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Large random tilings of a hexagon with periodic weightings

Arno Kuijlaars
KU Leuven, Belgium

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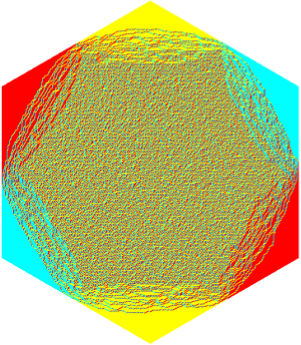
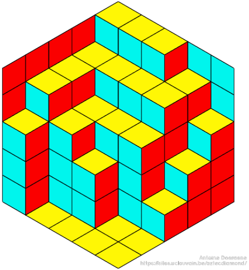
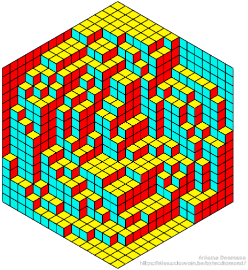
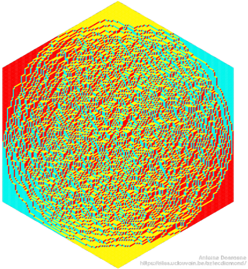
0 Outline

1. **Tilings of a hexagon**
2. **Exact formulas**
3. **Riemann HIlbert and MVOP**
4. **Asymptotic analysis (outline)**
5. **Spectral curve**
6. **Equilibrium measure**
7. **Further steps**

1. Tilings of a hexagon



1 Tilings of a hexagon: Arctic circle

Large random tilings of regular hexagon have Arctic Circle phenomenon:
Rigid pattern near corners and disorder in the middle.

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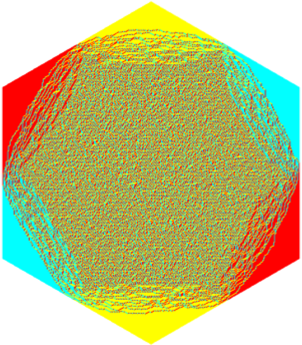
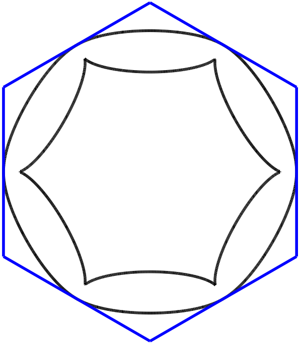
1 Tilings of a hexagon: periodic weighting

Different pictures in case of non-uniform probabilities





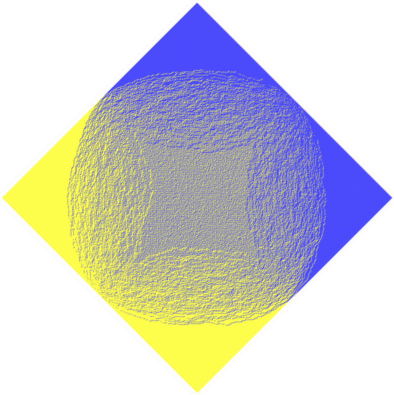
Three different regions: frozen, rough and smooth

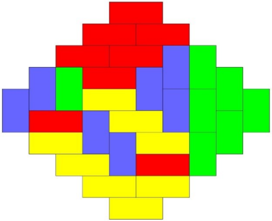
1 Related tiling model

Domino tilings of an Aztec diamond with periodic weights

- **Recent papers (selection)**

Chhita Johansson 2016
Duits Kuijlaars 2021
Berggren Borodin 2025





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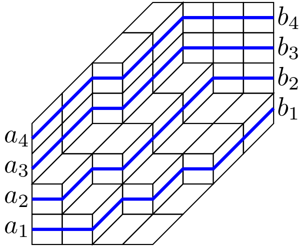
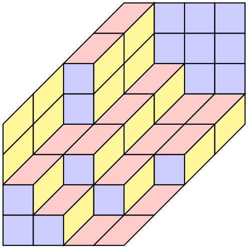
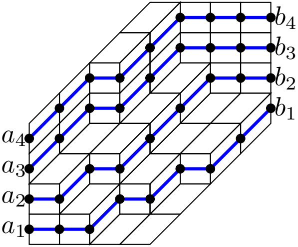
2. Exact formulas for finite size system



2 Non intersecting paths

Tiling of a hexagon is equivalent to a system of non-intersecting paths with prescribed starting and ending positions





