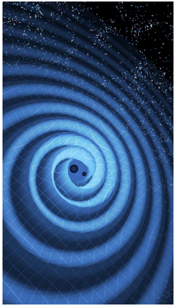
Perturbation Theory of Spinning Black Holes, leaving Vacuum GR



Simon Maenaut

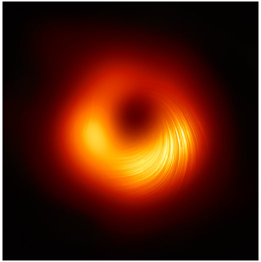
Be.HEP meeting

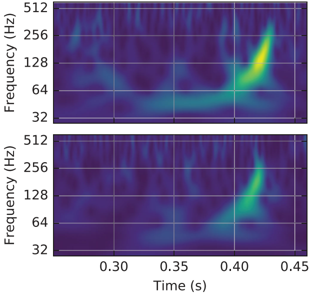
21 June 2024

KU LEUVEN

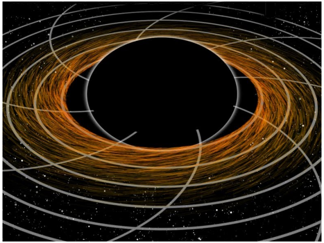
Black Holes

- Black holes are strange and special objects
- Uniquely defined by mass, rotation and charge*
- No additional characteristics (hairs) of a black hole
- Theoretically, several aspects still remain puzzling
- Astrophysical evidence grew in the last decades





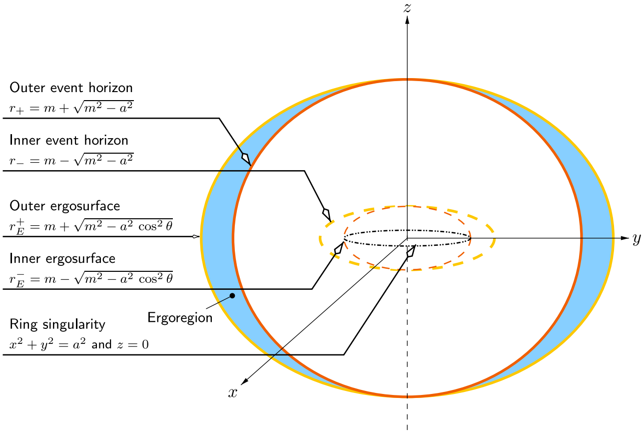
Spinning Black Holes



- First black hole found in 1916 by Karl Schwarzschild
  - A spherically symmetric and static black hole
- In 1963, Roy Kerr discovered the exact solution for a rotating black hole
  - An axially symmetric black hole that carries $J = a\,M$ angular momentum
  - Maximal value for the angular momentum $J \leq G\,M^2/c$ (extremal limit)

Kerr spacetime

- Ergoregion (outer and inner)
- Horizon (outer and inner)
- Ring singularity



Outer event horizon
$r_+ = m + \sqrt{m^2 - a^2}$

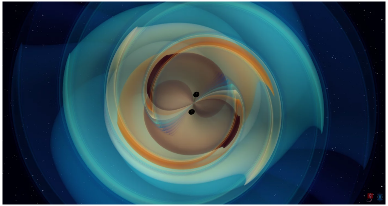
Inner event horizon
$r_- = m - \sqrt{m^2 - a^2}$

