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Large random tilings of a hexagon with periodic weightings: steepest descent analysis on the double amoeba

Based on joint work Arno Kuijlaars[†]

Max van Horssen[†]
[†] *KU Leuven*

Integrable Combinatorics: from exact enumeration to limit shapes
Louvain-la-Neuve, November 19, 2025

A 3×3-periodic hexagon tiling

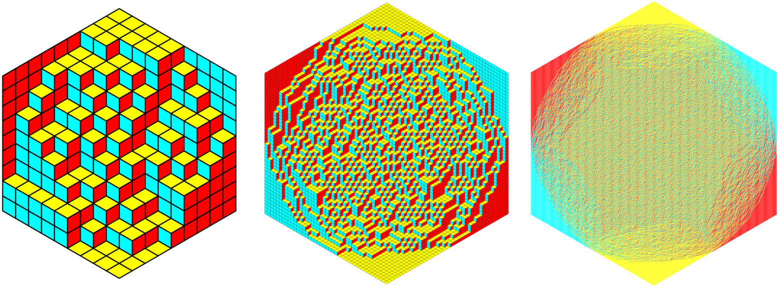


Figure: Created using a Python adaptation of Christophe Charlier's tiling program, developed by Lennart Hübner and MvH (publicly available on GitHub). Moreover, all figures in this talk are created using this program.

A 3×3-periodic hexagon tiling: three regions and arctic curves

- **Three regions** (or phases) appear: **solid**, **liquid**, and **gas**.
- The boundaries between these regions are called the **arctic curves**.
- The outer arctic curve resembles a **circle** and the inner has **six cusp points**.

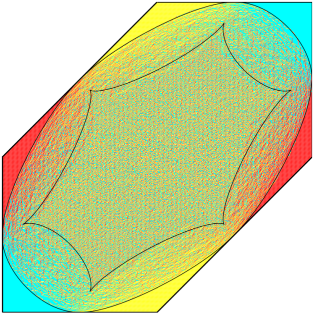
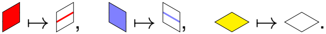


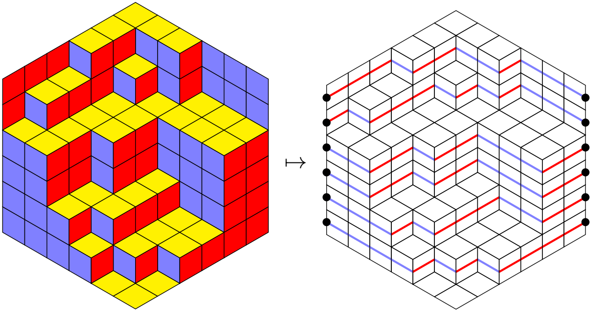
Figure: Large (skewed) hexagon tiling with the arctic curves.

Random point process I

- Given a **hexagon tiling** $\mathcal{T}$, we replace the red and blue tiles by lines:



- Result: a **non-intersecting path system** $\mathcal{P}$ connecting the left to the right points •.



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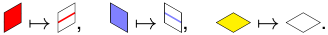
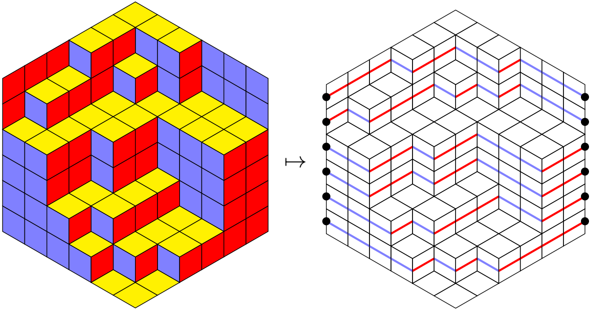


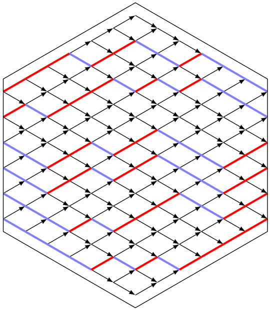
- Result: a **non-intersecting path system** $\mathcal{P}$ connecting the left to the right points $\bullet$.
- $\mathcal{T} \mapsto \mathcal{P}$ is a bijective correspondence.



Random point process II

- All path systems $\mathcal{P}$ are part of a directed graph (V, E).
- Introduce a **weight function** $\mathbf{w} : E \to \mathbb{R}^+$, and the **weight** of $\mathcal{P}$

$$\mathbf{w}(\mathcal{P}) = \prod_{e \in \mathcal{P}} \mathbf{w}(e).$$



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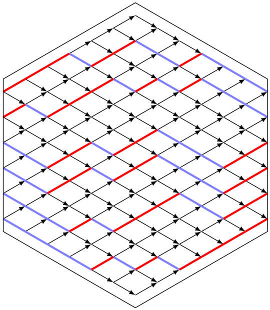

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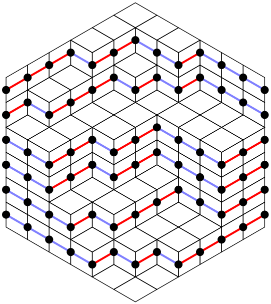
- Placing points • on ▱ and ▱ gives a **particle system**.
- This defines a **random point process** $\mathbf{Pr}$ on the hexagon.



Random point process III

- The weight function $\mathbf{w}$ is determined by $a_{j,k}$ and $b_{j,k}$.
- We assume $\mathbf{3} \times \mathbf{3}$**-periodicity** of $\mathbf{w}$, i.e.,

$$a_{j+3n,k+3m} = a_{j,k},$$
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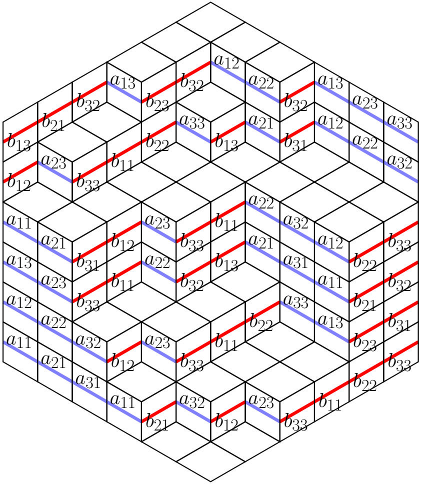

Figure: Taken from [Kuijlaars, '25].

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- **Simplifying assumptions** are needed to use [Kuijlaars, '25].
  $\Longrightarrow$ $a_{j,k}$ and $b_{j,k}$ are reduced to two parameters: α_1 and α_2.