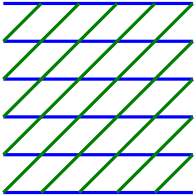
Particles on the paths are random particle system (in case of random tiling)

2 Non uniform probabilities

Paths are a graph $G = (\mathbb{Z}^2, E)$.
Assign weights to the edges $w : E \to \mathbb{R}^+$

Probability of non intersecting path system $\mathcal{P}$

$$\frac{1}{Z} \prod_{e \in \mathcal{P}} w(e)$$



Theorem (Eynard Mehta 1998)

Random particle system is determinantal. I.e., there exists $K : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$ ***such that for distinct vertices*** $v_1, \ldots, v_k$,

$$\text{Prob}\left[\begin{array}{l} \textit{there is particle at} \\ \textit{each } v_1, \ldots, v_k \end{array} \right] = \det \left[K(v_i, v_j) \right]_{i,j=1}^{k}$$

2 Determinantal point process

Theorem (Eynard Mehta 1998)

Random particle system is determinantal. I.e., there exists $K : V \times V \to \mathbb{R}$ ***such that for distinct vertices*** $v_1, \ldots, v_k$***,***

$$\mathrm{Prob}\left[\begin{array}{l}\textit{there is particle at}\\ \textit{each } v_1, \ldots, v_k\end{array}\right] = \det\left[K(v_i, v_j)\right]_{i,j=1}^{k}$$

- **All information is in the correlation kernel** K**.**
- **Eynard Mehta have a double sum formula for** K**.**

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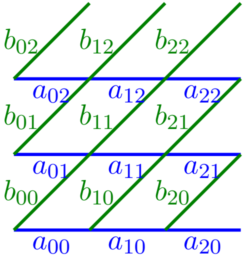
2 Periodic weights

Weights are periodic with period p if for every i, j,

$$a_{i,j} = a_{i+p,j} = a_{i,j+p}$$
$$b_{i,j} = b_{i+p,j} = b_{i,j+p}$$



Assign weights in fundamental domain, and extend periodically.

Transition matrix
$$T_j(z) = \begin{pmatrix} a_{j0} & b_{j0} & 0 & \cdots & 0 \\ 0 & a_{j1} & b_{j1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & a_{j,p-2} & b_{j,p-2} \\ b_{j,p-1}z & 0 & \cdots & 0 & a_{j,p-1} \end{pmatrix}$$

2 Transition matrices

$$T_j(z) = \begin{pmatrix} a_{j0} & b_{j0} & 0 & \cdots & 0 \\ 0 & a_{j1} & b_{j1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & a_{j,p-2} & b_{j,p-2} \\ b_{j,p-1} z & 0 & \cdots & 0 & a_{j,p-1} \end{pmatrix}$$

Products of transition matrices

$$\boxed{W(z) = T_0(z) T_1(z) \cdots T_{p-1}(z)}$$

Partial products

$$T_{0,j} = T_0 \cdot T_1 \cdots T_{j-1}, \qquad j = 0, \dots p-1.$$

2 Correlation kernel in case of periodic weights

Theorem (Duits K 2021 for hexagon of size $pN \times pBN \times pCN$)

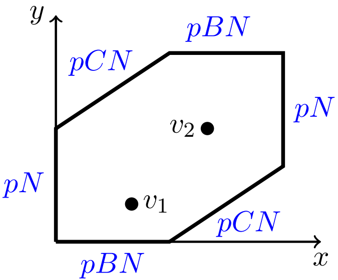
$$K_N(v_1,v_2)=\Bigg[-\frac{\chi_{px_2+j_2<px_1+j_1}}{2\pi i}\oint_{\mathbb{T}} T_{0,j_2}^{-1}(z)\frac{W^{x_1-x_2}(z)}{z^{y_1-y_2}}T_{0,j_1}(z)\frac{dz}{z}+$$

$$\frac{1}{(2\pi i)^2}\oint_{\mathbb{T}}\oint_{\mathbb{T}} T_{0,j_2}^{-1}(z_1)\frac{W^{(B+C)N-x_2}(z_1)}{z_1^{CN-y_2}}R_N(z_1,z_2)\frac{W^{x_1}(z_2)}{z_2^{y_1}}T_{0,j_1}(z_2)\frac{dz_1\,dz_2}{z_2}\Bigg]_{k_1,k_2}$$