Outer ergosurface
$r_E^+ = m + \sqrt{m^2 - a^2\,\cos^2\theta}$

Inner ergosurface
$r_E^- = m - \sqrt{m^2 - a^2\,\cos^2\theta}$

Ring singularity
$x^2 + y^2 = a^2$ and $z = 0$

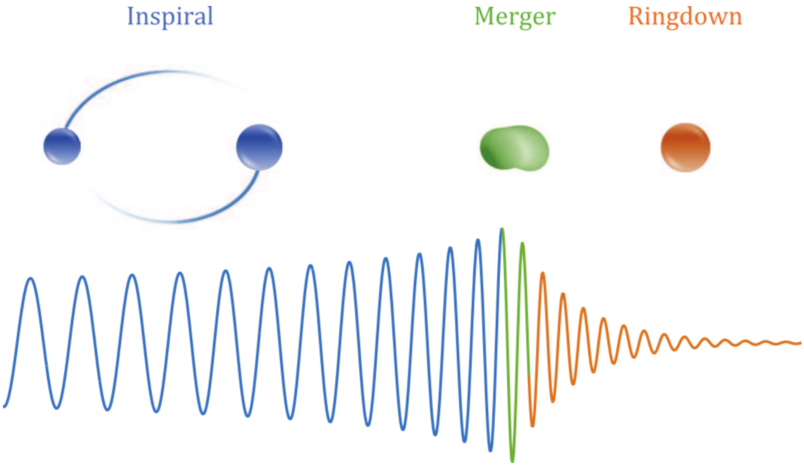
Gravitational wave signals



Ripples in the fabric of space-time

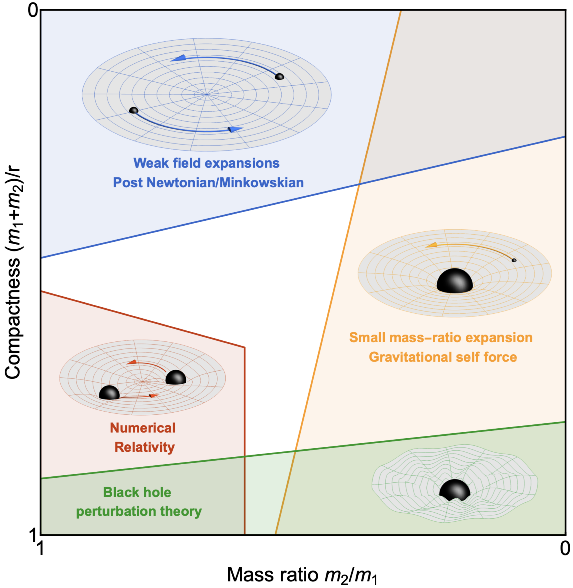
- Gravitational wave bursts (supernovae explosions)
- Compact binary coalescence (binary black holes / neutron stars)
- Continuous gravitational waves (deformed fast spinning neutron stars)
- Stochastic gravitational waves (astrophysical / cosmological)

Black Hole Mergers



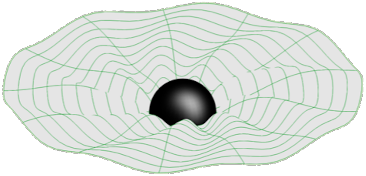
Theoretical modelling

- Post-Newtonian approximation
- Effective-one-body formalism
- Numerical Relativity
- BH perturbation theory
- Small mass-ratio expansion



Quasinormal modes

- Perturbations of black hole decay over time
- GWs come from space around the black hole
- Fluctuations of a damped harmonic oscillator
- Boundary conditions set a dissipative system
- Resonance modes have complex frequencies



Metric perturbations

Consider a small wave-like perturbation $(e^{-i\omega t})$ to the background metric

$$\boxed{g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon\, h_{\mu\nu} + \dots}$$

where $\bar{g}_{\mu\nu}$ is the background black hole metric and $\epsilon\, h_{\mu\nu}$ a small perturbation.

Solve the Einstein Field Equation for the first order corrections

$$\boxed{G_{\mu\nu} = 0 \quad \text{(in vacuum).}}$$

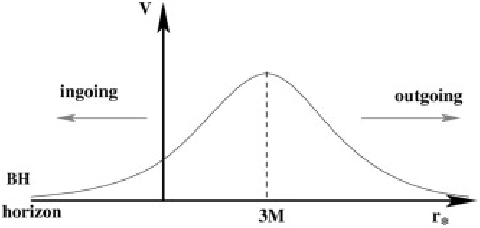
Schwarzschild quasinormal modes

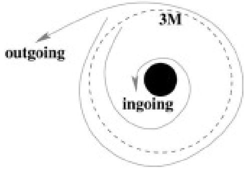
- Separate the metric perturbation in two parity modes $h^{\pm}_{\mu\nu}$
- This can be done with the Regge-Wheeler and Zerelli gauge
- Separate the radial and angular components (spherical harmonics)
- Fill the perturbed metric in the linearised Einstein Field Equations
- Reduce the equations to a single variable differential equation $\Psi_{\pm}$