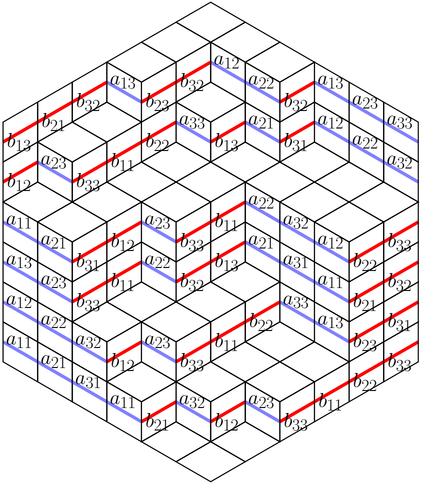
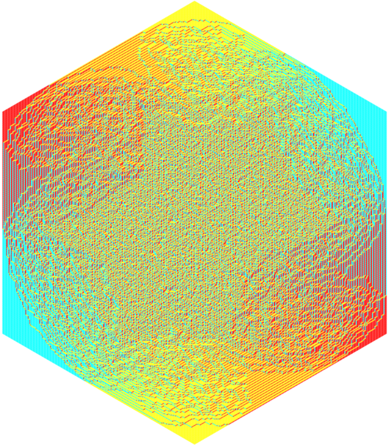


Figure: Taken from [Kuijlaars, '25].

Simplifying assumptions

- The simplifying assumptions exclude, among other things, the appearance of quasi-solid regions:



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Determinantal point process

- Define the **correlation function** ρ_n for distinct $v_1, \ldots, v_n \in V$:

$$\rho_n(v_1, \ldots, v_n) = \mathbf{Pr}(\{v_1, \ldots, v_n\} \subset \mathcal{P})$$

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$\big(=$ probability of $\mathcal{T}$ having  or  at $v_1, \ldots, v_n \big).$

- The random point process $\mathbf{Pr}$ is **determinantal**: there exists a function $K : V \times V \to \mathbb{R}$ such that

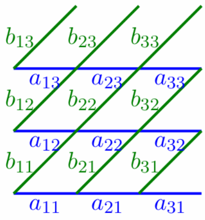
$$\rho_n(v_1, \ldots, v_n) = \det \big(K(v_i, v_j) \big)_{i,j=1}^{n}.$$

  The function K is called the **correlation kernel**. This follows from [Eynard–Mehta, '98], in which also an **explicit** K is given.

Transition matrices

- We encode the weight function $\mathbf{w}$ by the **transition matrices**

$$\widehat{T}_i(j,k) = \begin{cases} a_{i+1,j+1} & \text{if } k = j, \\ b_{i+1,j+1} & \text{if } k = j+1, \\ 0 & \text{otherwise,} \end{cases}$$



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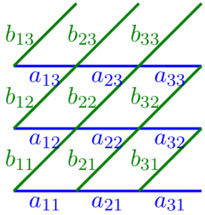

- which can be recovered from their symbols

$$T_0(z) = \begin{pmatrix} \alpha_1 & 1 & 0 \\ 0 & \alpha_2^{-1} & 1 \\ z & 0 & \alpha_1^{-1}\alpha_2 \end{pmatrix}, \quad T_1(z) = \begin{pmatrix} \alpha_1^{-1} & 1 & 0 \\ 0 & \alpha_2 & 1 \\ z & 0 & \alpha_1\alpha_2^{-1} \end{pmatrix}, \quad T_2(z) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ z & 0 & 1 \end{pmatrix}.$$

- Periodicity $\implies T_{3i+j} = T_j$.
- Notation: $T_{i,j} = T_i \cdots T_j$ and $W = T_0 T_1 T_2$.

Integral representation of the correlation kernel

- **Double contour integral** representation of K from [Duits–Kuijlaars, '21]:

$$
\begin{aligned}
&[K((3x_1+j_1,3y_1+k_1),(3x_2+j_2,3y_2+k_2))]_{k_1,k_2=0}^{2} = \\
&\qquad -\frac{\chi_{3x_1+j_1>3x_2+j_2}}{2\pi i}\oint_{\mathbb{T}} T_{0,j_2}^{-1}(z)\frac{W(z)^{x_1-x_2}}{z^{y_1-y_2}}T_{0,j_1}(z)\frac{dz}{z} \\
&\qquad +\frac{1}{(2\pi i)^2}\oint_{\mathbb{T}}\oint_{\mathbb{T}} T_{0,j_2}^{-1}(z_1)\frac{W(z_1)^{2N-x_2}}{z_1^{2N-y_2}}R_N(z_1,z_2)\frac{W(z_2)^{x_1}}{z_2^{y_1}}T_{0,j_1}(z_2)\frac{dz_1dz_2}{z_2}.
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- $\mathbb{T}$ = unit circle.
- $W(z)^{2N}/z^{2N}$ = a non-Hermitian 3×3-matrix-weight.
- $R_N(z_1,z_2)$ = **reproducing kernel** of the MVOP $P_N(z)$.
- **Existence** of $P_N(z)$ comes from the random tiling.

Riemann surface

- The **spectral curve** is defined by

$$P(z, \lambda) := \det(\lambda I_3 - W(z)) = 0.$$

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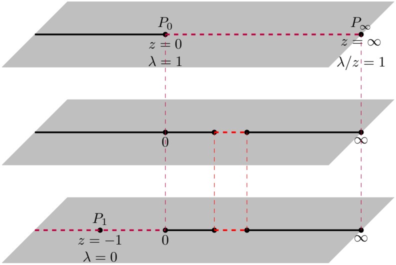
- Explicitly,

$$P(z, \lambda) = (\lambda - z - 1)^3 - 27(1 + \beta)\lambda z = 0,$$

with

$$\beta = \frac{(1 + \alpha_1 + \alpha_2)^3}{27\alpha_1\alpha_2} - 1.$$

- We represent the associated **compact Riemann surface** $\mathcal{R}$ as follows:



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Main theorem

- Take (ξ_1, ξ_2) in the **liquid region** and suppose that $\xi_{1,N}$ and $\xi_{2,N}$ vary with N such that $\xi_{1,N} \to \xi_1$ and $\xi_{2,N} \to \xi_2$ (and $N\xi_{1,N}, N\xi_{2,N} \in \mathbb{Z}$). Define the scaling coordinates:

$$v_{1,N} = (3N(1+\xi_{1,N}), 3N(1+\xi_{2,N})) + v_1$$
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Theorem

The large-N limit of the correlation kernel is given by

$$\lim_{N\to\infty} K(v_{1,N}, v_{2,N}) = K_{(\xi_1,\xi_2)}(v_1, v_2).$$

The 3×3-periodic limit kernel $K_{(\xi_1,\xi_2)} : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$ equals

$$\left[K_{(\xi_1,\xi_2)}(v_1,v_2)\right]_{k_1,k_2=0}^2 = -\frac{\chi_{3x_1+j_1>3x_2+j_2}}{2\pi i}\oint_{\mathbb{T}} \frac{T_{0,j_2}^{-1}(z)W(z)^{x_1-x_2}T_{0,j_1}(z)}{z^{y_1-y_2}}\frac{dz}{z}$$
$$+\frac{1}{2\pi i}\int_{\gamma_{(\xi_1,\xi_2)}} \frac{\lambda(p)^{x_1-x_2}}{z(p)^{y_1-y_2}} T_{0,j_2}^{-1}(z(p))\Xi(p)T_{0,j_1}(z(p)),$$

with $v_1 = (3x_1 + j_1, 3y_1 + k_1)$ and $v_2 = (3x_2 + j_2, 3y_2 + k_2)$. Here $x_1, y_1, x_2, y_2 \in \mathbb{Z}$ and $j_1, k_1, j_2, k_2 \in \{0, 1, 2\}$.