Coordinates

$$v_1 = (px_1 + j_1, py_1 + k_1)$$
$$v_2 = (px_2 + j_2, py_2 + k_2)$$

with $j_1, j_2, k_1, k_2 \in \{0, 1, \dots, p-1\}$



3. Riemann Hilbert problem and MVOP

3 Riemann-Hilbert problem

Double integral contains

$$R_N(z_1, z_2) = \frac{1}{z_2 - z_1} \begin{pmatrix} 0_p & I_p \end{pmatrix} \mathbf{Y}^{-1}(z_1)\mathbf{Y}(z_2) \begin{pmatrix} I_p \\ 0_p \end{pmatrix}$$

where $\mathbf{Y}$ solves the Riemann Hilbert problem (RH problem)

- $\mathbf{Y} : \mathbb{C} \setminus \mathbb{T} \to \mathbb{C}^{2p\times 2p}$ **is analytic,**
- $\mathbf{Y}_+(z) = \mathbf{Y}_-(z) \begin{pmatrix} I_p & \frac{W(z)^{(B+C)N}}{z^{(1+C)N}} \\ 0_p & I_p \end{pmatrix}$ **for** $z \in \mathbb{T}$**,**
- $\mathbf{Y}(z) = (I_{2p} + O(z^{-1})) \begin{pmatrix} z^N I_p & 0_p \\ 0_p & z^{-N} I_p \end{pmatrix}$ **as** $z \to \infty$**.**

$\mathbf{Y}$ is given in terms of matrix valued orthogonal polynomials

3 Matrix valued orthogonal polynomials

$$\mathbf{P}_N = \begin{pmatrix} I_p & 0_p \end{pmatrix} \mathbf{Y} \begin{pmatrix} I_p \\ 0_p \end{pmatrix}$$

is matrix valued polynomial of degree N satisfying

$$\frac{1}{2\pi i} \oint_{\mathbb{T}} \mathbf{P}_N(z) \frac{W(z)^{(B+C)N}}{z^{(1+C)N}} z^k dz = 0_p, \qquad k = 0, \ldots, N-1$$

- **$\mathbb{T}$ can be replaced by any contour going once around the origin.**
- **non-hermitian orthogonality with varying weight.**

3 Large N limit: two steps

$$\frac{1}{(2\pi i)^2}\oint_{\mathbb{T}}\oint_{\mathbb{T}} T_{0,j_2}^{-1}(z_1)\frac{W^{(B+C)N-x_2}(z_1)}{z_1^{CN-y_2}} \times R_N(z_1,z_2)\frac{W^{x_1}(z_2)}{z_2^{y_1}}T_{0,j_1}(z_2)\frac{dz_1 dz_2}{z_2}$$

First step:

- **Analyze the RH problem with the Deift-Zhou method of steepest descent**

Second step:

- **Classical steepest descent analysis for the double integral.**

4. RH steepest descent analysis (outline)

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4 RH steepest descent

Deift-Zhou method of steepest descent is a sequence of transformations

$$\mathbf{Y} \mapsto \mathbf{X} \mapsto \mathbf{T} \mapsto \mathbf{S} \mapsto \mathbf{R}$$

leading to R that satisfies a **small norm RH problem**.

The method was applied first to **orthogonal polynomials** by
Deift Kriecherbauer McLaughlin Venakides Zhou 1999

- $\mathbf{Y} \mapsto \mathbf{X}$ is preliminary transformation
- $\mathbf{X} \mapsto \mathbf{T}$ uses **equilibrium measure**
- $\mathbf{T} \mapsto \mathbf{S}$ is deformation step (opening of lenses)
- $\mathbf{S} \mapsto \mathbf{R}$ is approximation step (parametrices)

4 Equilibrium measure

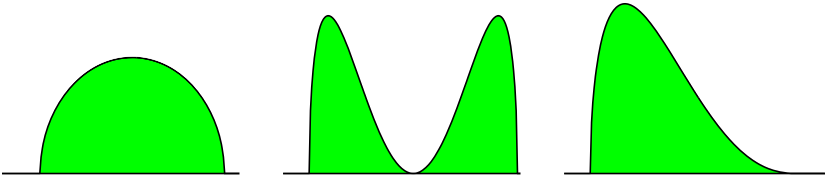
OPs on the real line (with varying weight) satisfy

$$\int_{\mathbb{R}} P_N(x) e^{-NV(x)} x^k dx = 0, \qquad k = 0, 1, \ldots, N-1$$

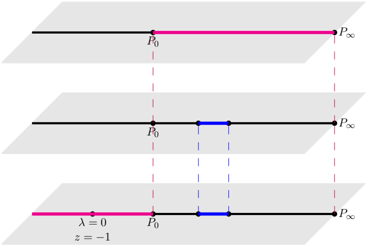
Equilibrium measure μ_{eq} minimizes

$$\iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int V(x) d\mu(x)$$

among probability measures μ on $\mathbb{R}$.