Master equation

$$\frac{\mathrm{d}^2\Psi_\pm}{\mathrm{d}{r_*}^2} + \left[\omega^2 - V_\pm(r)\right] = 0$$

- Solve wave-equation with correct boundary cond.
- A discrete spectrum of complex QNM frequencies
- Surprisingly, polar and axial modes are isospectral





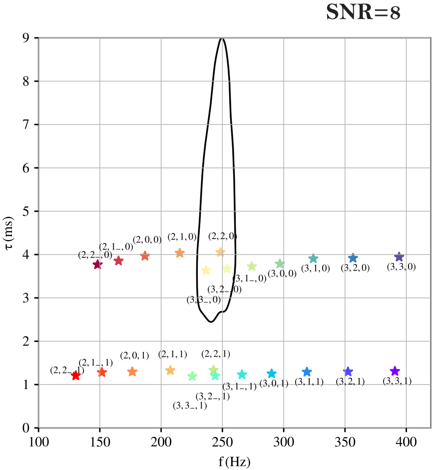
$$\sum A_k \exp(-i\omega_k t)$$

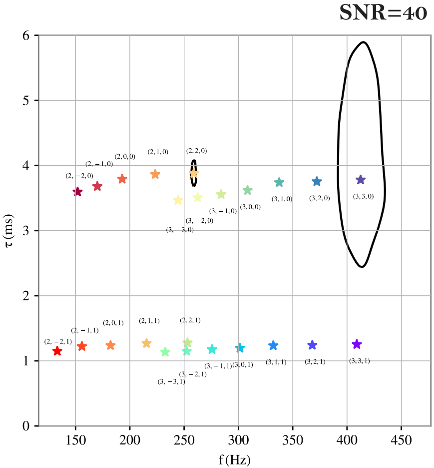
Kerr quasinormal modes

- Kerr metric perturbations do not separate → Teukolsky formalism
- Based on Newman-Penrose decomposition with the Kinnersley Tetrad
- Angular equation → Solved in terms of spin-weighted spheroidal harmonics
- Radial equation → Complex, long ranged potential for wave equation
- Frequencies ω_{lmn} depend uniquely on mass M and spin a

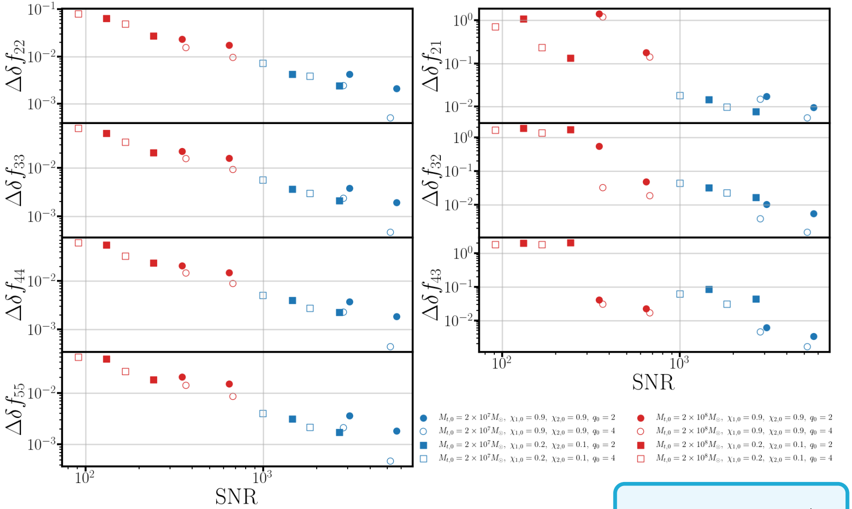
$$\boxed{\Delta^{-s+1}\frac{\mathrm{d}}{\mathrm{d}r}\left[\Delta^{s+1}\frac{\mathrm{d}\,{}_sR_{lm}}{\mathrm{d}r}\right]+V(r)\,{}_sR_{lm}=0}$$

