, v_n)$ is a Laurent polynomial in u_1 of lower degree $\geq -n+1$ & upper degree $\leq n-1$ (*Laurent polynomial condition*).
- (iii) If $u_1 v_1^{-1} = q^{\pm 1}$, then

$$X(u_1, \dots, u_n, v_1, \dots, v_n) = \frac{\prod_{i=2}^n \sigma(q^{\pm 1} u_1 v_i^{-1}) \sigma(q^{\pm 1} u_i v_1^{-1})}{\sigma(q^2)^{2n-2}} X(u_2, \dots, u_n, v_2, \dots, v_n).$$
 (*Recurrence condition*).
- (iv) $X(u_1, \dots, u_n, v_1, \dots, v_n)$ is symmetric in $v_1, \dots, v_n$ (*Symmetry condition*).

(b) Show that LHS of Izergin–Korepin formula satisfies conditions (i)–(iv).

(c) Show that RHS of Izergin–Korepin formula satisfies conditions (i)–(iv).

Subsections of Proof of Izergin–Korepin Formula

(a) If $X(u_1, \dots, u_n, v_1, \dots, v_n)$ satisfies the following conditions, then it is uniquely determined:

(i) $X(u_1, v_1) = 1$.

(ii) $X(u_1, \dots, u_n, v_1, \dots, v_n)$ is a Laurent polynomial in u_1 of lower degree $\geq -n+1$ & upper degree $\leq n-1$.

(iii) If $u_1 v_1^{-1} = q^{\pm 1}$, then

$$X(u_1, \dots, u_n, v_1, \dots, v_n) = \frac{\prod_{i=2}^n \sigma(q^{\pm 1} u_1 v_i^{-1}) \sigma(q^{\pm 1} u_i v_1^{-1})}{\sigma(q^2)^{2n-2}} X(u_2, \dots, u_n, v_2, \dots, v_n).$$

(iv) $X(u_1, \dots, u_n, v_1, \dots, v_n)$ is symmetric in $v_1, \dots, v_n$.

Proof:

By (ii), $X(u_1, \dots, u_n, v_1, \dots, v_n)$ is uniquely determined if it is known at $2n-1$ values of u_1 .

Combining (iii) & (iv) gives expressions at $2n$ values of u_1 , i.e.

$$X(q^{\pm 1} v_i, u_2, \dots, u_n, v_1, \dots, v_n) = \dots X(u_2, \dots, u_n, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n) \quad \text{for } i = 1, \dots, n.$$

Required result now follows using recursion on n , together with (i). □

(b) The LHS of the Izergin-Korepin formula, i.e. the partition function $Z(u_1, \dots, u_n, v_1, \dots, v_n)$, satisfies:

(i) $Z(u_1, v_1) = 1$.

(ii) $Z(u_1, \dots, u_n, v_1, \dots, v_n)$ is a Laurent polynomial in u_1 of lower degree $\geq -n+1$ & upper degree $\leq n-1$.

(iii) If $u_1 v_1^{-1} = q^{\pm 1}$, then

$$Z(u_1, \dots, u_n, v_1, \dots, v_n) = \frac{\prod_{i=2}^n \sigma(q^{\pm 1} u_1 v_i^{-1}) \sigma(q^{\pm 1} u_i v_1^{-1})}{\sigma(q^2)^{2n-2}} Z(u_2, \dots, u_n, v_2, \dots, v_n).$$

(iv) $Z(u_1, \dots, u_n, v_1, \dots, v_n)$ is symmetric in $v_1, \dots, v_n$.

Proof:

(i) & (ii) follow straightforwardly from definitions of weights & partition function.

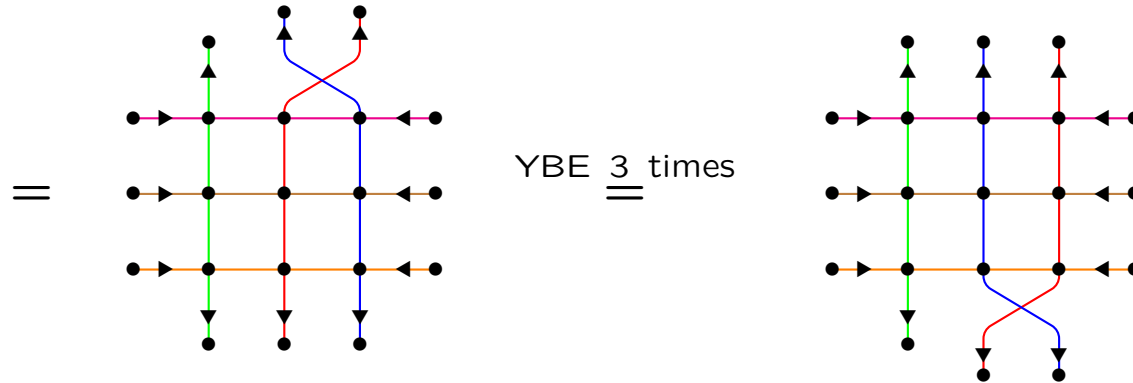
As an example of (iii), let $n = 3$ & $u_1 v_1^{-1} = q^{-1}$.

Then $Z(u_1, u_2, u_3, v_1, v_2, v_3)$

$$\begin{aligned}
 &= \text{Diagram 1} = \text{Diagram 2} \\
 &= \frac{\sigma(q^{-1} u_1 v_2^{-1}) \sigma(q^{-1} u_1 v_3^{-1}) \sigma(q^{-1} u_2 v_1^{-1}) \sigma(q^{-1} u_3 v_1^{-1})}{\sigma(q^2)^4} Z(u_2, u_3, v_2, v_3).
 \end{aligned}$$

As an example of (iv), symmetry of $Z(u_1, u_2, u_3, v_1, v_2, v_3)$ in v_2 & v_3 follows by using Yang-Baxter equation 3 times:

$$W(\begin{array}{c} \uparrow \\ \rightarrow \\ \leftarrow \\ \downarrow \end{array}, qv_2^{-1}v_3) Z(u_1, u_2, u_3, v_1, v_2, v_3)$$



$$= W(\begin{array}{c} \uparrow \\ \rightarrow \\ \leftarrow \\ \downarrow \end{array}, qv_2^{-1}v_3) Z(u_1, u_2, u_3, v_1, v_3, v_2).$$

(c) The RHS of the Izergin–Korepin formula, i.e.