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- $\Xi = (\eta_{j,k})_{j,k=0}^2$ is a matrix with meromorphic differentials on $\mathcal{R}$.
- $\gamma(\xi_1, \xi_2)$ is a contour on the double cover of $\mathcal{R}$ depending on (ξ_1, ξ_2).

Strategy

- For simplicity, we take $x_1 = x_2 = x$, $y_1 = y_2 = y$ and $j_1 = j_2 = 0$, i.e.,

$$[K(v_1, v_2)]^2_{k_1,k_2=0} = \frac{1}{(2\pi i)^2} \oint_{\mathbb{T}} \oint_{\mathbb{T}} \frac{W(z_1)^{2N-x}}{z_1^{2N-y}} R_N(z_1, z_2) \frac{W(z_2)^x}{z_2^y} \frac{dz_1 dz_2}{z_2}.$$

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- Perform steepest descent analysis (or saddle point analysis). A special role will be played by a saddle point s of $\Phi(p)$.
- Study the level set $\operatorname{Re}\Phi(p) = \operatorname{Re}\Phi(s)$, and use these to prove the existence of steepest descent/ascent curves.

First transformation $Y \mapsto X$

From the previous talk, we recall the following:

- Riemann–Hilbert problem:

$$Y_+(z) = Y_-(z) \begin{pmatrix} I_3 & W(z)^{2N}/z^{2N} \\ 0_3 & I_3 \end{pmatrix}, \quad z \in \mathbb{T}.$$

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- Christoffel–Darboux formula:

$$R_N(z_1, z_2) = \frac{1}{z_2 - z_1} \begin{pmatrix} 0_3 & I_3 \end{pmatrix} Y^{-1}(z_1) Y(z_2) \begin{pmatrix} I_3 \\ 0_3 \end{pmatrix}, \quad z_1 \neq z_2.$$

Note that $R_N(z_1, z_2)$ is a polynomial of degree $N-1$ in both variables and that the right-hand side still makes sense even when $z_1 = z_2$ due to a zero-pole cancellation.

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- First transformation:

$$X(z) = Y(z) \begin{pmatrix} E(z) & 0_3 \\ 0_3 & E(z) \end{pmatrix} \qquad \text{with } W(z) = E(z)\Lambda(z)E^{-1}(z).$$

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  Note that $R_N(z_1, z_2)$ is a polynomial of degree $N-1$ in both variables and that the right-hand side still makes sense even when $z_1 = z_2$ due to a zero-pole cancellation.

- First transformation:

$$X(z) = Y(z) \begin{pmatrix} E(z) & 0_3 \\ 0_3 & E(z) \end{pmatrix} \qquad \text{with } W(z) = E(z)\Lambda(z)E^{-1}(z).$$

- After substitution:

$$R_N(z_1, z_2) = \frac{1}{z_2 - z_1} E(z_1) \begin{pmatrix} 0_3 & I_3 \end{pmatrix} X^{-1}(z_1) X(z_2) \begin{pmatrix} I_3 \\ 0_3 \end{pmatrix} E^{-1}(z_2).$$

First transformation $Y \mapsto X$

- After contour deformation, we have

$$[K(v_1, v_2)]_{k_1,k_2=0}^{2} = \frac{1}{(2\pi i)^2} \oint_{\mathbb{T}_{1-\delta}} \oint_{\mathbb{T}} \frac{W(z_1)^{2N-x}}{z_1^{2N-y}} R_N(z_1, z_2) \frac{W(z_2)^x}{z_2^y} \frac{dz_1 dz_2}{z_2}.$$

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- Thus

$$[K(v_1,v_2)]^2_{k_1,k_2=0} = \frac{1}{(2\pi i)^2} \oint_{\mathbb{T}_{1-\delta}} E(z_1) \frac{\Lambda(z_1)^{2N-x}}{z_1^{2N-y}} \begin{pmatrix} 0_3 & I_3 \end{pmatrix} X^{-1}(z_1)$$
$$\times \oint_{\mathbb{T}} X(z_2) \begin{pmatrix} I_3 \\ 0_3 \end{pmatrix} \frac{\Lambda(z_2)^x}{z_2^y} \frac{E^{-1}(z_2)}{z_2 - z_1} \frac{dz_1 dz_2}{z_2},$$

where we have used that

$$R_N(z_1,z_2) = \frac{1}{z_2 - z_1} E(z_1) \begin{pmatrix} 0_3 & I_3 \end{pmatrix} X^{-1}(z_1) X(z_2) \begin{pmatrix} I_3 \\ 0_3 \end{pmatrix} E^{-1}(z_2).$$

First transformation $Y \mapsto X$: a balancing trick

- A consequence of the jump condition for X is:

$$X(z)\begin{pmatrix} I_3 \\ 0_3 \end{pmatrix}\frac{\Lambda(z)^{2N}}{z^{2N}} = X_+(z)\begin{pmatrix} 0_3 \\ I_3 \end{pmatrix} - X_-(z)\begin{pmatrix} 0_3 \\ I_3 \end{pmatrix}, \quad z \in \mathbb{T}.$$

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- We use this to split the integral:

$$\begin{aligned}[K(v_1, v_2)]^2_{k_1,k_2=0} = \frac{1}{(2\pi i)^2} \oint_{\mathbb{T}_{1-\delta}} E(z_1) \frac{\Lambda(z_1)^{2N-x}}{z_1^{2N-y}} \begin{pmatrix} 0_3 & I_3 \end{pmatrix} X^{-1}(z_1) \\ \times \left(\oint_{\mathbb{T}_{1-\delta/2}} X(z_2) \begin{pmatrix} 0_3 \\ I_3 \end{pmatrix} \frac{\Lambda(z_2)^{-(2N-x)}}{z_2^{-(2N-y)}} \frac{E^{-1}(z_2)}{z_2 - z_1} \frac{dz_1 dz_2}{z_2} \right. \\ \left. - \oint_{\mathbb{T}_{1+\delta/2}} X(z_2) \begin{pmatrix} 0_3 \\ I_3 \end{pmatrix} \frac{\Lambda(z_2)^{-(2N-x)}}{z_2^{-(2N-y)}} \frac{E^{-1}(z_2)}{z_2 - z_1} \frac{dz_1 dz_2}{z_2} \right).\end{aligned}$$

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- **Disclaimer:** we should treat λ_3 differently due to $\lambda_3(-1) = 0$, but we choose to ignore this to simplify the exposition.