Future sensitivity



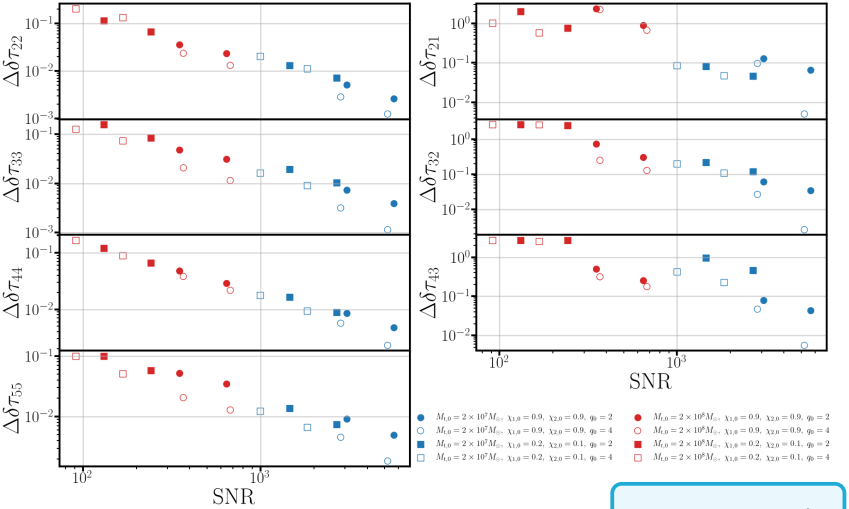


LISA



$$\omega = 2\pi f - i/\tau$$

LISA



$$\omega = 2\pi f - i/\tau$$

Perturbative corrections

- Non-vacuum background (environmental effects):
    - Accretion discs, dark matter spikes, scalar clouds, . . .
- Corrections to General Relativity (beyond GR effects):
    - In 4 dimensions, GR is unique under some assumptions (Lovelock's theorem)
    - Breaking one (or more) of these opens a whole zoo of beyond GR theories

For static black holes, the Regge-Wheeler formalism can be adapted

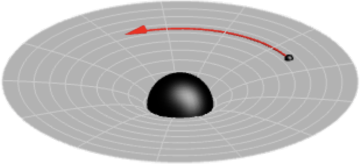
For spinning black holes, since recently, a modified Teukolsky approach exists for solutions that are a small correction to Kerr, by using metric reconstruction

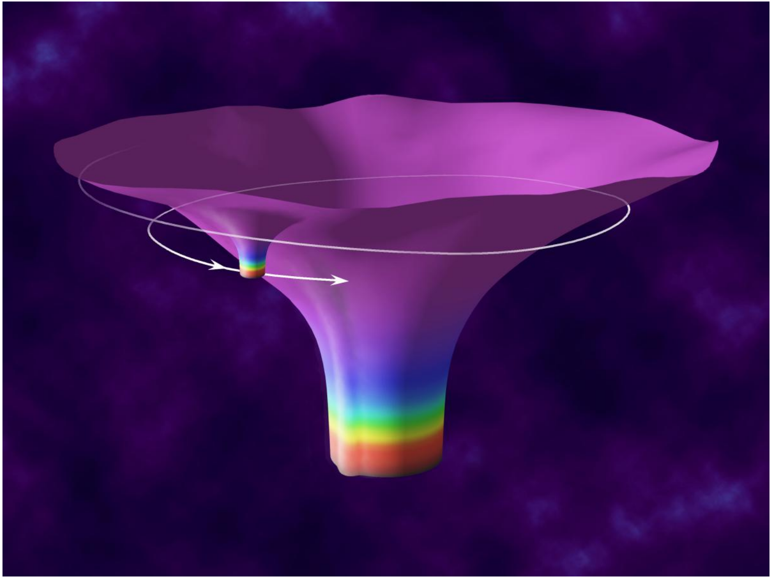
Black hole spectroscopy

- Remnant compact object nature;
  Are we really observing black holes?
- General Relativity predictions for spectral emission;
  Is General Relativity a correct description of gravity at high curvatures?
- Black Hole Uniqueness Theorems;
  Do non-extremal black holes have additional hairs?