$$Y(u_1, \dots, u_n, v_1, \dots, v_n) := \frac{\prod_{i,j=1}^n \sigma(q u_i v_j^{-1}) \sigma(q u_i^{-1} v_j)}{\sigma(q^2)^{n(n-1)} \prod_{1 \leq i < j \leq n} \sigma(u_i u_j^{-1}) \sigma(v_i^{-1} v_j)} \det_{1 \leq i, j \leq n} \left(\frac{1}{\sigma(q u_i v_j^{-1}) \sigma(q u_i^{-1} v_j)} \right),$$

satisfies:

- (i) $Y(u_1, v_1) = 1$.
- (ii) $Y(u_1, \dots, u_n, v_1, \dots, v_n)$ is a Laurent polynomial in u_1 of lower degree $\geq -n+1$, upper degree $\leq n-1$.
- (iii) If $u_1 v_1^{-1} = q^{\pm 1}$, then
$$Y(u_1, \dots, u_n, v_1, \dots, v_n) = \frac{\prod_{i=2}^n \sigma(q^{\pm 1} u_1 v_i^{-1}) \sigma(q^{\pm 1} u_i v_1^{-1})}{\sigma(q^2)^{2n-2}} Y(u_2, \dots, u_n, v_2, \dots, v_n).$$
- (iv) $Y(u_1, \dots, u_n, v_1, \dots, v_n)$ is symmetric in $v_1, \dots, v_n$.

Proof:

- (i) is trivial.
- (ii) follows from expansion of determinant.
- (iii) holds since, by multiplying first row of matrix by $\sigma(q u_1 v_1^{-1}) \sigma(q u_1^{-1} v_1)$ from prefactor, & then setting $u_1 v_1^{-1} = q^{\pm 1}$, this row becomes $(1, 0, \dots, 0)$.
- (iv) holds since if v_i & v_j (for $i \neq j$) are interchanged, then determinant simply changes sign, since columns i & j of the matrix are swapped, & prefactor also changes sign. $\square$

Evaluation of RHS of Izergin–Korepin Formula

with $u_1 = \dots = u_n = v_1 = \dots = v_n = 1$ & $q = e^{i\pi/3}$

- Recall that $Z(\underbrace{1, \dots, 1}_{2n})|_{q=e^{i\pi/3}} = (\# \text{ of } n \times n \text{ ASMs})$.
- Cannot immediately evaluate RHS of Izergin–Korepin formula at $u_1 = \dots = u_n = v_1 = \dots = v_n = 1$ since setting $u_i = u_j$ or $v_i = v_j$ for $i \neq j$ gives $0/0$.
- Instead will first set $q = e^{i\pi/3}$ & then use a result of Okada.
[This differs from the method used by Kuperberg.]
- $Z(u_1, \dots, u_n, v_1, \dots, v_n)|_{q=e^{i\pi/3}} =$

$$3^{-n(n-1)/2} \left(\prod_{i=1}^n u_i v_i \right)^{-n+1} s_{(n-1, n-1, \dots, 2, 2, 1, 1)}(u_1^2, \dots, u_n^2, v_1^2, \dots, v_n^2) \quad (\text{Okada 2006})$$

[where $s_\lambda(x_1, \dots, x_k) = \text{Schur function}$].
- A proof of this will be sketched soon.
- Now set $u_1 = \dots = u_n = v_1 = \dots = v_n = 1$.
- Then apply standard fact for Schur function with all variables 1:

$$s_\lambda(\underbrace{1, \dots, 1}_k) = \text{SSYT}_\lambda(k)$$

[where $\text{SSYT}_\lambda(k) = (\# \text{ of semistandard Young tableaux of shape } \lambda \text{ with entries } \leq k)$].

- Finally, apply standard product formula for # of semistandard Young tableaux:

$$\text{SSYT}_{\lambda}(k) = \frac{\prod_{1 \leq i < j \leq k} (\lambda_i - \lambda_j - i + j)}{\prod_{i=1}^{k-1} i!}.$$

- Gives (# of $n \times n$ ASMs) = $3^{-n(n-1)/2} \text{SSYT}_{(n-1, n-1, \dots, 2, 2, 1, 1)}(2n)$

$$= \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} \text{ as required.}$$

Sketch of Okada's proof that

$$Z(u_1, \dots, u_n, v_1, \dots, v_n)|_{q=e^{i\pi/3}} = 3^{-n(n-1)/2} \left(\prod_{i=1}^n u_i v_i \right)^{-n+1} s_{(n-1, n-1, \dots, 2, 2, 1, 1)}(u_1^2, \dots, u_n^2, v_1^2, \dots, v_n^2):$$

- Set $q = e^{i\pi/3}$ in Izergin–Korepin formula.
- Apply identity which relates certain $n \times n$ determinant to certain $2n \times 2n$ determinant:

$$\det_{1 \leq i, j \leq n} \left(\frac{a_i - b_j}{x_i - y_j} \right) = \frac{(-1)^{n(n+1)/2}}{\prod_{i,j=1}^n (x_i - y_j)} \det \begin{pmatrix} 1 & a_1 & x_1 & a_1 x_1 & \dots & x_1^{n-1} & a_1 x_1^{n-1} \\ 1 & b_1 & y_1 & b_1 y_1 & \dots & y_1^{n-1} & b_1 y_1^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_n & x_n & a_n x_n & \dots & x_n^{n-1} & a_n x_n^{n-1} \\ 1 & b_n & y_n & b_n y_n & \dots & y_n^{n-1} & b_n y_n^{n-1} \end{pmatrix} \quad (Okada 1998)$$

with appropriate a_i , b_i , x_i & y_i .

- Use standard determinantal expression for Schur function:

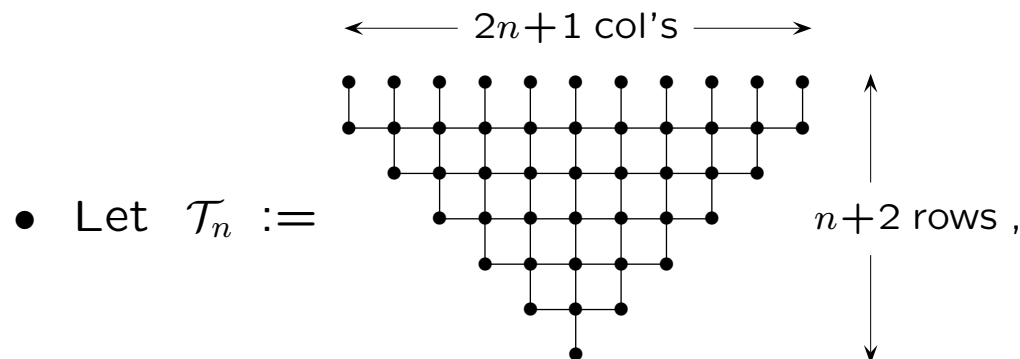
$$s_\lambda(x_1, \dots, x_n) = \frac{\det_{1 \leq i, j \leq n} (x_i^{\lambda_j + n - j})}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}.$$

II'. Outline of Proof of Odd-Order DASASM Formula

$$(\# \text{ of } (2n+1) \times (2n+1) \text{ DASASMs}) = \prod_{i=0}^n \frac{(3i)!}{(n+i)!}$$

- (1) Obtain bijection between $(2n+1) \times (2n+1)$ DASASMs & configurations of six-vertex model on a certain isosceles triangle.
- (2) Introduce parameter-dependent bulk weights, *boundary weights* & associated partition function.
- (3) Use Yang–Baxter equation, *reflection equation* & other properties to prove formula for partition function involving *sum of two* $(n+1) \times (n+1)$ determinants.
- (4) Evaluate determinantal formula at values of parameters for which all weights are 1 (homogeneous limit).