5. Eigenvalues and spectral curve

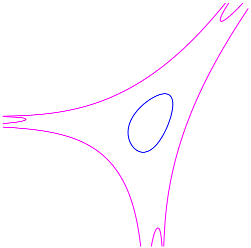


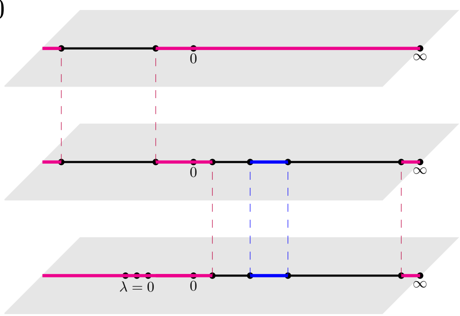
5 Algebraic curve and amoeba

$\det(\lambda I_p - W(z)) = 0$ is a **Harnack curve**

Kenyon Okounkov Sheffield 2006

- **The amoeba map** $(z, \lambda) \mapsto (\log|z|, \log|\lambda|)$ **is at most** 2**-to-**1 **on the algebraic curve.**
- **For** $z \in \mathbb{C} \setminus \mathbb{R}$ **the eigenvalues** $\lambda_j(z)$ **can be ordered such that** $|\lambda_1(z)| > \cdots > |\lambda_p(z)| > 0$





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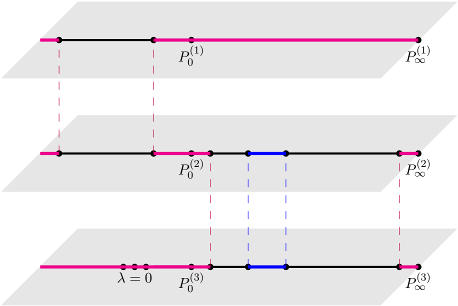
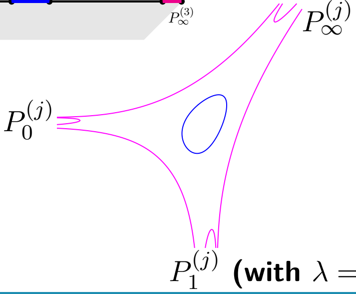
5 Sheet structure of Riemann surface $\mathcal{R}$



Real locus has two parts

- **The unbounded oval, containing all points where z or λ are 0 or ∞,**
- **The bounded oval**
- **Points at infinity $P_\infty^{(j)}$,** $j = 1, \dots, p$



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5 First transformation $\mathbf{Y} \mapsto \mathbf{X}$

E **is the matrix of eigenvectors of** W

$$W = E\Lambda E^{-1}, \qquad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$$

Definition

$$\mathbf{X} = \mathbf{Y} \begin{pmatrix} E & 0_p \\ 0_p & E \end{pmatrix}$$

New jumps

- $$\mathbf{X}_+ = \mathbf{X}_- \times \begin{cases} \begin{pmatrix} I_p & \frac{\Lambda^{(B+C)N}}{z^{(1+C)N}} \\ 0_p & I_p \end{pmatrix} & \text{on } \mathbb{T}, \\ \begin{pmatrix} J_{\mathcal{R}} & 0_p \\ 0_p & J_{\mathcal{R}} \end{pmatrix} & \text{on } \mathbb{R}, \end{cases}$$
- $J_{\mathcal{R}}$ **is the permutation matrix that models the sheet structure of the Riemann surface.**

6. Equilibrium measure on $\mathcal{R}$

6 Potential theory on $\mathcal{R}$

- **We do not have the logarithmic kernel $\log \frac{1}{|x-y|}$ on $\mathcal{R}$**

Potential theory on $\mathcal{R}$ uses the bipolar Green's kernel

$$G_P(p,q) \qquad p,q, \in \mathcal{R}, \qquad \text{with singularity at } P \in \mathcal{R},$$