Small mass-ratio

- Asymmetric binaries with mass-ratio $\epsilon = m/M \ll 1$
- Perturbation on the background black hole metric
- Secondary will deviate from a geodesic motion
- This effect is a result of the gravitational self-force
- Applicable for $10^{-7} \leq \epsilon \leq 10^{-4}$ (EMRIs) and larger (IMRIs)



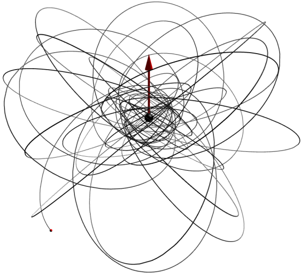


Gravitational self-force

- Expand the metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon\, h_{\mu\nu} + \dots$
  - $\bar{g}_{\mu\nu}$ the background of the primary black hole(Schwarzschild or Kerr metric)
  - $h_{\mu\nu}$ forces secondary away from a geodesic of $\bar{g}_{\mu\nu}$ (gravitational self-force)
- Solve the Einstein Eq. and the trajectory of the secondary order-by-order in ϵ
  - Secondary is described as point particle (puncture), with its multipole moments;
  - The perturbation $h_{\mu\nu}$ can be recovered via matched asymptotic expansions
  - The regular part of $h_{\mu\nu}$ is calculated with black hole perturbation theory

Orbits around Kerr

- GSF computations can be performed in the two-timescale expansion
  - Fast orbital time $\mathcal{O}(1)$, to calculate the phase evolution
  - Slow inspiral time $\mathcal{O}(1/\epsilon)$, to evolve orbital parameters
- Point particles around a Schwarzschild remain in a plane
- Around Kerr the orbit is not confined to a single plane
- Instead these orbits can follow chaotic-looking paths

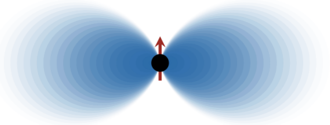


Superradiance

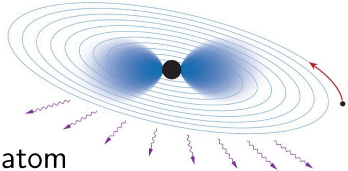
- Consider the presence of a very light boson field ($10^{-15} - 10^{-21}$ eV)
- Quasi-bound states can be amplified around astrophysical black holes
- If the Compton wavelength is comparable to the size of a spinning black hole and the black hole spins fast enough at the horizon such that $\omega < m\,\Omega_{\mathrm{H}}$

$$\boxed{\lambda_C = \hbar/\mu\,c \quad \gtrsim \quad r_{\mathrm{BH}} = GM/c^2}$$

- The "cloud" extracts energy and angular momentum
- Limits the black hole spin for certain mass ranges