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Second transformation $X \mapsto T$

- Recall second transformation:

$$T(z) = \begin{pmatrix} e^{2N\ell} I_3 & 0_3 \\ 0_3 & I_3 \end{pmatrix} X(z) \begin{pmatrix} G_+^N(z) & 0_3 \\ 0_3 & G_-^N(z) \end{pmatrix} \begin{pmatrix} e^{-2N\ell} I_3 & 0_3 \\ 0_3 & I_3 \end{pmatrix},$$

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$$\begin{aligned} [K(v_1, v_2)]^2_{k_1,k_2=0} = \frac{1}{(2\pi i)^2} \oint_{\mathbb{T}_{1-\delta}} E(z_1) G_+^N(z_1) \frac{\Lambda(z_1)^{2N-x}}{z_1^{2N-y}} \begin{pmatrix} 0_3 & I_3 \end{pmatrix} T^{-1}(z_1) \\ \times \left(\oint_{\mathbb{T}_{1-\delta/2}} T(z_2) \begin{pmatrix} 0_3 \\ I_3 \end{pmatrix} G_-^N(z_2) \frac{\Lambda(z_2)^{-(2N-x)}}{z_2^{-(2N-y)}} \frac{E^{-1}(z_2)}{z_2 - z_1} \frac{dz_1 dz_2}{z_2} \right. \\ \left. - \oint_{\mathbb{T}_{1+\delta/2}} T(z_2) \begin{pmatrix} 0_3 \\ I_3 \end{pmatrix} G_-^N(z_2) \frac{\Lambda(z_2)^{-(2N-x)}}{z_2^{-(2N-y)}} \frac{E^{-1}(z_2)}{z_2 - z_1} \frac{dz_1 dz_2}{z_2} \right). \end{aligned}$$

Remaining transformations $T \mapsto S \mapsto M \mapsto R$

- Outside the lens, we have

$$T(z) = \begin{pmatrix} L^N & 0_3 \\ 0_3 & L^{-N} \end{pmatrix} S(z) = \begin{pmatrix} L^N & 0_3 \\ 0_3 & L^{-N} \end{pmatrix} R(z)M(z)$$

for some constant lower-triangular matrix L and

$$R(z) = I_6 + \mathcal{O}\left(\frac{e^{-cN}}{1+|z|}\right) \qquad \text{for some } c > 0.$$

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- Therefore, for a small enough lens, we may take

$$T(z) = \begin{pmatrix} L^N & 0_3 \\ 0_3 & L^{-N} \end{pmatrix} M(z),$$

  which only contributes an **exponentially small error** (in N).

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Remaining transformations $T \mapsto S \mapsto R$

- Finally, we arrive at

$$[K(v_1, v_2)]^2_{k_1,k_2=0} = \frac{1}{(2\pi i)^2} \oint_{\mathbb{T}_{1-\delta}} \oint_{\mathbb{T}_{1-\delta/2}} \frac{A(z_1)B(z_2)}{z_2 - z_1} \frac{dz_1 dz_2}{z_2} - \frac{1}{(2\pi i)^2} \oint_{\mathbb{T}_{1-\delta}} \oint_{\mathbb{T}_{1+\delta/2}} \frac{A(z_1)B(z_2)}{z_2 - z_1} \frac{dz_1 dz_2}{z_2},$$

with

$$A(z) = E(z) G_+^N(z) \frac{\Lambda(z)^{2N-x}}{z^{2N-y}} \begin{pmatrix} 0_3 & I_3 \end{pmatrix} M^{-1}(z)$$

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- $A(z)$ and $B(z)$ only have a jump on $\mathbb{T}$.

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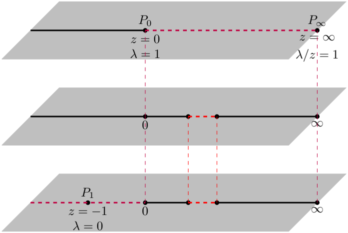
Functions on the Riemann surface

- Writing

$$\Lambda(z) = \begin{pmatrix} \lambda_1(z) & & \\ & \lambda_2(z) & \\ & & \lambda_3(z) \end{pmatrix},$$

we find a function λ on the Riemann surface by defining

$$\lambda(p) = \begin{cases} \lambda_1(z(p)) & \text{if } p \text{ lies on the first sheet;} \\ \lambda_2(z(p)) & \text{if } p \text{ lies on the second sheet;} \\ \lambda_3(z(p)) & \text{if } p \text{ lies on the third sheet.} \end{cases}$$



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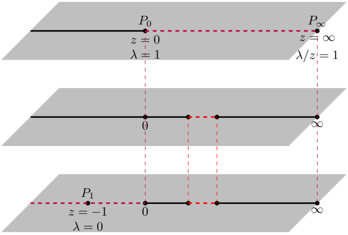

- Generically there are three points $p_0(z)$, $p_1(z)$, and $p_2(z)$ above z, i.e., $z(p_j(z)) = z$, so

$$\Lambda(z) = \sum_{j=0}^{2} \lambda(p_j(z))\mathrm{Sh}(p_j(z)), \qquad \mathrm{Sh}(p) = \begin{cases} \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} & \text{if } p \text{ lies on the first sheet;} \\ \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} & \text{if } p \text{ lies on the second sheet;} \\ \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} & \text{if } p \text{ lies on the third sheet.} \end{cases}$$

Interpreting the integrals on the Riemann surface

- Similarly, we have

$$G_{\pm}(z) = \sum_{j=0}^{2} \exp(\pm g(p_j(z)))\mathsf{Sh}(p_j(z)),$$

so that

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$$\frac{1}{(2\pi i)^2} \oint_{\mathbb{T}_{1-\delta}} \oint_{\mathbb{T}_{1\mp\delta/2}} \frac{A(z_1)B(z_2)}{z_2 - z_1} \frac{dz_1 dz_2}{z_2}$$

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Interpreting the integrals on the Riemann surface

- As promised, we arrive at

$$[K(v_1,v_2)]_{k_1,k_2=0}^2 = \oint_{\gamma_1} \oint_{\gamma_{2,+}\cup\overline{\gamma}_{2,-}} \frac{\exp(N(\Phi(p_1)-\Phi(p_2)))}{z(p_1)-z(p_2)} \Big(\text{bounded in } N\Big) \frac{dz(p_1)dz(p_2)}{z(p_2)}$$

with $\Phi(p) = g(p) + (1-\xi_1)\log\lambda(p) - (1-\xi_2)\log z(p)$.

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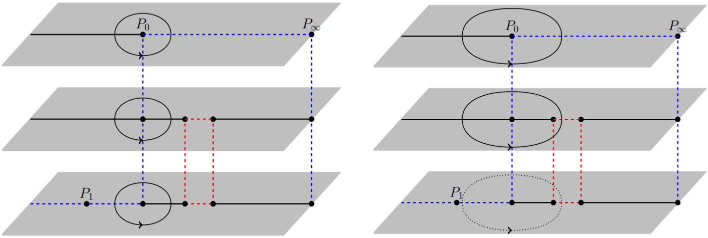

Figure: γ_1 (left) and $\gamma_{2,\pm}$ (right).

Curves on the amoeba

- The curve $P(z, \lambda) = 0$ has the **Harnack property**:

$$\log : (z, \lambda) \mapsto (\log |z|, \log |\lambda|)$$

  is at most 2-to-1.

