# Feynman Integrals: From Mathematics to Experiment

Ben Page

University of Ghent

When the M meets the P (UCLouvain)

GHENT UNIVERSITY

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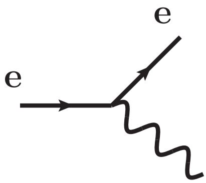
[based on work with Abreu, De Laurentis, Chicherin, Dormans, Febres Cordero, Figueiredo, Ita, Klinkert, Kraus, Monni, Moriello, Poegel, Reina, Sotnikov, Tschernow, Usovitsch, Zoia (many papers!)]

# The Standard Model of Particle Physics

In Nature, Fundamental interactions described by Quantum Field Theory.



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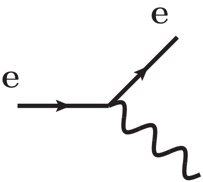


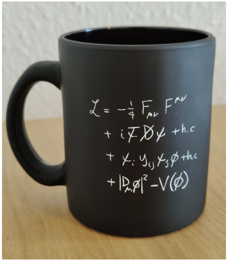


19 parameters $(m_\nu = 0)$

- 3 coupling strengths.
- 9 fermion masses $m_f$.
- 4 CKM parameters.
- 2 Higgs parameters.
- QCD vacuum angle.

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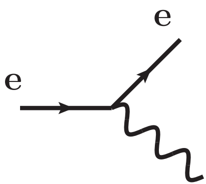


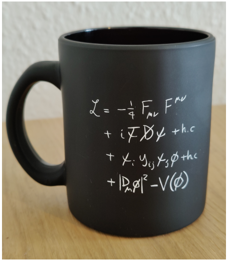


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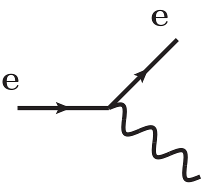


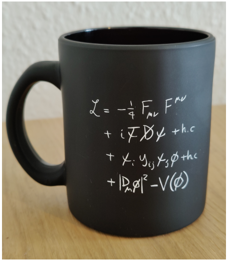

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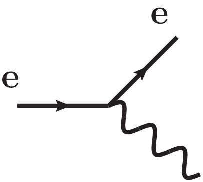


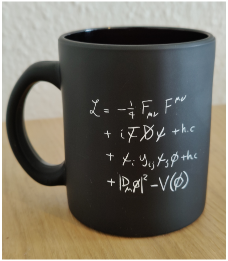


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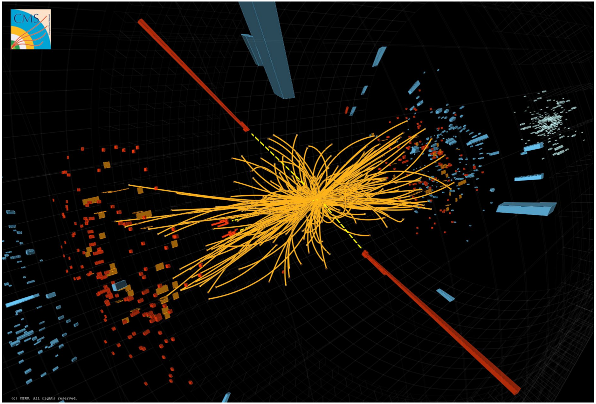
- Do we need to modify the SM?

$$\mathcal{L}_{\text{true}} \stackrel{?}{=} \mathcal{L}_{\text{SM}} + \frac{1}{\Lambda}\mathcal{L}_5 + \frac{1}{\Lambda^2}\mathcal{L}_6 + \cdots .$$