Configurations of Six-Vertex Model on Isosceles Triangle

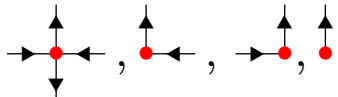
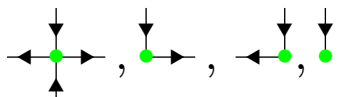
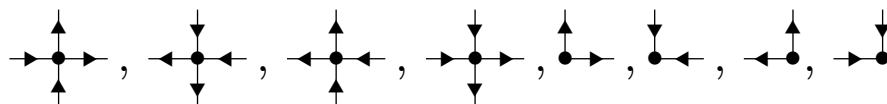


$$6\text{VDA}(n) := \left\{ \begin{array}{l} \text{orientations of} \\ \text{edges of } \mathcal{T}_n \end{array} \left| \begin{array}{l} \bullet \text{ 2 in \& 2 out arrows at each degree 4 vertex} \\ \quad (\Rightarrow 6 \text{ cases}) \\ \bullet \text{ no restriction at degree 2 vertices } (\Rightarrow 4 \text{ cases}) \\ \bullet \text{ all arrows upward on top boundary} \\ \bullet \text{ no restriction on single bottom edge } (\Rightarrow 2 \text{ cases}) \end{array} \right. \right\}.$$

• e.g. $6\text{VDA}(1) = \left\{ \begin{array}{c} \begin{array}{c} \uparrow \uparrow \uparrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \\ \uparrow \end{array}, \begin{array}{c} \uparrow \uparrow \uparrow \\ \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \\ \downarrow \end{array}, \begin{array}{c} \uparrow \uparrow \uparrow \\ \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \\ \uparrow \end{array} \end{array} \right\}.$

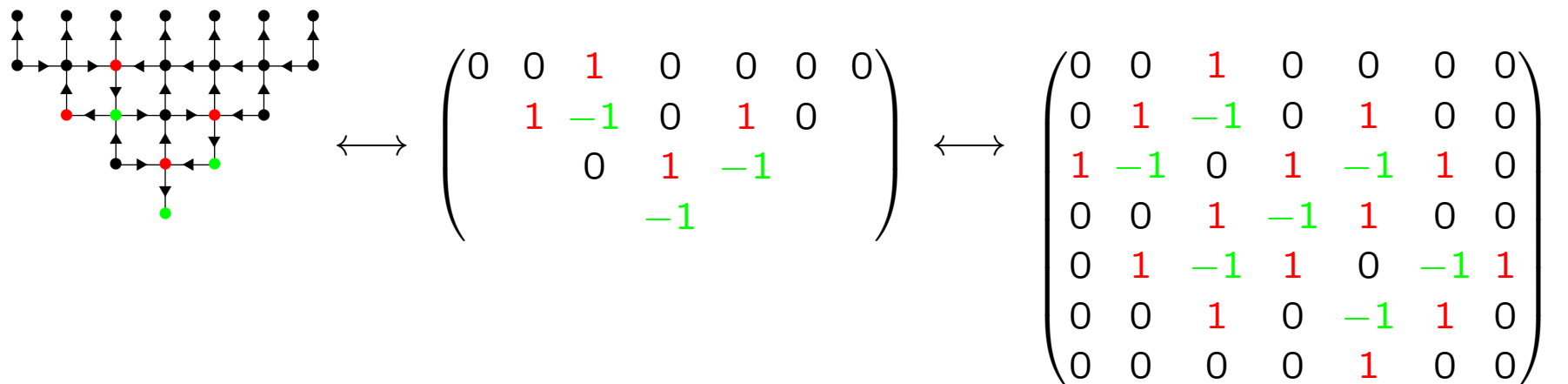
• e.g. $\in 6\text{VDA}(3).$

Six-Vertex Model Configuration – DASASM Bijection

<u>six-vertex model</u>		<u>DASASM</u>
	$\longleftrightarrow$	1
	$\longleftrightarrow$	-1
	$\longleftrightarrow$	0

- Also use reflections in diagonal & antidiagonal.
- Gives bijection between $6VDA(n)$ & $\{(2n+1) \times (2n+1) \text{ DASASM}\}$.

- e.g.



Vertex Weights

- $\sigma(x) := x - x^{-1}$.
- $u = \text{'spectral parameter'}$.
- $q = \text{global parameter}$.

- Bulk weights (with q from previous ASM case replaced by q^2):

$$W\left(\begin{array}{c} \uparrow \\ \rightarrow \bullet \rightarrow \\ \downarrow \end{array}, u\right) = W\left(\begin{array}{c} \downarrow \\ \leftarrow \bullet \leftarrow \\ \uparrow \end{array}, u\right) = \frac{\sigma(q^2 u)}{\sigma(q^4)},$$

$$W\left(\begin{array}{c} \uparrow \\ \leftarrow \bullet \leftarrow \\ \downarrow \end{array}, u\right) = W\left(\begin{array}{c} \downarrow \\ \rightarrow \bullet \rightarrow \\ \uparrow \end{array}, u\right) = \frac{\sigma(q^2 u^{-1})}{\sigma(q^4)},$$

$$W\left(\begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array}, u\right) = W\left(\begin{array}{c} \downarrow \\ \bullet \\ \uparrow \end{array}, u\right) = 1.$$

- Left boundary weights: $W\left(\begin{array}{c} \uparrow \\ \bullet \rightarrow \end{array}, u\right) = W\left(\begin{array}{c} \downarrow \\ \bullet \leftarrow \end{array}, u\right) = \frac{\sigma(q u)}{\sigma(q)},$

$$W\left(\begin{array}{c} \uparrow \\ \bullet \leftarrow \end{array}, u\right) = W\left(\begin{array}{c} \downarrow \\ \bullet \rightarrow \end{array}, u\right) = 1.$$