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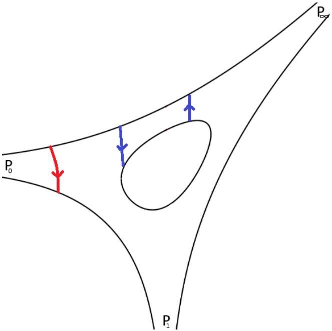
  is at most 2-to-1.
- The image is known as the **amoeba**:



Curves on the amoeba

- The function $\Phi(p)$ still has a jump on $\Gamma_1 \cup \Gamma_2$:

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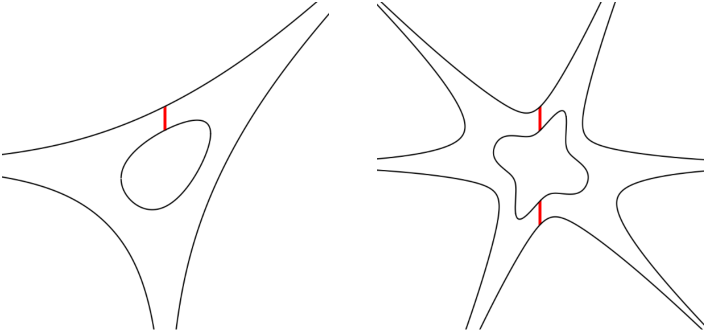
- It has an analytic continuation to the **double cover** of the Riemann surface.
- We represent the double cover by the **double amoeba**:



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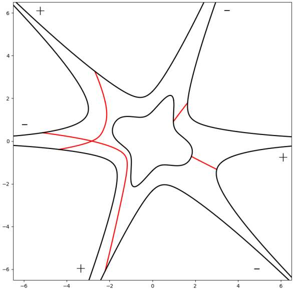
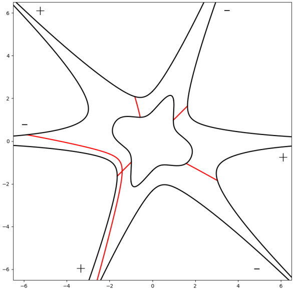
Level sets on the double amoeba

- Two possible configurations of the level set $\mathrm{Re}\,\Phi(p) = \mathrm{Re}\,\Phi(s)$ are:





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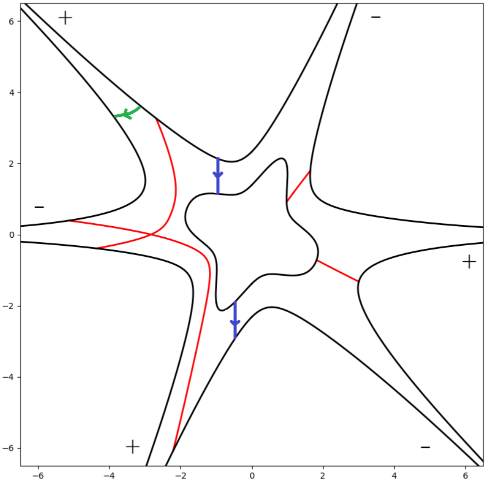
Contour deformations on the double amoeba



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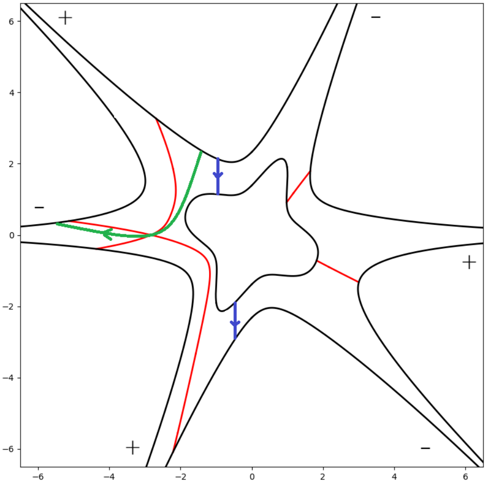

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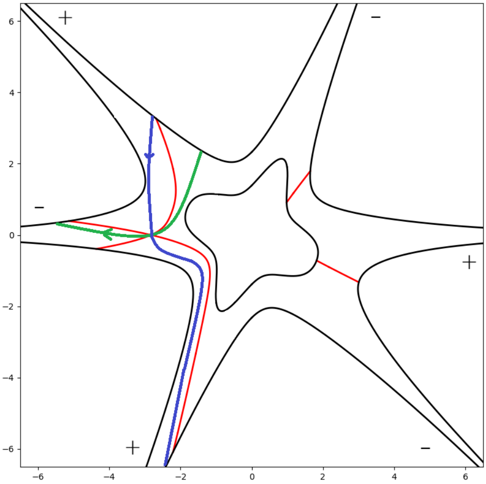

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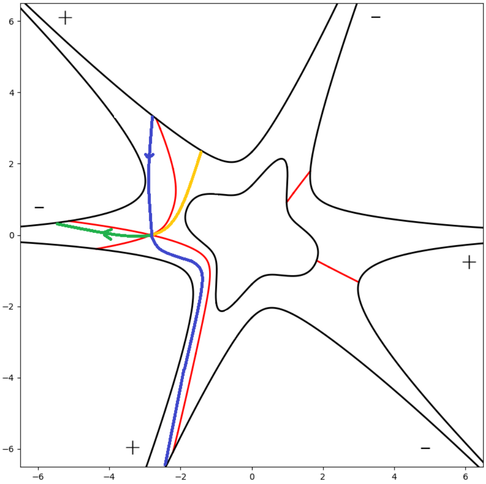

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Steepest descent analysis

- As promised, we arrive at

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$$+\frac{1}{2\pi i} \oint_{\gamma(\xi_1,\xi_2)} E(z(p))\mathrm{Sh}(p)E^{-1}(z(p)) \frac{dz(p)}{z(p)}$$

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- The double contour integral tends to 0 as $N \to \infty$ outside a small neighborhood of s.
- Local analysis shows that the remaining part also vanishes.

Main Theorem

Theorem

The large-N limit of the correlation kernel is given by

$$\lim_{N\to\infty} K(v_{1,N}, v_{2,N}) = K_{(\xi_1,\xi_2)}(v_1, v_2).$$

The 3×3-periodic limit kernel $K_{(\xi_1,\xi_2)} : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$ equals

$$\left[K_{(\xi_1,\xi_2)}(v_1, v_2)\right]_{k_1,k_2=0}^{2} = -\frac{\chi_{3x_1+j_1>3x_2+j_2}}{2\pi i} \oint_{\mathbb{T}} \frac{T_{0,j_2}^{-1}(z) W(z)^{x_1-x_2} T_{0,j_1}(z)}{z^{y_1-y_2}} \frac{dz}{z}$$
$$+ \frac{1}{2\pi i} \int_{\gamma_{(\xi_1,\xi_2)}} \frac{\lambda(p)^{x_1-x_2}}{z(p)^{y_1-y_2}} T_{0,j_2}^{-1}(z(p)) \Xi(p) T_{0,j_1}(z(p)),$$

with $v_1 = (3x_1 + j_1, 3y_1 + k_1)$ and $v_2 = (3x_2 + j_2, 3y_2 + k_2)$. Here $x_1, y_1, x_2, y_2 \in \mathbb{Z}$ and $j_1, k_1, j_2, k_2 \in \{0, 1, 2\}$.