# A Collision at the LHC



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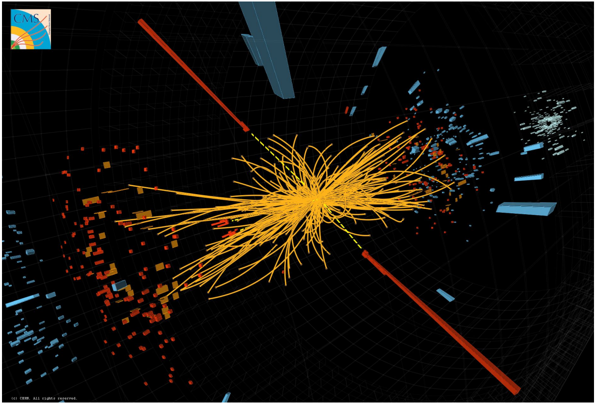
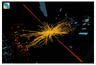
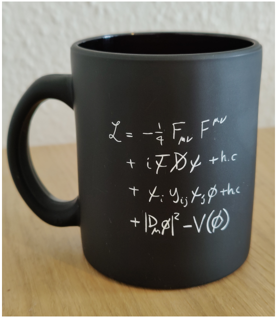


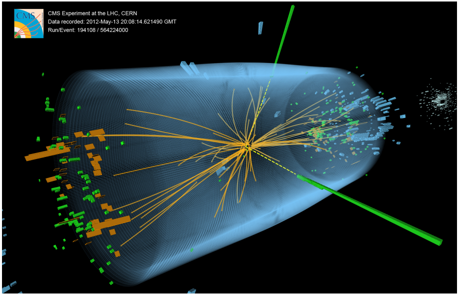

Experimentalists measure these “cross sections” by counting events:

$$\sigma \sim \sum_i \Big[$$  $$\Big]_i .$$

# The Theorist's Task




$\longrightarrow$

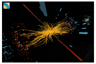



- LHC experiments gather enormous statistics $\Rightarrow$ precise measurements.
- Precise theoretical predictions needed to match experimental error.

# Precise Perturbative Predictions

- So just compute the cross-section!

$$\sum_i \Bigg[$$



$$\Bigg]_i \sim \int \underbrace{\mathrm{d}\phi_n}_{\text{phase space}} \Bigg[ \underbrace{|\mathcal{A}|^2}_{\text{amplitude}} \Bigg].$$

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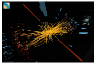
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- Perturbation theory is our major tool for making predictions.

$$\sigma \;\sim\; \sigma_{\mathrm{LO}} + \alpha_S \delta\sigma_{\mathrm{NLO}} + \alpha_S^2 \delta\sigma_{\mathrm{NNLO}} + \mathcal{O}(\alpha_S^3).$$

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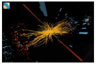
$$\sum_i \Big[$$  $$\Big]_i \sim \int \underbrace{\mathrm{d}\phi_n}_{\text{phase space}} \Big[ \underbrace{|\mathcal{A}|^2}_{\text{amplitude}} \Big].$$

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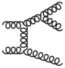
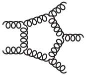
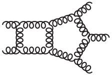
- LHC precision requires high order amplitude calculations!

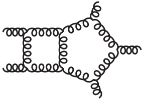
$$A_5 = \Big[$$  $$+ \cdots \Big] + \alpha_S \Big[$$  $$+ \cdots \Big] + \alpha_S^2 \Big[$$  $$+ \cdots \Big] + \mathcal{O}(\alpha_s^3).$$

# Feynman Diagrams for the LHC

# Feynman Diagram Basics

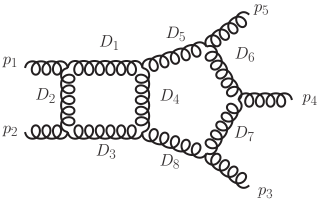
- Textbook: A loop amplitude is the sum of Feynman diagrams.

$$A_{5g}^{(2)} =$$



$$+\,\mathcal{O}(10000) \text{ diagrams.}$$

- Each diagram describes a collection of Feynman integrals:

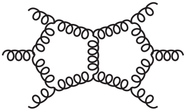



$$= \int_{\mathbb{R}^{4\times 2}} \mathrm{d}^4\ell_1 \mathrm{d}^4\ell_2 \frac{N(\ell_1, \ell_2)}{D_1 D_2 D_3 D_4 D_5 D_6 D_7 D_8}.$$

- Computation is demanding mathematical problem that hides physics.

# Algebra for Feynman Diagrams

- Consider the 6 gluon amplitude at two loops.