- Right boundary weights: $W\left(\begin{array}{c} \uparrow \\ \leftarrow \bullet \end{array}, u\right) = W\left(\begin{array}{c} \downarrow \\ \rightarrow \bullet \end{array}, u\right) = \frac{\sigma(q u^{-1})}{\sigma(q)},$

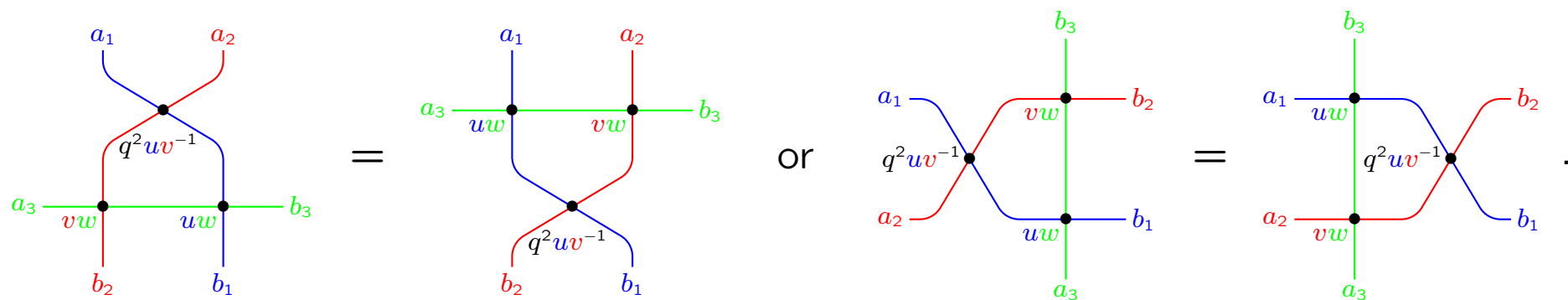
$$W\left(\begin{array}{c} \uparrow \\ \rightarrow \bullet \end{array}, u\right) = W\left(\begin{array}{c} \downarrow \\ \leftarrow \bullet \end{array}, u\right) = 1.$$

- At $u = 1$ & $q = e^{i\pi/6}$: $W(c, 1)|_{q=e^{i\pi/6}} = 1$ for all c .

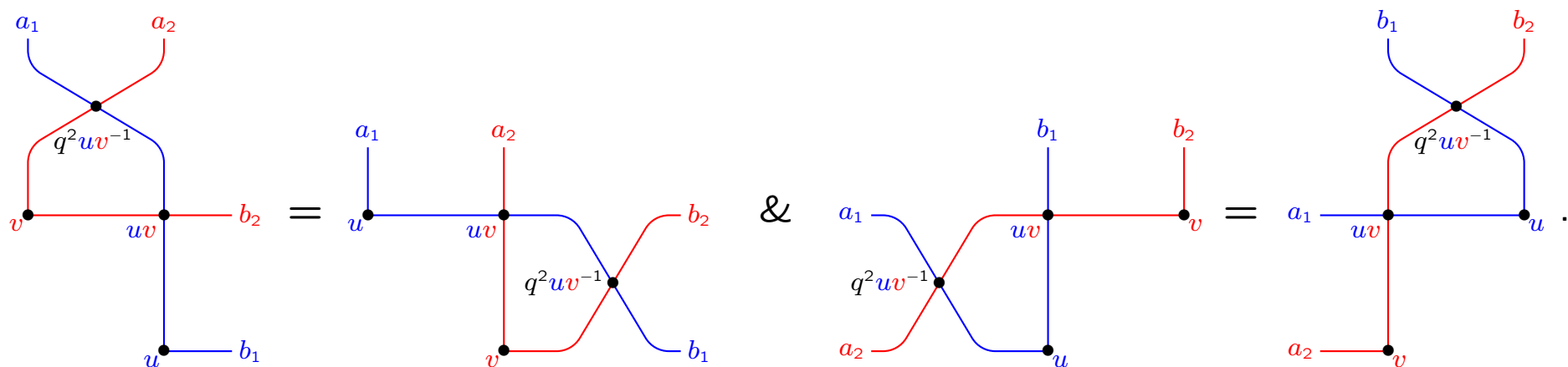
- At $u = q^{\pm 2}$: $W\left(\begin{array}{c} d \\ a \bullet c \\ b \end{array}, q^{-2}\right) = \delta_{ab'} \delta_{cd'}$ & $W\left(\begin{array}{c} d \\ a \bullet c \\ b \end{array}, q^2\right) = \delta_{ad'} \delta_{bc'},$

where a' denotes reversal of orientation a (i.e. in $\leftrightarrow$ out).

- Yang–Baxter equation:



- Left & right reflection equations (boundary Yang–Baxter equations):

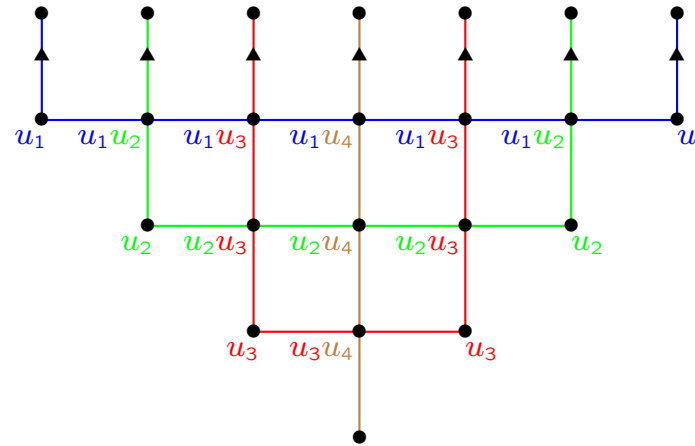


Odd-Order DASASM Partition Function

- $$Z(u_1, \dots, u_{n+1}) := \sum_{C \in 6\text{VDA}(n)} \prod_{i=1}^n W(C_{ii}, u_i) \left(\prod_{j=i+1}^{2n+1-i} W(C_{ij}, u_i u_{\min(j, 2n+2-j)}) \right) W(C_{i, 2n+1-i}, u_i)$$

[where C_{ij} = local configuration at vertex in row i & column j of $\mathcal{T}_n$].