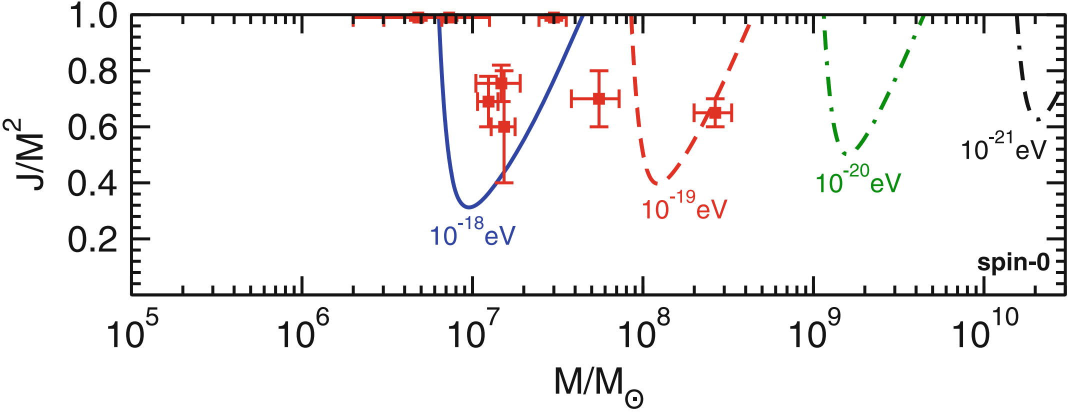


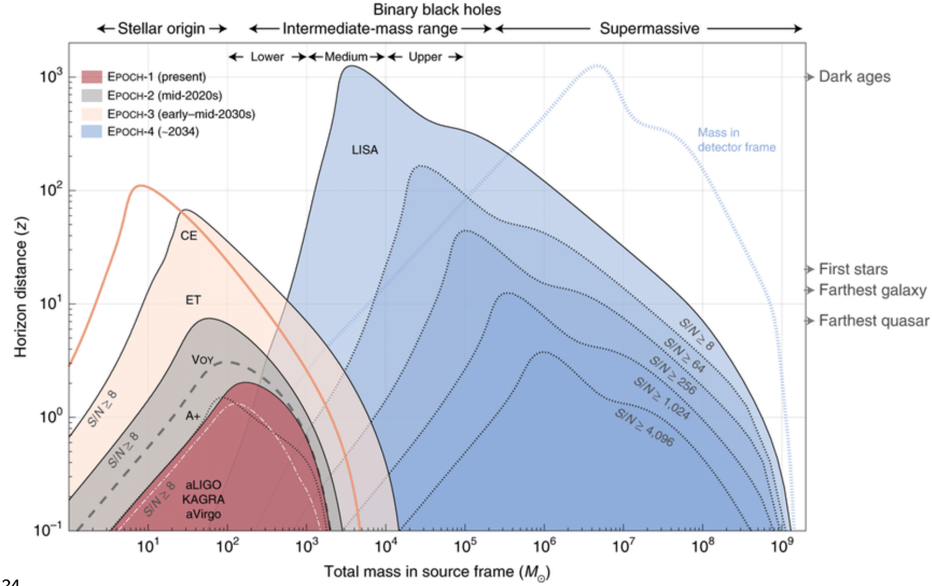
Gravitational Atom



- Structure of the bound states resembles the Hydrogen atom
- A binary companion can induce resonant transitions between bound states
- The back-reaction on the binary's orbit leads to characteristic signatures
- These could be detectable if many cycles of the inspiral are observed
- The mass of the primary black hole determines the sensitive mass range
    - Stellar mass black holes $5 - 10^2\,\mathrm{M}_\odot$ probe fields with $m \sim 10^{-11} - 10^{-13}\,\mathrm{eV}$
    - Intermediate black holes $10^2 - 10^5\,\mathrm{M}_\odot$ probe fields with $m \sim 10^{-13} - 10^{-16}\,\mathrm{eV}$
    - Supermassive black holes $> 10^5\,\mathrm{M}_\odot$ probe lighter fields down to $10^{-21}\,\mathrm{eV}$

Existing constraints




Binary black holes
Stellar origin
Intermediate-mass range
Supermassive
Lower
Medium
Upper
Epoch-1 (present)
Epoch-2 (mid-2020s)
Epoch-3 (early–mid-2030s)
Epoch-4 (~2034)
LISA
CE
ET
Voy
A+
aLIGO
KAGRA
aVirgo
S/N ≥ 8
S/N ≥ 8
S/N ≥ 8
S/N ≥ 8
S/N ≥ 64
S/N ≥ 256
S/N ≥ 1,024
S/N ≥ 4,096
Mass in detector frame
Dark ages
First stars
Farthest galaxy
Farthest quasar
Horizon distance (z)
10³
10²
10¹
10⁰
10⁻¹
10¹
10²
10³
10⁴
10⁵
10⁶
10⁷
10⁸
10⁹
Total mass in source frame (M⊙)

Some open challenges

- Quasi-normal modes
    - Amplitudes of quasi-normal mode perturbations
    - QNMs with additional degrees of freedom beyond GR
- Extreme-mass ratio inspirals
    - Generic second-order self-force calculations on Kerr
    - First-order corrections of non-vacuum contributions

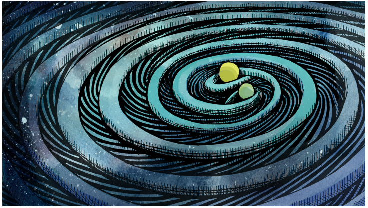
Summary

- We can study to study black hole astrophysics with perturbation theory
- Allowing for rotation enriches the phenomenology of black holes
- There are still theoretical challenges regarding spinning black holes
- Including non-vacuum or beyond GR effects remains a non-trivial matter
- Future GW detectors will detect many more black holes in a wider mass range

Thank you for your attention!