 $\sim$ 100MB, $\qquad \mathcal{A}_{gg \to gggg} \sim$ 3TB

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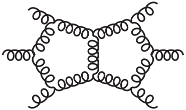
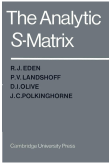
 $\sim$ 100MB, $\quad \mathcal{A}_{gg\to gggg} \sim$ 3TB

- Just the expressions for Feynman diagrams is a huge amount of data!

$\mathcal{A}_{gg\to gggg} \sim 3\,000\,000 \times$ 


$\sim$1MB

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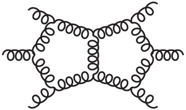

$\sim 100\text{MB}, \qquad \mathcal{A}_{gg\to gggg} \sim 3\text{TB}$

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$\mathcal{A}_{gg\to gggg} \;\sim\; 10\,000 \times$

$\sim$300MB

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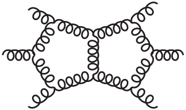 $\sim$ 100MB, $\qquad \mathcal{A}_{gg\to gggg} \sim$ 3TB

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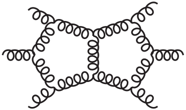 $\sim$ 100MB, $\mathcal{A}_{gg \to gggg} \sim$ 3TB

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The two-loop six-gluon amplitude fills half of Royal Library of Belgium!

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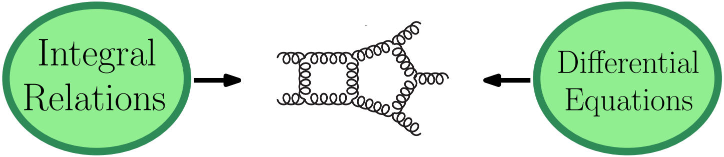

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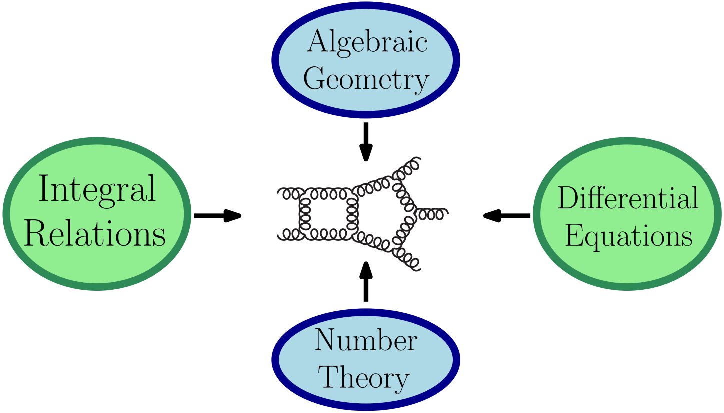

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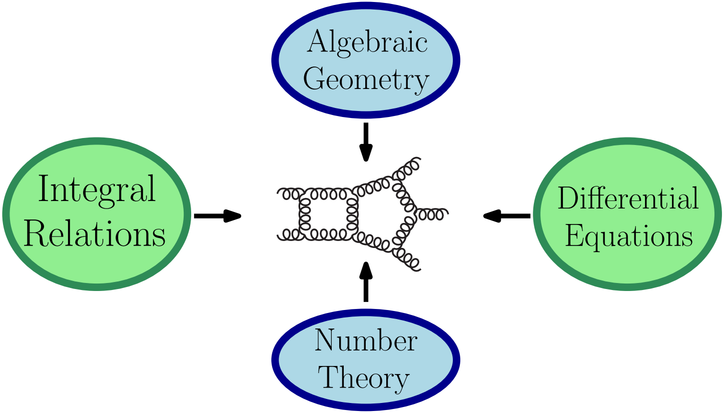


Mathematics tells us the important questions. Physics controls the answers.