- e.g. $Z(u_1, u_2, u_3, u_4) =$



- Therefore $Z(\underbrace{1, \dots, 1}_{n+1})|_{q=e^{i\pi/6}} = |6\text{VDA}(n)| = (\# \text{ of } (2n+1) \times (2n+1) \text{ DASASMs}).$

Sum of Determinants Formula for Partition Function

$$\begin{aligned}
 Z(u_1, \dots, u_{n+1}) = & \frac{\sigma(q^2)^n}{\sigma(q)^{2n} \sigma(q^4)^{n^2}} \prod_{i=1}^n \frac{\sigma(u_i) \sigma(qu_i) \sigma(qu_i^{-1}) \sigma(q^2 u_i u_{n+1}) \sigma(q^2 u_i^{-1} u_{n+1}^{-1})}{\sigma(u_i u_{n+1}^{-1})} \prod_{1 \leq i < j \leq n} \left(\frac{\sigma(q^2 u_i u_j) \sigma(q^2 u_i^{-1} u_j^{-1})}{\sigma(u_i u_j^{-1})} \right)^2 \\
 & \times \left(\det_{1 \leq i, j \leq n+1} \left(\begin{cases} \frac{q^2 + q^{-2} + u_i^2 + u_j^{-2}}{\sigma(q^2 u_i u_j) \sigma(q^2 u_i^{-1} u_j^{-1})}, & i \leq n \\ \frac{u_{n+1}^{-1} - 1}{u_j^2 - 1}, & i = n+1 \end{cases} \right) + \det_{1 \leq i, j \leq n+1} \left(\begin{cases} \frac{q^2 + q^{-2} + u_i^{-2} + u_j^2}{\sigma(q^2 u_i u_j) \sigma(q^2 u_i^{-1} u_j^{-1})}, & i \leq n \\ \frac{u_{n+1}^{-1} - 1}{u_j^{-2} - 1}, & i = n+1 \end{cases} \right) \right).
 \end{aligned}$$

Proof outline:

- Show that a function $X(u_1, \dots, u_{n+1})$ which satisfies the following properties is uniquely determined:
 - (i) $X(u_1) = 1$.
 - (ii) $X(u_1, \dots, u_{n+1})$ a Laurent polynomial in u_{n+1} of lower degree $\geq -n$, upper degree $\leq n$.
 - (iii) $X(u_1, \dots, u_{n+1})|_{u_1 u_{n+1} = q^2} = \frac{\sigma(qu_1) (\sigma(qu_1^{-1}) + \sigma(q)) \prod_{i=2}^n \sigma(q^2 u_1 u_i) \sigma(q^2 u_i u_{n+1})}{\sigma(q)^2 \sigma(q^4)^{2n-2}} X(u_2, \dots, u_n, u_1)$.
 - (iv) $X(u_1, \dots, u_{n+1})$ symmetric in $u_1, \dots, u_n$.
 - (v) $X(u_1^{-1}, \dots, u_{n+1}^{-1}) = X(u_1, \dots, u_{n+1})$.
 - (vi) $X(u_1, \dots, u_{n+1})$ even in u_i , for $i = 1, \dots, n$.
- Show that LHS & RHS of required formula both satisfy all these properties.

Sections of Proof

- If a function $X(u_1, \dots, u_{n+1})$ satisfies following properties, then it is uniquely determined:
 - (i) $X(u_1) = 1$.
 - (ii) $X(u_1, \dots, u_{n+1})$ a Laurent polynomial in u_{n+1} of lower degree $\geq -n$, upper degree $\leq n$.
 - (iii) $X(u_1, \dots, u_{n+1})|_{u_1 u_{n+1} = q^2} = \frac{\sigma(q u_1) (\sigma(q u_1^{-1}) + \sigma(q)) \prod_{i=2}^n \sigma(q^2 u_1 u_i) \sigma(q^2 u_i u_{n+1})}{\sigma(q)^2 \sigma(q^4)^{2n-2}} X(u_2, \dots, u_n, u_1)$.
 - (iv) $X(u_1, \dots, u_{n+1})$ symmetric in $u_1, \dots, u_n$.
 - (v) $X(u_1^{-1}, \dots, u_{n+1}^{-1}) = X(u_1, \dots, u_{n+1})$.
 - (vi) $X(u_1, \dots, u_{n+1})$ even in u_i , for $i = 1, \dots, n$.

Proof:

By (ii), $X(u_1, \dots, u_{n+1})$ is uniquely determined if known at $2n+1$ values of u_{n+1} .

Combining (iii) with (iv)–(vi) gives expressions at $4n$ ($\geq 2n+1$) values of u_{n+1} , i.e.

$X(u_1, \dots, u_n, q^{\pm 2} u_i^{-1})$ expressed in terms of $X(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n, u_i)$ &

$X(u_1, \dots, u_n, -q^{\pm 2} u_i^{-1})$ expressed in terms of $X(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n, -u_i)$, $i = 1, \dots, n$.

Result follows using recursion on n , together with (i). □

• The partition function $Z(u_1, \dots, u_{n+1})$ satisfies:

(i) $Z(u_1) = 1$.

(ii) $Z(u_1, \dots, u_{n+1})$ a Laurent polynomial in u_{n+1} of lower degree $\geq -n$, upper degree $\leq n$.

(v) $Z(u_1^{-1}, \dots, u_{n+1}^{-1}) = Z(u_1, \dots, u_{n+1})$.

(vi) $Z(u_1, \dots, u_{n+1})$ even in u_i , for $i = 1, \dots, n$.

Proof:

(i), (ii) & (vi) follow straightforwardly from definition of weights & partition function.

(v) follows from reflection of configurations in central vertical line of graph $\mathcal{T}_n$. □