Joint Theoretical Gravitational Waves Seminars

- The series started earlier this year in February
- Scheduled for a Friday afternoon once per month
- In Brussels, currently at the University Foundation
- Introduction of the topic by local PhD students
- One or two expert speakers around a theme



Mailing list:

https://ls.kuleuven.be/cgi-bin/wa?SUBED1=GWSEMINARS&A=1

Newman-Penrose Formalism

- Introduce a tetrad frame of four null vectors $l^\mu, n^\mu, m^\mu, \bar{m}^\mu$
- Require that the tangent space metric is specifically set
- Two of these null vectors have to be complex $m^\mu, \bar{m}^\mu$
- Final metric is real so these two are complex conjugate
- The real tetrads l^μ, n^μ label in- and outgoing null directions

$$\boxed{g_{\mu\nu} = -l_\mu n_\nu - n_\mu l_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu}$$

Weyl Scalars

- Traceless part of the Riemann tensor, is the Weyl tensor $C_{\alpha\beta\mu\nu}$
- Can be written in terms of 5 independent complex components
- Invariant under diffeomorphisms, dependant on the tetrad basis

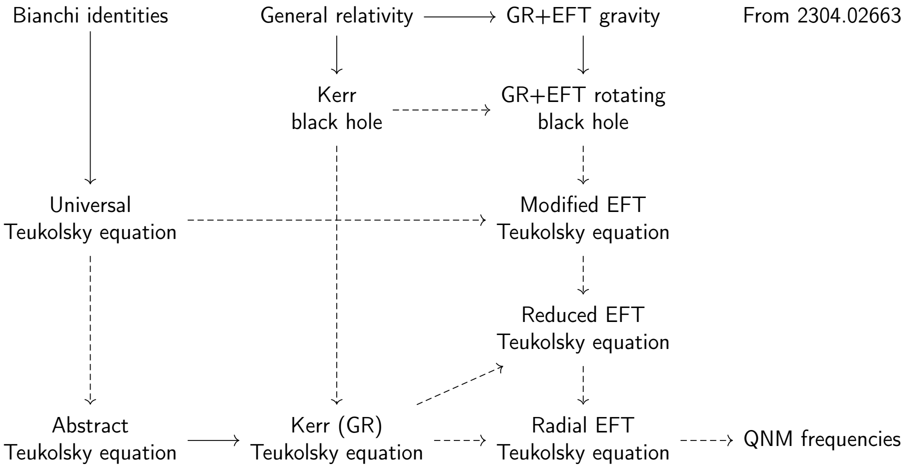
$$\Psi_0 = C_{\alpha\beta\mu\nu} l^\alpha m^\beta l^\mu m^\nu \quad \Psi_1 = C_{\alpha\beta\mu\nu} l^\alpha n^\beta l^\mu m^\nu \quad \Psi_2 = C_{\alpha\beta\mu\nu} l^\alpha m^\beta \bar{m}^\mu n^\nu$$

$$\Psi_3 = C_{\alpha\beta\mu\nu} l^\alpha n^\beta \bar{m}^\mu n^\nu \quad \Psi_4 = C_{\alpha\beta\mu\nu} n^\alpha \bar{m}^\beta n^\mu \bar{m}^\nu$$

Teukolsky Equations

- Kerr metric is special (Petrov type D), especially with the Kinnersley tetrad
- Its only non-vanishing Weyl scalar is $\Psi_2 = -M/\zeta$ with $\zeta = r - ia\cos\theta$
- Perturbations can be solved in terms of $\delta\Psi_0$ or $\delta\Psi_4$ equivalently
- The equations for these two perturbed Weyl scalars are separable
- Physically, $\delta\Psi_0$ and $\delta\Psi_4$ describe the two polarization modes of GWs

$$\boxed{\psi = R^s_{lm}(r) S^s_{lm}(\cos\theta) \exp(-i\omega t + im\phi)}$$



Bianchi identities

General relativity

GR+EFT gravity

From 2304.02663

Kerr
black hole

GR+EFT rotating
black hole

Universal
Teukolsky equation

Modified EFT
Teukolsky equation

Reduced EFT
Teukolsky equation

Abstract
Teukolsky equation

Kerr (GR)
Teukolsky equation

Radial EFT
Teukolsky equation

QNM frequencies