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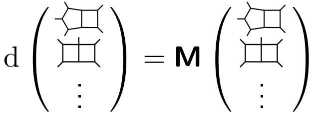
- Must only compute basis modulo relations. (Cohomology classes).

$$\underbrace{\{\overline{\mathcal{I}}_1, \overline{\mathcal{I}}_2, \ldots\}}_{100\,000\text{s of integrals}} \xrightarrow{\overline{\mathcal{I}}_i = \mathcal{R}_{ij}\mathcal{I}_j} \underbrace{\{\mathcal{I}_1, \mathcal{I}_2, \ldots\}}_{\mathcal{O}(100)\text{ master integrals}}$$

- Derivatives of Feynman integrals are again Feynman integrals!

[Kotikov '91; Remiddi '97; Gehrmann, Remiddi '01]



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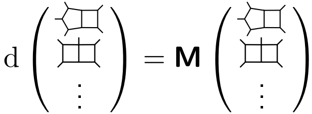

- Multiparticle physics all wrapped up into single object

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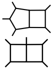
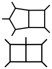
d (  ⋮ ) = **M** (  ⋮ )

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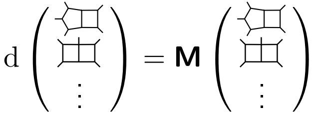


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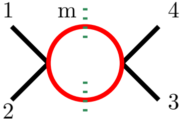
- Algebraic matrix $\mathbf{M}$. Singularities at (generalized) thresholds.




$$\Rightarrow \qquad \mathbf{M} \sim \frac{\mathbf{M}_{-1}}{s_{12} - 4m^2} + \mathcal{O}(s_{12} - 4m^2)^0.$$

# From Theory to Calculation

# The Philosophy

> "Calculus is hard. Algebra is easy."
>
> — Unknown (heard from David Kosower)

# The Reality

"Algebra is also hard."

— Your average high schooler/precision physicist

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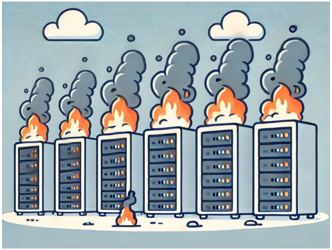
Practical Challenges:

- Ludicrously large expressions.
- Many variables to manipulate.

$\longrightarrow$



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$$\begin{pmatrix} \omega(\phi_n^{(1)}) \\ \omega(\phi_n^{(2)}) \end{pmatrix} = \begin{pmatrix} \mathfrak{a}_1(\phi_n^{(1)}) & \cdots & \mathfrak{a}_N(\phi_n^{(1)}) \\ \mathfrak{a}_1(\phi_n^{(2)}) & \cdots & \mathfrak{a}_2(\phi_n^{(2)}) \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_N \end{pmatrix}.$$

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$$\begin{pmatrix} 0.680327 \\ \vdots \\ 0.0901156 \end{pmatrix} = \begin{pmatrix} 0.453731 & \cdots & 0.757166 \\ \vdots & \ldots & \vdots \\ 0.419842 & \cdots & 0.122505 \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_N \end{pmatrix}.$$

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- Precision/arithmetic issues avoided by working modulo a large prime.

$$c_i \longleftrightarrow c_i \mod p \qquad \mathbb{Q} \longleftrightarrow \mathbb{F}_p.$$

[Schabinger, von Manteuffel '14; Peraro '16]

# The Finite Field Ansatz Approach

- Consider an Ansatz for a function/form $\omega$, that we can evaluate.

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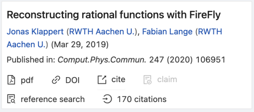
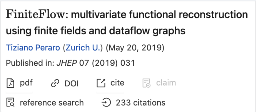
We reconstruct $\omega$ from evaluations, bypassing computer algebra!

# The Finite Field/Ansatz Revolution

- Now multiple public tools implementing this approach


Reconstructing rational functions with FireFly
Jonas Klappert (RWTH Aachen U.), Fabian Lange (RWTH Aachen U.) (Mar 29, 2019)
Published in: *Comput.Phys.Commun.* 247 (2020) 106951
pdf DOI cite claim
reference search 170 citations


FiniteFlow: multivariate functional reconstruction using finite fields and dataflow graphs
Tiziano Peraro (Zurich U.) (May 20, 2019)
Published in: *JHEP* 07 (2019) 031
pdf DOI cite claim
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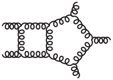
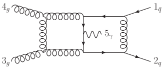
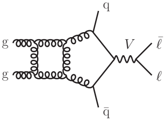
- Approach powering many computations of frontier amplitudes.