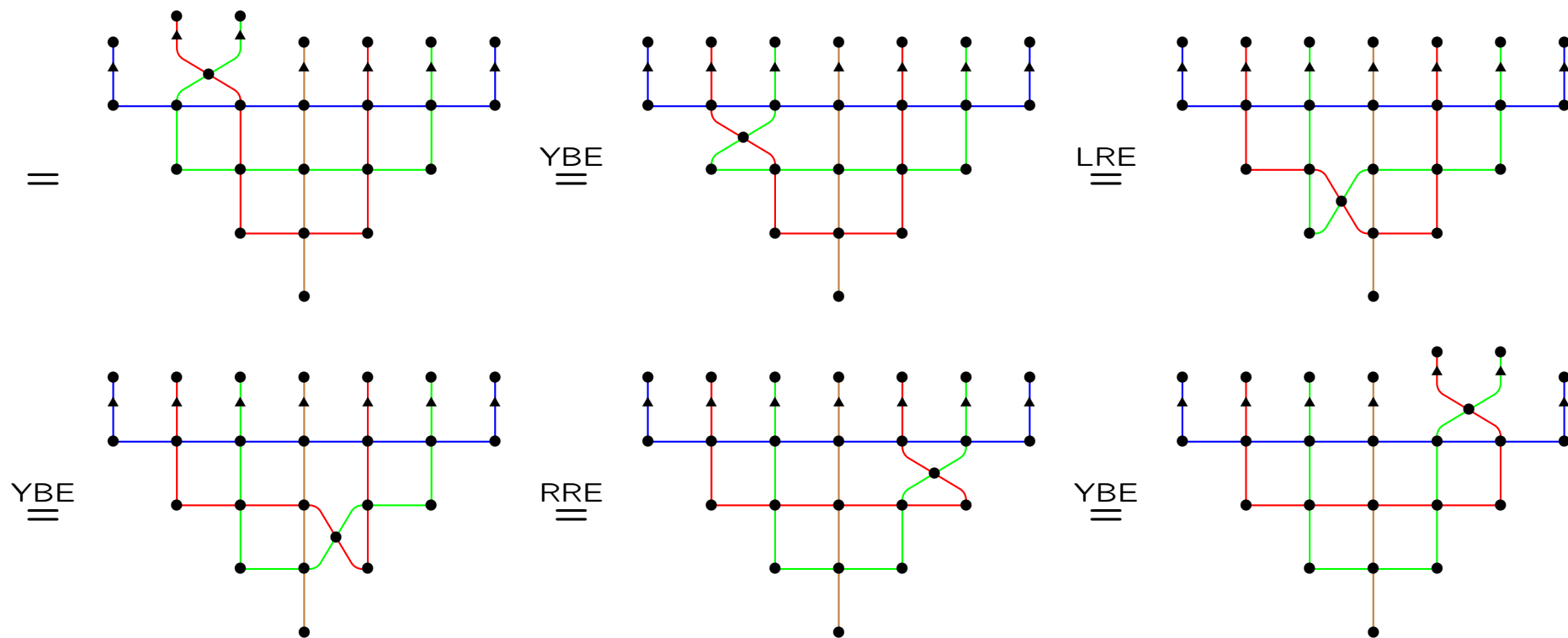
- (iv) $Z(u_1, \dots, u_{n+1})$ is symmetric in $u_1, \dots, u_n$.

Proof:

Use Yang–Baxter & reflection equations (YBE, LRE, RRE) to show that $Z(u_1, \dots, u_{n+1})$ is symmetric in u_i & u_{i+1} , $i = 1, \dots, n-1$.

e.g. $n = 3$ & $i = 2$:

$$W(\begin{array}{c} \uparrow \\ \leftarrow \rightarrow \\ \downarrow \end{array}, q^2 u_2^{-1} u_3) Z(u_1, u_2, u_3, u_4)$$



$$= W(\begin{array}{c} \uparrow \\ \leftarrow \rightarrow \\ \downarrow \end{array}, q^2 u_2^{-1} u_3) Z(u_1, u_3, u_2, u_4).$$

□

- (iii) $Z(u_1, \dots, u_{n+1})|_{u_1 u_{n+1} = q^2} = \frac{\sigma(q u_1) (\sigma(q u_1^{-1}) + \sigma(q)) \prod_{i=2}^n \sigma(q^2 u_1 u_i) \sigma(q^2 u_i u_{n+1})}{\sigma(q)^2 \sigma(q^4)^{2n-2}} Z(u_2, \dots, u_n, u_1).$

Proof:

e.g. $n = 3$:

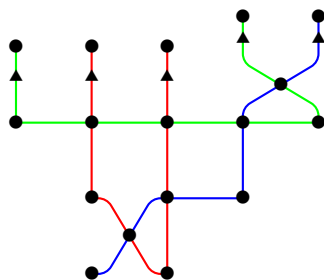
$$Z(u_1, u_2, u_3, u_4)|_{u_1 u_4 = q^2} =$$

$$= \frac{\sigma(q u_1) \sigma(q^2 u_1 u_2) \sigma(q^2 u_1 u_3)}{\sigma(q) \sigma(q^4)^2}$$

$$= \frac{\sigma(q u_1) \sigma(q^2 u_1 u_2) \sigma(q^2 u_1 u_3)}{\sigma(q) \sigma(q^4)^2}$$

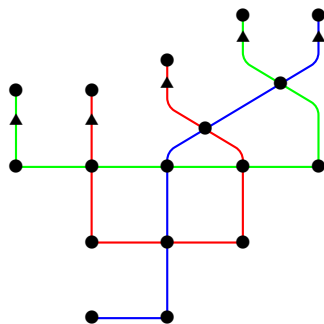
$$\text{YBE, RRE}$$

$$\frac{\sigma(qu_1)\sigma(q^2u_1u_2)\sigma(q^2u_1u_3)}{\sigma(q)\sigma(q^4)^2}$$



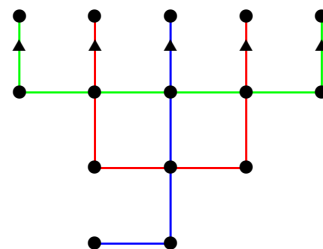
$$\text{RRE, YBE}$$

$$\frac{\sigma(qu_1)\sigma(q^2u_1u_2)\sigma(q^2u_1u_3)}{\sigma(q)\sigma(q^4)^2}$$



==

$$\frac{\sigma(q\mathbf{u}_1) \sigma(q^2\mathbf{u}_1\mathbf{u}_2) \sigma(q^2\mathbf{u}_2\mathbf{u}_4) \sigma(q^2\mathbf{u}_1\mathbf{u}_3) \sigma(q^2\mathbf{u}_3\mathbf{u}_4)}{\sigma(q) \sigma(q^4)^4}$$



$$=$$

$$\frac{\sigma(qu_1) \sigma(q^2u_1u_2) \sigma(q^2u_2u_4) \sigma(q^2u_1u_3) \sigma(q^2u_3u_4)}{\sigma(q) \sigma(q^4)^4} \left(\frac{\sigma(qu_1^{-1})}{\sigma(q)} + 1 \right) (Z_{-}(u_2, u_3, u_1) + Z_{+}(u_2, u_3, u_1))$$

==

$$\frac{\sigma(qu_1)(\sigma(qu_1^{-1})+\sigma(q))\sigma(q^2u_1u_2)\sigma(q^2u_2u_4)\sigma(q^2u_1u_3)\sigma(q^2u_3u_4)}{\sigma(q)^2\sigma(q^4)^4}Z(u_2, u_3, u_1).$$