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- $\Xi = (\eta_{j,k})_{j,k=0}^2$ is a matrix with meromorphic differentials on $\mathcal{R}$.
- $\gamma(\xi_1, \xi_2)$ is a contour on the double cover of $\mathcal{R}$ that connects s to $\bar{s}$ in a specific way.

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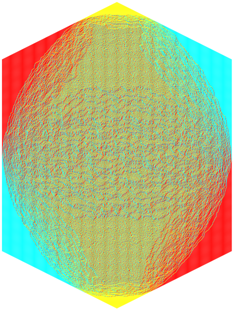
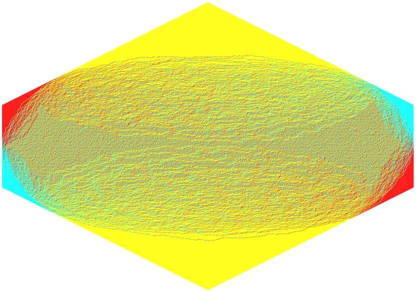
Outlook: non-regular random hexagon tilings

- The 3×3-periodic tiling model is **very rich**; even for the specialized weights.
- Keeping $A = 1$ and varying $B = C \in (0, 1) \cup (1, \infty)$, we observe that the **gas phase splits** in two different ways.
- The inner arctic curve consists of **two curves** with **four cusp points** (cf. one curve with six cusp points).





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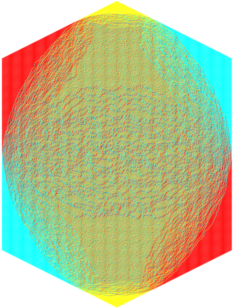
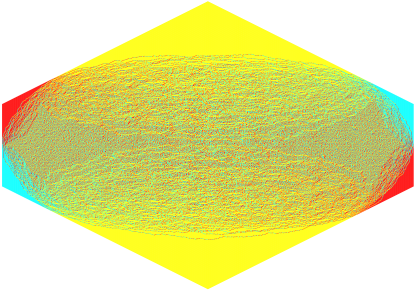
- Merging occurs at either one or two cusp points.
- The geometry is captured by the **double cover**.

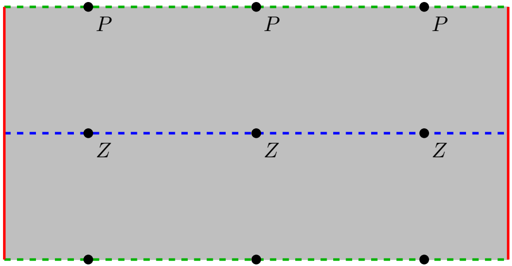




KU LEUVEN

What happens to the equilibrium measure?

- In the regular case, the support of the equilibrium measure equals $\Gamma_1 \cup \Gamma_2$.
- The support is also a **trajectory** of a quadratic differential $Q(z,\lambda)dz^2$.
- The "root of $Q(z,\lambda)dz^2$" extends to a **meromorphic differential** on the double cover.



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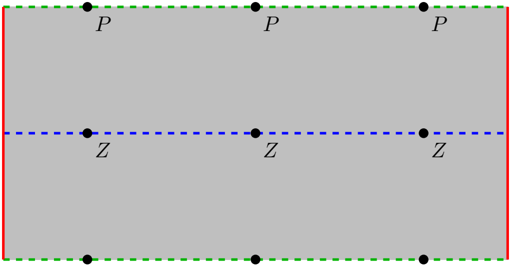
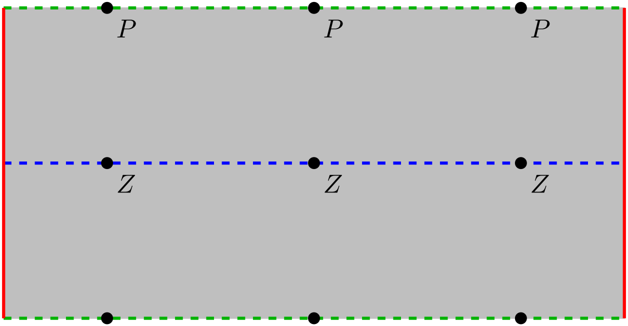
- Recall that:

$$\begin{aligned} \text{number of zeros} - \text{number of poles} &= 2 \cdot \text{genus} - 2 \\ 6 \quad - \quad 6 &= 2 \cdot 1 - 2. \quad \checkmark \end{aligned}$$



Moving zeros: track & trace ($B = C > 1$)



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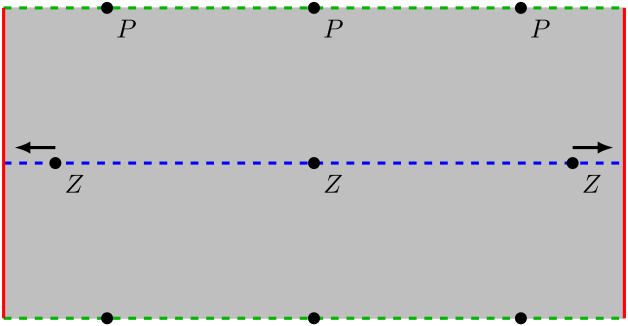

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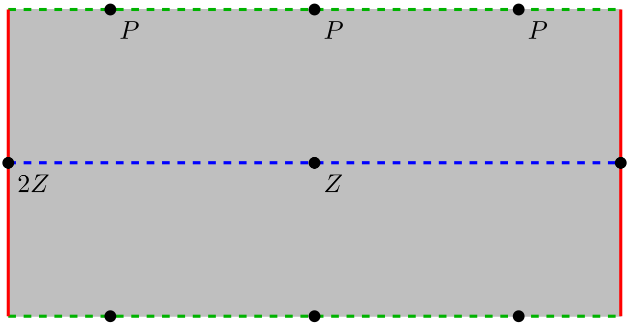

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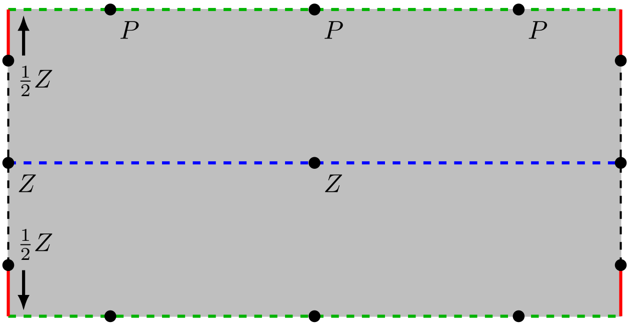

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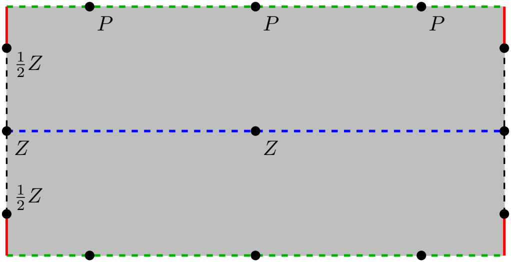
Ramified double cover

- For $B = C = c > 1$, the support is some subarc of $\Gamma_1 \cup \Gamma_2$.
- The support is a trajectory of an **explicit** quadratic differential $Q_c(z, \lambda)dz^2$.
- The double cover becomes **ramified** for large $c > 1$.



