UNIVERSITÉ CATHOLIQUE DE LOUVAIN
INSTITUT DE RECHERCHE EN MATHÉMATIQUE ET PHYSIQUE

When the M meets the P

MODELS OF A LAWVERE THEORY

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What is a group?

A group is a set G with

- a binary operation

$$\begin{aligned} m_G\colon G\times G &\longrightarrow G \\ (x,y) &\longmapsto x\cdot y \end{aligned}$$

- a 0-ary operation

$$\begin{aligned} e_G\colon \{\star\} &\longrightarrow G \\ \star &\longmapsto 1 \end{aligned}$$

- an unary operation

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satisfying the following axioms :

- associativity

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

- unity

$$x \cdot 1 = x = 1 \cdot x$$

- inverse

$$x \cdot x^{-1} = 1 = x^{-1} \cdot x$$

Axioms as commutative diagrams

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$$\begin{array}{ccc} G \times G \times G & \xrightarrow{m_G \times \mathrm{id}_G} & G \times G \\ {\scriptstyle \mathrm{id}_G \times m_G}\downarrow & & \downarrow{\scriptstyle m_G} \\ G \times G & \xrightarrow[m_G]{} & G \end{array}$$

Axioms as commutative diagrams

Associativity :

$$
\begin{array}{ccccc}
(x, y, z) & \longrightarrow & & \longrightarrow & (x \cdot y, z) \\
\Big\downarrow & G \times G \times G & \xrightarrow{m_G \times \mathrm{id}_G} & G \times G & \Big\downarrow \\
& \mathrm{id}_G \times m_G \Big\downarrow & & \Big\downarrow m_G & \\
& G \times G & \xrightarrow[m_G]{} & G & \\
(x, y \cdot z) & \longrightarrow & x \cdot (y \cdot z) = (x \cdot y) \cdot z & &
\end{array}
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Lawvere theory

What do I need to express the rules of a group?

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Objects $T^0, T^1, \cdots, T^n, \cdots$
Arrows $m\colon T^2 \to T$, $e\colon T^0 \to T$, $i\colon T^1 \to T^1$

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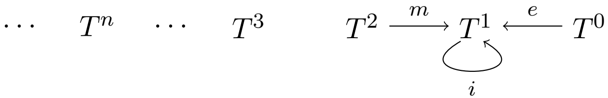

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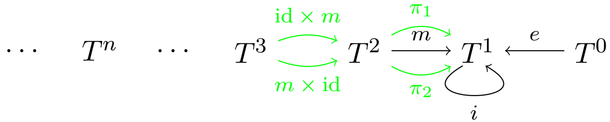

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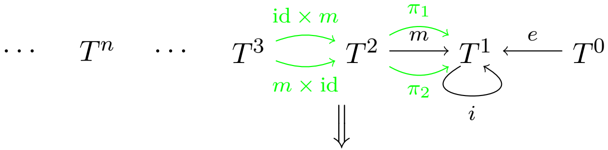

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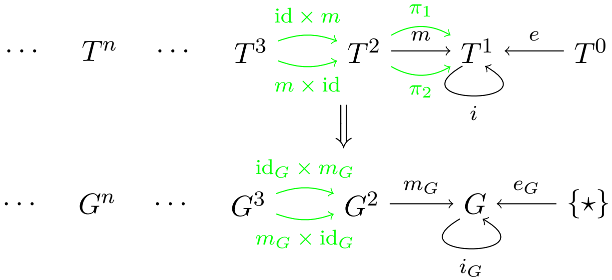

Models as functors

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A Lawvere theory is a category $\mathcal{T}$ with an object T and all its powers T^n for $N \in \mathbb{N}$.

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A model of a Lawvere theory $\mathcal{T}$ is a product-preserving functor from the Lawvere theory to the category of sets

$$\begin{aligned} F\colon \mathcal{T} &\longrightarrow \mathcal{S}et \\ T^0 &\longmapsto \{\star\} \\ T^1 &\longmapsto FT \\ T^n &\longmapsto FT^n \\ (f\colon T^n \to T) &\longmapsto Ff\colon (FT^n \to FT) \end{aligned}$$

Back to the example of groups

Let $\mathcal{T}$ be the Lawvere theory of groups.

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$$\mathsf{TopGrp} \cong \mathsf{Fun}_{\mathsf{prod}}[\mathcal{T}, \mathsf{Top}]$$

A functor from $\mathcal{T}$ to the category Top of topological spaces is a topological group.

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What if we replace $\mathcal{S}et$ with another category ?

$$\mathsf{LieGrp} \cong \mathsf{Fun}_{\mathsf{prod}}[\mathcal{T}, \mathsf{Diff}]$$

A functor from $\mathcal{T}$ to the category Diff of smooth manifolds is a Lie group.

Hopf algebras

A Hopf algebra over a field $\mathcal{K}$ is a functor from the Lawvere theory of groups to the category Coalg of cocommutative coalgebras over $\mathcal{K}$.

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\begin{aligned}
\mathcal{H} \colon \mathcal{T} &\longrightarrow \mathsf{Coalg} \\
T &\longmapsto H \\
m \colon T^2 \to T &\longmapsto m_H \colon H \otimes H \to H \\
e \colon T^0 \to T &\longmapsto u_H \colon \mathcal{K} \to H \\
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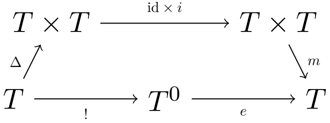

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$$\begin{aligned} \mathcal{H}\colon \mathcal{T} &\longrightarrow \mathsf{Coalg} \\ T &\longmapsto H \\ m\colon T^2 \to T &\longmapsto m_H\colon H\otimes H \to H \\ e\colon T^0 \to T &\longmapsto u_H\colon \mathcal{K} \to H \\ i\colon T \to T &\longmapsto S_H\colon H \to H \end{aligned}$$

$$\begin{array}{ccccc} T\times T & & \xrightarrow{\mathrm{id}\times i} & & T\times T \\ {\scriptstyle\Delta}\nearrow & & & & \searrow{\scriptstyle m} \\ T & \xrightarrow[!]{} & T^0 & \xrightarrow[e]{} & T \end{array}$$

$$x\cdot x^{-1} = 1$$

$$\begin{array}{ccccc} H\otimes H & & \xrightarrow{\mathrm{id}\otimes S_H} & & H\otimes H \\ {\scriptstyle\Delta_H}\nearrow & & & & \searrow{\scriptstyle m_H} \\ H & \xrightarrow[!]{} & \mathcal{K} & \xrightarrow[u_H]{} & H \end{array}$$

$$m_H(a_1\otimes S_H(a_2)) = u_H(\epsilon_H(a))$$

My research

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$$\begin{aligned} \mathsf{Hopf}_{\mathsf{coc}} &\cong \mathsf{Fun}_{\mathsf{prod}}(\mathcal{T}_{\mathsf{Grp}}, \mathsf{Coalg}) \\ \mathsf{HBr} &\cong \mathsf{Fun}_{\mathsf{prod}}(\mathcal{T}_{\mathsf{SKB}}, \mathsf{Coalg}) \\ \mathsf{HRadRng} &\cong \mathsf{Fun}_{\mathsf{prod}}(\mathcal{T}_{\mathsf{RadRng}}, \mathsf{Coalg}) \\ \mathsf{HDiGrp} &\cong \mathsf{Fun}_{\mathsf{prod}}(\mathcal{T}_{\mathsf{DiGrp}}, \mathsf{Coalg}) \end{aligned}$$