☐

- $$Y(u_1, \dots, u_{n+1}) := \frac{\sigma(q^2)^n}{\sigma(q)^{2n} \sigma(q^4)^{n^2}} \prod_{i=1}^n \frac{\sigma(u_i) \sigma(qu_i) \sigma(qu_i^{-1}) \sigma(q^2 u_i u_{n+1}) \sigma(q^2 u_i^{-1} u_{n+1}^{-1})}{\sigma(u_i u_{n+1}^{-1})} \prod_{1 \leq i < j \leq n} \left(\frac{\sigma(q^2 u_i u_j) \sigma(q^2 u_i^{-1} u_j^{-1})}{\sigma(u_i u_j^{-1})} \right)^2$$

$$\times \left(\det_{1 \leq i, j \leq n+1} \left(\begin{cases} \frac{q^2 + q^{-2} + u_i^2 + u_j^{-2}}{\sigma(q^2 u_i u_j) \sigma(q^2 u_i^{-1} u_j^{-1})}, & i \leq n \\ \frac{u_{n+1} - 1}{u_j^2 - 1}, & i = n+1 \end{cases} \right) + \det_{1 \leq i, j \leq n+1} \left(\begin{cases} \frac{q^2 + q^{-2} + u_i^{-2} + u_j^2}{\sigma(q^2 u_i u_j) \sigma(q^2 u_i^{-1} u_j^{-1})}, & i \leq n \\ \frac{u_{n+1}^{-1} - 1}{u_j^{-2} - 1}, & i = n+1 \end{cases} \right) \right)$$

satisfies:

- (i) $Y(u_1) = 1$.
- (ii) $Y(u_1, \dots, u_{n+1})$ a Laurent polynomial in u_{n+1} of lower degree $\geq -n$, upper degree $\leq n$.
- (iii) $Y(u_1, \dots, u_{n+1})|_{u_1 u_{n+1} = q^2} = \frac{\sigma(qu_1) (\sigma(qu_1^{-1}) + \sigma(q)) \prod_{i=2}^n \sigma(q^2 u_1 u_i) \sigma(q^2 u_i u_{n+1})}{\sigma(q)^2 \sigma(q^4)^{2n-2}} Y(u_2, \dots, u_n, u_1)$.
- (iv) $Y(u_1, \dots, u_{n+1})$ symmetric in $u_1, \dots, u_n$.
- (v) $Y(u_1^{-1}, \dots, u_{n+1}^{-1}) = Y(u_1, \dots, u_{n+1})$.
- (vi) $Y(u_1, \dots, u_{n+1})$ even in u_i , for $i = 1, \dots, n$.

Proof:

Reasonably straightforward, using standard properties of determinants. □

Partition Function at $q = e^{i\pi/6}$

$$Z(u_1, \dots, u_{n+1})|_{q=e^{i\pi/6}} = 3^{-n(n-1)/2} \left(\frac{u_{n+1}^n}{u_{n+1}+1} s_{(n,n-1,n-1,\dots,2,2,1,1)}(u_1^2, u_1^{-2}, \dots, u_n^2, u_n^{-2}, u_{n+1}^{-2}) \right. \\ \left. + \frac{u_{n+1}^{-n}}{u_{n+1}^{-1}+1} s_{(n,n-1,n-1,\dots,2,2,1,1)}(u_1^2, u_1^{-2}, \dots, u_n^2, u_n^{-2}, u_{n+1}^2) \right)$$

[where $s_\lambda(x_1, \dots, x_k) =$ Schur function].

Proof outline:

- Substitute $q = e^{i\pi/6}$ into determinantal expression for partition function.
- Apply general determinant identity

$$\det_{1 \leq i, j \leq k} \left(\frac{a_i - b_j}{x_i - y_j} \right) = \frac{(-1)^{k(k+1)/2}}{\prod_{i,j=1}^k (x_i - y_j)} \det \begin{pmatrix} 1 & a_1 & x_1 & a_1 x_1 & \dots & x_1^{k-1} & a_1 x_1^{k-1} \\ 1 & b_1 & y_1 & b_1 y_1 & \dots & y_1^{k-1} & b_1 y_1^{k-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_k & x_k & a_k x_k & \dots & x_k^{k-1} & a_k x_k^{k-1} \\ 1 & b_k & y_k & b_k y_k & \dots & y_k^{k-1} & b_k y_k^{k-1} \end{pmatrix} \quad (Okada 1998)$$

$$\text{with } a_i = \begin{cases} u_i^{\pm 2} + u_i^{\pm 4}, & i \leq n \\ -1, & i = n+1, \end{cases} \quad x_i = \begin{cases} u_i^{\pm 6}, & i \leq n \\ 1, & i = n+1, \end{cases} \quad b_j = u_j^{\mp 2} + u_j^{\mp 4}, \quad y_j = u_j^{\mp 6}.$$

- Use standard determinantal expression for Schur function:

$$s_\lambda(x_1, \dots, x_k) = \det_{1 \leq i, j \leq k} (x_i^{\lambda_j + k - j}) / \prod_{1 \leq i < j \leq k} (x_i - x_j).$$

of Odd-Order DASASMs

$$\begin{aligned}
 (\# \text{ of } (2n+1) \times (2n+1) \text{ DASASMs}) &= 3^{-n(n-1)/2} \text{SSYT}_{(n, n-1, n-1, \dots, 2, 2, 1, 1)}(2n+1) \\
 &= \prod_{i=0}^n \frac{(3i)!}{(n+i)!}
 \end{aligned}$$

[where $\text{SSYT}_{\lambda}(k) = (\# \text{ of semistandard Young tableaux of shape } \lambda \text{ with entries } \leq k)$].

Proof:

- Recall $(\# \text{ of } (2n+1) \times (2n+1) \text{ DASASMs}) = Z(\underbrace{1, \dots, 1}_{n+1})|_{q=e^{i\pi/6}}$.
- Set $u_1 = \dots = u_{n+1} = 1$ in Schur function expression for $Z(u_1, \dots, u_{n+1})|_{q=e^{i\pi/6}}$.
- Apply standard fact for Schur function with all variables 1:

$$s_{\lambda}(\underbrace{1, \dots, 1}_k) = \text{SSYT}_{\lambda}(k).$$

- Apply standard product formula for # of semistandard Young tableaux:

$$\text{SSYT}_{\lambda}(k) = \frac{\prod_{1 \leq i < j \leq k} (\lambda_i - \lambda_j - i + j)}{\prod_{i=1}^{k-1} i!}.$$

of Odd-Order DASASMs with Fixed Central Entry

$$\frac{\# \text{ of } (2n+1) \times (2n+1) \text{ DASASMs with central entry } -1}{\# \text{ of } (2n+1) \times (2n+1) \text{ DASASMs with central entry } 1} = \frac{n}{n+1}.$$

Proof outline:

- Introduce partition functions $Z_{\pm}(u_1, \dots, u_{n+1})$ for $(2n+1) \times (2n+1)$ DASASMs with central entry ± 1 .
- Show that $Z_{\pm}(u_1, \dots, u_{n+1}) = \frac{1}{2} (Z(u_1, \dots, u_n, u_{n+1}) \pm (-1)^n Z(u_1, \dots, u_n, -u_{n+1}))$.
- Use previous results for $Z(u_1, \dots, u_{n+1})$.