3 jet,



$\gamma\gamma$ + jet,



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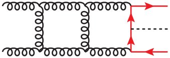
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... many more

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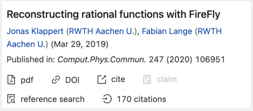


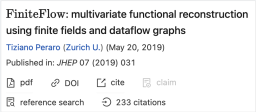


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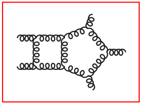

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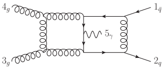

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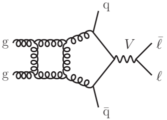


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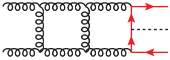

Higgs $+b\bar{b}$

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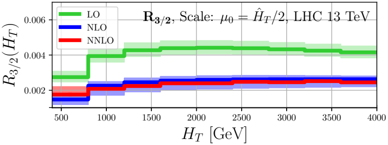
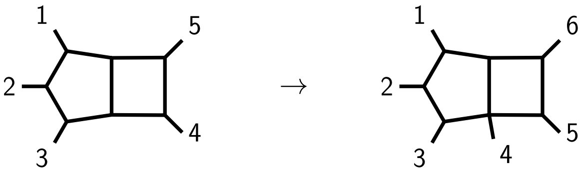
- Highlight: three-jets. Allowed for precise 3-jet/$\alpha_S$ measurements.



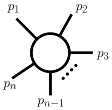

[Abreu, BP, Febres Cordero, Ita, Sotnikov '21; Czakon, Mitov, Poncelet '21; ATLAS '23; ATLAS '24]

# New Frontiers



# Phase Space at Higher Multiplicity

- Amplitudes are functions defined on momentum space:




$$\rightarrow \qquad \sum_{i=1}^{n} p_i^\mu = 0, \qquad p_i^2 = m_i^2.$$

- Feynman integrals are Lorentz-invariant. Use Mandelstam variables:

$$s_{ij} = (p_i + p_j)^2, \qquad \mathrm{d} = \mathrm{d}s_{12} \frac{\partial}{\partial s_{12}} + \cdots.$$

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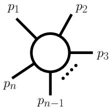


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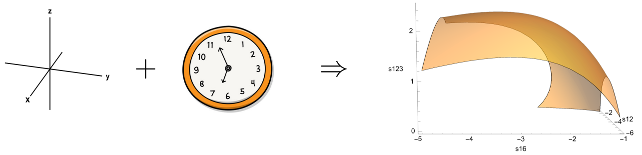
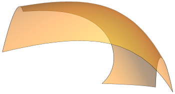
- Space-time is 4 dimensional $\Rightarrow$ Mandelstams constrained for $n \geq 6$.



- Mandelstam polynomials live in "coordinate ring" of this surface.

$p(\vec{s}) \in \mathbb{C}$ [  ] .    $p_1 \sim p_2 \Leftrightarrow p_1|$  $= p_2|$  .

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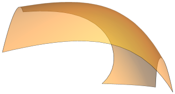
$p(\vec{s}) \in \mathbb{C}\Big[$  $\Big]$.  $p_1 \sim p_2 \Leftrightarrow p_1|$ $= p_2|$.

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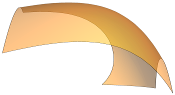
$p(\vec{s}) \in \mathbb{C}\Big[$  $\Big]$.  $p_1 \sim p_2 \;\Leftrightarrow\; p_1|$ $= p_2|$.

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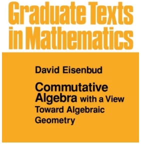
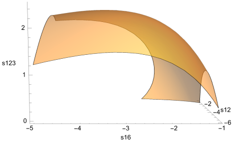


:  "You're working with $\mathbb{C}[$$][\sqrt{G(p_1, p_2, p_3, p_4)}]$ and you don't understand its algebra!"