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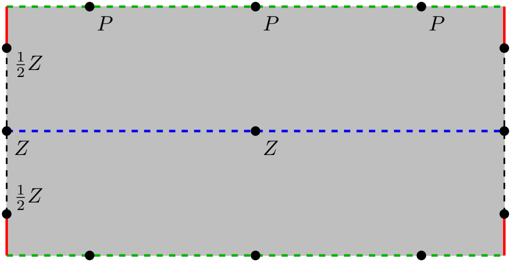

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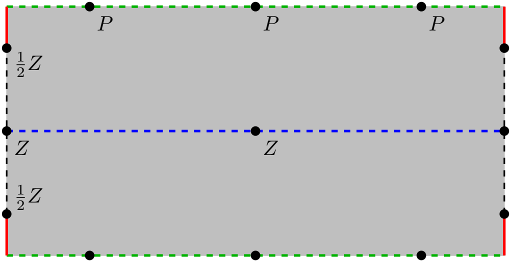
- **2 extra zeros** are created by dz because of the ramification:

$$dz = d(z^{\frac{1}{2}})^2 = 2z^{\frac{1}{2}} dz^{\frac{1}{2}} \qquad (z^{\frac{1}{2}} \text{ local coordinate}).$$



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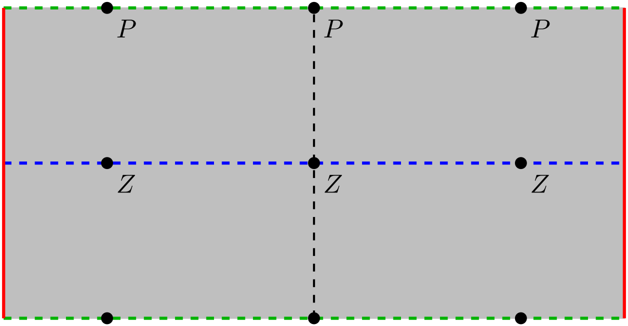

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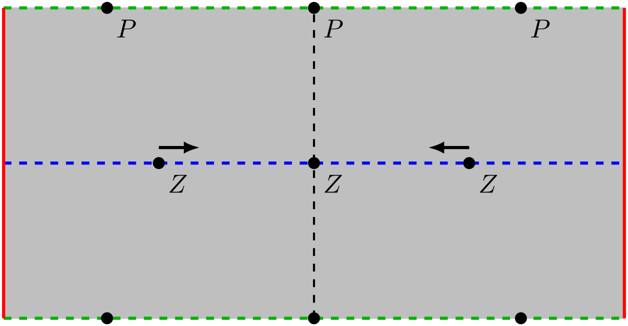

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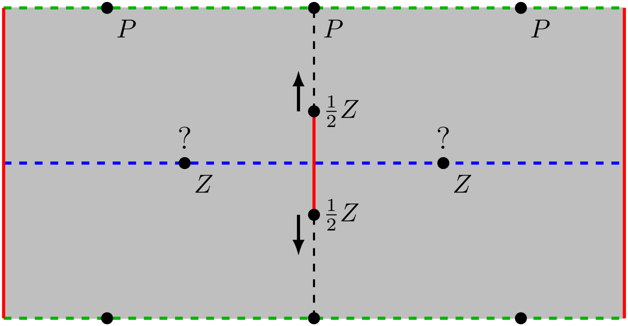

KU LEUVEN

Computing arctic curves

- For $(\xi_1, \xi_2) \in \mathcal{L}$, there is a unique zero $s_c(\xi_1, \xi_2)$ inside $\widetilde{\mathcal{R}}_{c,+}$ of

$$Q_c^{\frac{1}{2}} dz + \xi_1 d \log \lambda - \xi_2 d \log z.$$

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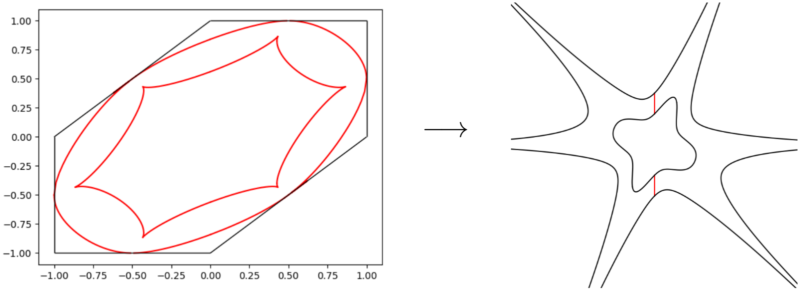

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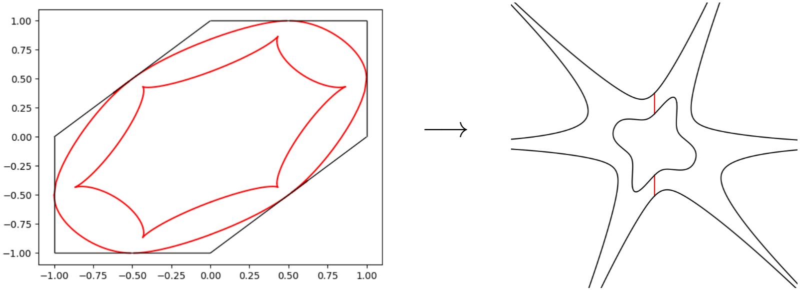
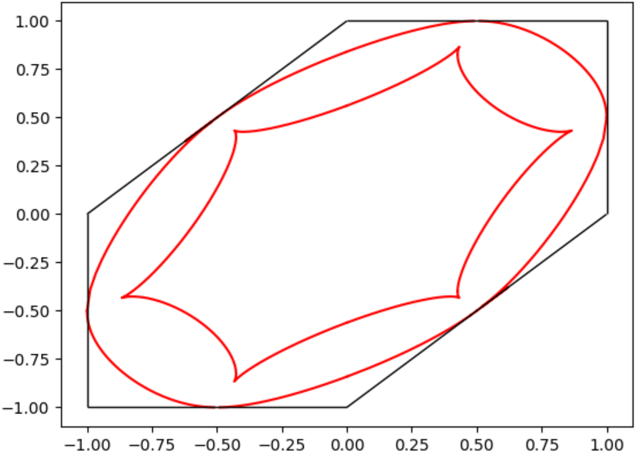
- The arctic curves are given by

$$\Omega_c^{-1}(\text{(un)bounded ovals}).$$



Splitting of the gas phase ($B = C > 1$)



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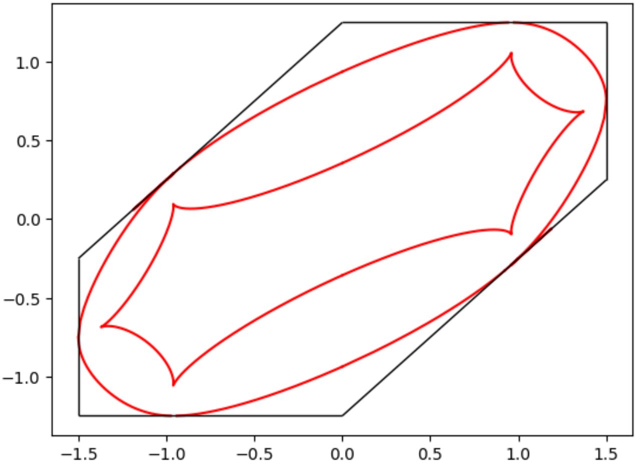