- $p \mapsto G_P(p,q)$ **is harmonic on $\mathcal{R} \setminus \{P,q\}$**
- $G_P(p,q) = \log |z_P(p)| + \mathcal{O}(1)$ **as $p \to P$,**
  if z_P is local coordinate at P,
- $G_P(p,q) = -\log |z_q(p)| + \mathcal{O}(1)$ **as $p \to q$,**
  if z_q is local coordinate at q,
- $G_P(p,q) = G_P(q,p)$

6 Max min problem

Equilibrium problem is Max-min problem

$$\max_{\Gamma} \min_{\mu \text{ on } \Gamma} \left[\sum_j \iint G_{P_\infty^{(j)}}(p,q) d\mu(p) d\mu(q) + \int \operatorname{Re} V d\mu \right]$$

$$V = (B+C)\log z - (1+C)\log\lambda$$

- **Maximize over closed contours Γ that go around $z = 0$ on each sheet.**
- **Minimize over probability measures μ on Γ.**

6 Special case

Equilibrium problem can be solved in special case.

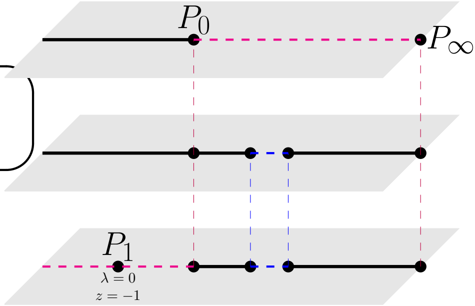
K 2025, arXiv:2412.03115

Assume $p = 3$, $B = C = 1$,

$$\det(\lambda I_3 - W(z)) = (\lambda - 1 - z)^3 - 27(1+\beta)\lambda z$$

for some $\beta > 0$

Special points P_0, P_1, P_∞



6 Equilibrium measure

Optimal $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ is union of unit circles on all sheets

Support of equilibrium measure is on first two sheets

$$\mathrm{supp}(\mu_{eq}) = \Gamma_1 \cup \Gamma_2$$

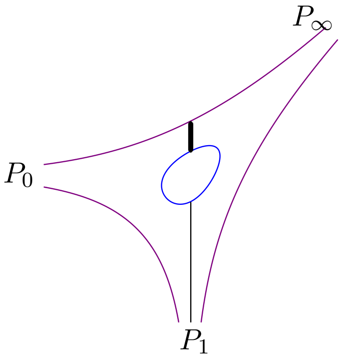
Explict formula

$$d\mu_{eq} = \frac{1}{3\pi i} \left(\frac{c_0 \lambda + z + 1}{3(1+\beta)\lambda} \right)^{\frac{1}{2}} \frac{c_1 \lambda + z + 1}{2\lambda + z + 1} \frac{dz}{z}$$

for certain $c_0, c_1 > 0$

As balayage measure

$$\mu_{eq} = \mathrm{Bal}\left(\delta_{P_0} - \delta_{P_1} + \delta_{P_\infty}, \Gamma_1 \cup \Gamma_2\right)$$



7. Further steps in RH analysis

7 Second transformation $\mathbf{X} \mapsto \mathbf{T}$

The g-function on the Riemann surface

$$g(p) = 3 \int G_{P_\infty}(p,q) d\mu_{eq}(q)$$

Let g_j be its restriction to the jth sheet.

Definition

$$\mathbf{T} = L^N \mathbf{X} \, \mathrm{diag}\left(e^{-Ng_1}, e^{-Ng_2}, e^{-Ng_3}, e^{Ng_1}, e^{Ng_2}, e^{Ng_3}\right) L^{-N}$$

where L is some constant diagonal matrix.

Further steps $\mathbf{T} \mapsto \mathbf{S} \mapsto \mathbf{R}$ in the RH analysis. Outcome is that $\mathbf{T}$ and $\mathbf{T}^{-1}$ remain uniformly bounded as $N \to \infty$

7 Zeros of $\det \mathbf{P}_N$

Another outcome of the RH analysis is asymptotic formula for $\mathbf{P}_N$ as $N \to \infty$. It implies in particular

Theorem (K 2025)

The weak limit of the zeros of $\det \mathbf{P}_N$ is equal to the pushforward of the equilibrium measure under the projection map $(z, \lambda) \mapsto z$.