# Algebraic Geometry of the Square Root

- Introducing root gave us new variable defining self-intersecting variety

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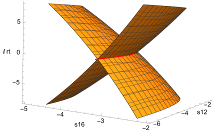

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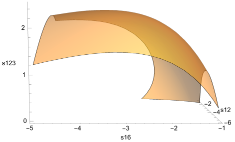


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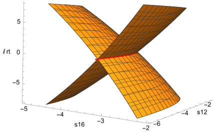


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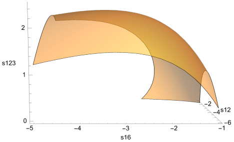


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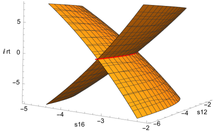


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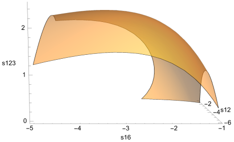


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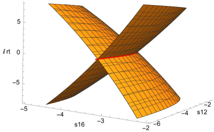


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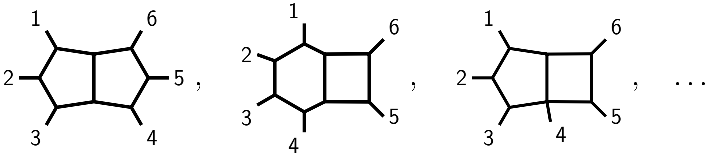
Computation controlled by physics!

# Towards Six-Point Scattering at Two Loops

- Algebraic geometry of phase space key for 6-point computations.




[Abreu, Monni, BP, Usovitsch '24; Henn, Matijasic, Miczajka, Peraro, Xu, Zhang '24, '25]

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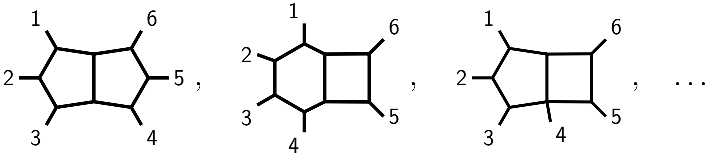


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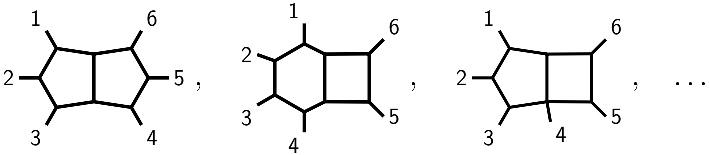


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- First steps towards era of precise predictions for six-point processes:
  - $pp \rightarrow 4j$
  - $pp \rightarrow t\bar{t}b\bar{b}$
  - $pp \rightarrow t\bar{t}t\bar{t}$
  - . . . .

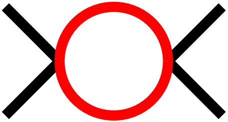
# Summary

- Precise understanding of fundamental physics at colliders leads to the challenging problem of computing Feynman integrals.

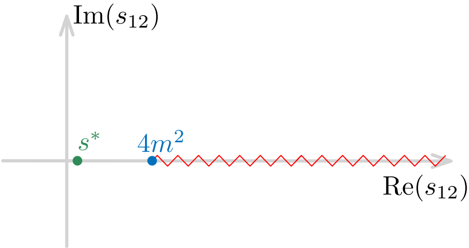
- Mathematical insights from algebraic geometry and number theory power cutting edge computations, physics makes them possible.

- As we push towards precise high-multiplicity scattering at the LHC, we uncover new computational and mathematical challenges.

# Feynman Integrals: Complex Analytic Functions



$[s^*] \quad \sim$



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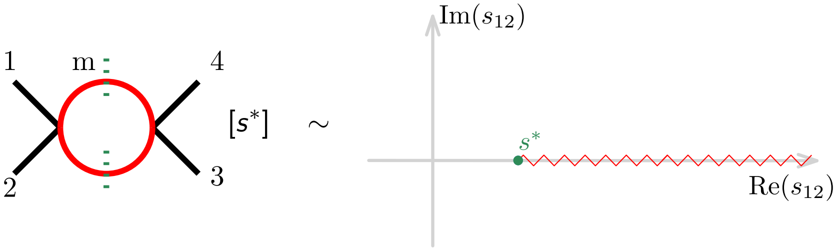


- Logarithmic and algebraic branch points, e.g.
  - when intermediate particles go "on-shell".