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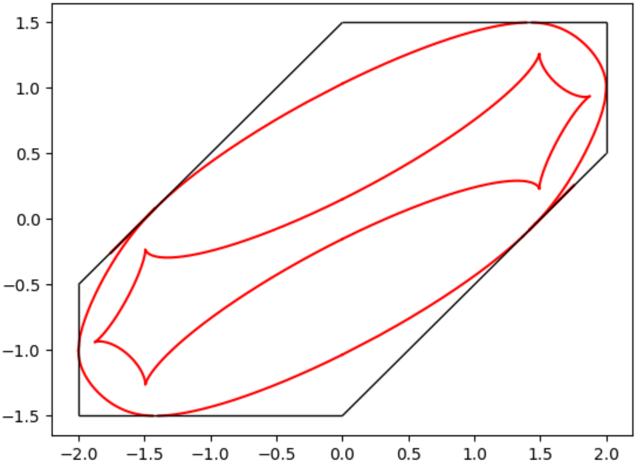

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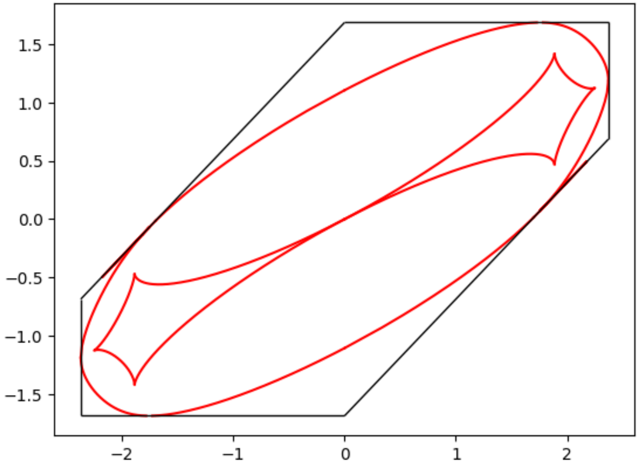

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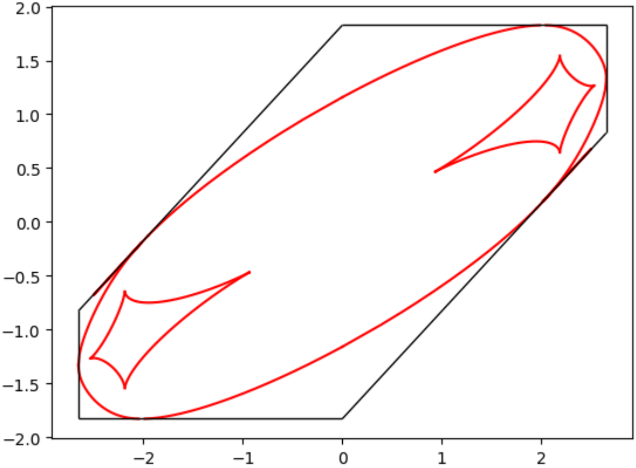

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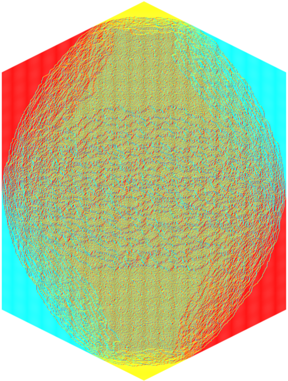

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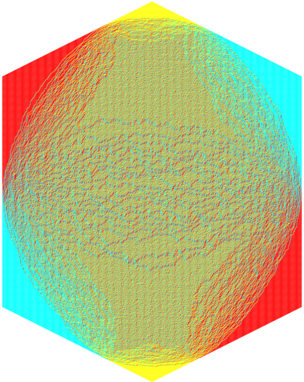

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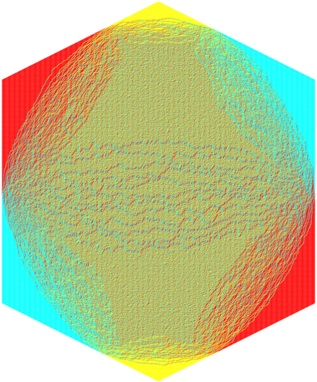

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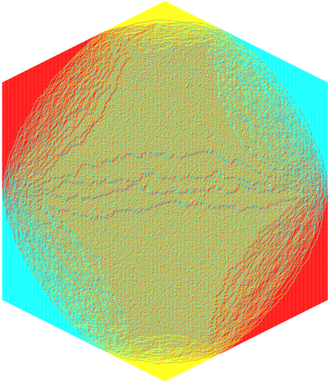

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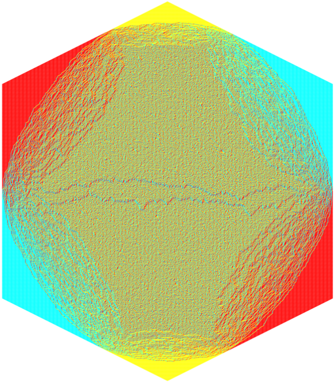

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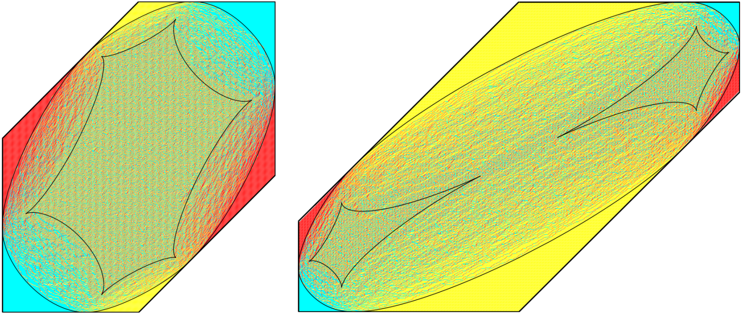
Final gallery



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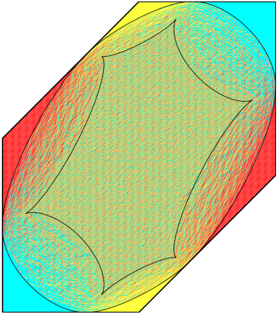
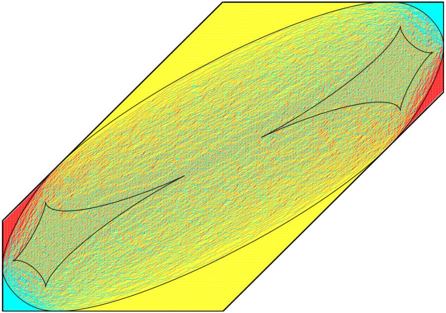
Thank you for your attention!





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