Special Features of Odd-Order DASASM Proof

- Six-vertex model considered on *triangle* instead of square.
- Only a *single* set $u_1, \dots, u_{n+1}$ of spectral parameters used.
- Last spectral parameter u_{n+1} plays *special* role.
- Yang–Baxter *and* reflection equation needed (with certain boundary weights).
- Partition function formula involves *sum* of *two* determinantal terms.

III'. Outline of Proof of DSASM Formula

(# of $n \times n$ DSASMs)

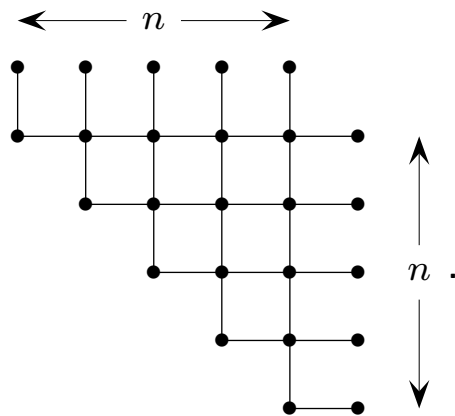
$$= \text{Pfaffian}_{0 \text{ or } 1 \leq i, j \leq n-1} \left(\sum_{k=0}^{\min(i,j)} (3 - \delta_{k,0}) \left(\binom{i+j-2k-1}{i-k} - \binom{i+j-2k-1}{j-k} \right) \right)$$

[range for i, j starts at 0 for n even, 1 for n odd]

- (1) Obtain bijection between $n \times n$ DSASMs & configurations of six-vertex model on a certain triangle.
 - (2) Introduce parameter-dependent bulk weights, *left boundary weights* & associated partition function.
 - (3) Use Yang–Baxter equation, *left reflection equation* & other properties to prove formula for partition function involving Pfaffian.
 - (4) Evaluate Pfaffian formula at values of parameters for which all weights are 1 (homogeneous limit).
- Steps (1)–(3) are, to some extent, similar to steps (1)–(3) for odd-order DASASMs.
 - In steps (2) & (3), the partition function was independently defined & the Pfaffian formula was independently obtained by *Garbali, de Gier, Mead, Wheeler 2023*.

A Few Details of DSASM Proof

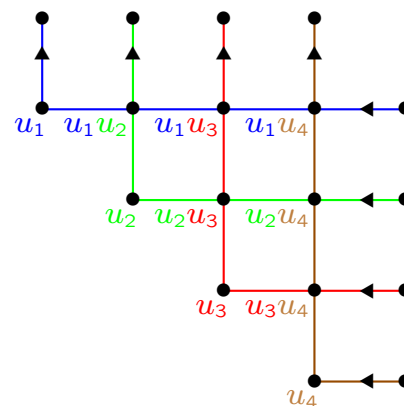
- Let $\mathcal{E}_n :=$



- $6\text{VD}(n) = \left\{ \begin{array}{l} \text{orientations of} \\ \text{edges of } \mathcal{E}_n \end{array} \left| \begin{array}{l} \bullet \text{ 2 in \& 2 out arrows at each degree 4 vertex} \\ \quad (\Rightarrow 6 \text{ cases}) \\ \bullet \text{ no restriction at degree 2 vertices } (\Rightarrow 4 \text{ cases}) \\ \bullet \text{ all arrows upward on top boundary} \\ \bullet \text{ all arrows leftward on right boundary} \end{array} \right. \right\}.$

- e.g. $6\text{VD}(3) = \left\{ \begin{array}{c} \text{Diagram 1}, \text{Diagram 2}, \text{Diagram 3}, \text{Diagram 4}, \text{Diagram 5} \end{array} \right\}.$

- Partition function: e.g. $Z(u_1, u_2, u_3, u_4) =$



- Symmetry of partition function: e.g.

$$W(\begin{array}{c} \uparrow \\ \downarrow \end{array}, q^2 u_2^{-1} u_3) Z(u_1, u_2, u_3, u_4) =$$

$$\stackrel{\text{YBE}}{=} \begin{array}{c} \uparrow \\ \downarrow \end{array}$$

$$\stackrel{\text{LRE}}{=} \begin{array}{c} \uparrow \\ \downarrow \end{array} = W(\begin{array}{c} \uparrow \\ \downarrow \end{array}, q^2 u_2^{-1} u_3) Z(u_1, u_3, u_2, u_4).$$

- Pfaffian formula for partition function:

$$Z(u_1, \dots, u_n) = \prod_{1 \leq i < j \leq n} \frac{\sigma(q^2 u_i u_j) \sigma(q^2 u_i^{-1} u_j^{-1})}{\sigma(q^4) \sigma(u_i u_j^{-1})} \text{Pfaffian}_{0 \text{ or } 1 \leq i < j \leq n} \left(\begin{array}{ll} Z(u_j), & i = 0 \\ \frac{\sigma(q^4) \sigma(u_i u_j^{-1}) Z(u_i, u_j)}{\sigma(q^2 u_i u_j) \sigma(q^2 u_i^{-1} u_j^{-1})}, & i \geq 1 \end{array} \right)$$

(RB, Fischer, Koutschan 2023; Garbali, de Gier, Mead, Wheeler 2023).

- Range for i, j starts at 0 for n odd, 1 for n even.
- Only strictly upper triangular part of skew-symmetric matrix is shown for Pfaffian.
- $Z(u_j)$ & $Z(u_i, u_j)$ in Pfaffian can easily be written explicitly using sums over 6VD(1) & 6VD(2).
- Formula remains valid for general six-vertex model boundary weights.
- In step (4) of proof, evaluation of RHS of Pfaffian formula at $u_1 = \dots = u_n = 1$ & $q = e^{i\pi/6}$ involves application of general result for homogeneous limit of certain multivariate Pfaffian expressions.
- In this case, $Z(u_1, \dots, u_n)|_{q=e^{i\pi/6}}$ is not a Schur function.

Final Messages

- ASMs are interesting **combinatorial** objects.
- Many results involving ASMs have been proved using certain connections with **integrability**.
- Most proofs are indirect & technical.
- Various aspects of ASMs are still not properly understood.

“These conjectures are of such compelling simplicity that it is hard to know how any mathematician can bear the pain of living without understanding why they are true . . . I expect that these problems will remain with us for some time.”

(Robbins 1991, “The story of 1, 2, 7, 42, 429, 7436, . . .”)