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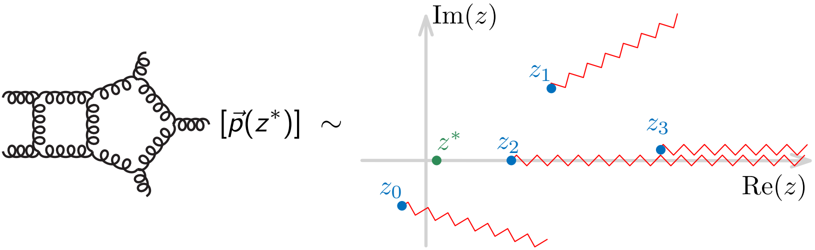


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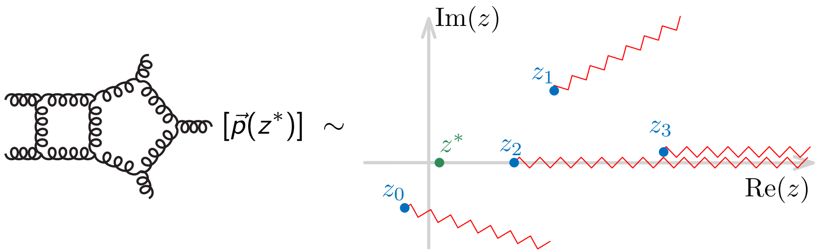


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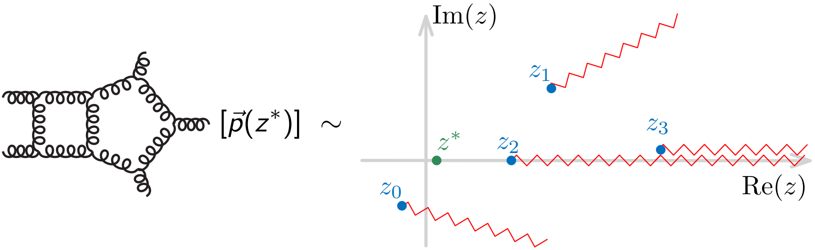


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Highly complicated analytic functions.

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- Let $f$ be $\mathcal{C}_k$ or $M_{jk}$. Typically algebraic and Lorentz invariant.

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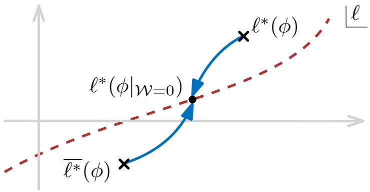


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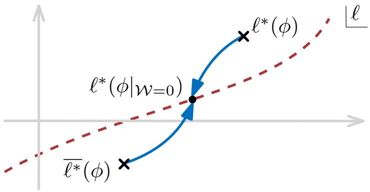


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- Problem reduced to understanding numerator polynomial.

$$f(\vec{s}) = \frac{\mathcal{N}(\vec{s}, \sqrt{\mathcal{W}_{\text{root}}}, \ldots)}{\prod_k \mathcal{W}_k(\vec{s})^{q_k}}.$$

# Solving the Differential Equation

- Conjecture: exists "special" basis in dim reg.

$$\vec{I} = U\vec{J} \qquad : \qquad \mathrm{d}\vec{J} = \epsilon \mathbf{M}(\vec{s})\vec{J}.$$

- Solve $\vec{J}$ as series expansion in dim-reg $\epsilon$

$$\vec{J} = \mathbb{P} \exp \left[ \epsilon \int_\gamma \mathbf{M} \right] \vec{J}_0$$

- Entries of $\mathbf{M}$ determine special functions, e.g.

$$\mathrm{d}\log(x-a) \qquad \sim$$