Summary of activity: The Nonlinear Analysis and Differential Equations group at UCLouvain

Visit of the scientific committee @IRMP
Louvain-la-Neuve, 26 April 2024

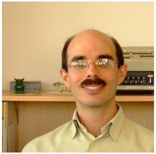
UCLouvain

People

Academic permanent members of the Nonlinear Analysis and Differential Equations group:

- Jean Van Schaftingen (since 2006/07)

- Augusto Ponce (since 2008)



- Heiner Olbermann (since 2018)

Overview

- In the evaluation period, there have been 6 postdocs and 5 PhD students in the Analysis/PDE group
- Major results in the following research areas (as listed in the report):
    - Choquard equation
    - Discrete-to-continuum limits
    - Ginzburg-Landau-problems
    - Lake equations
    - Magnetic problems
    - Manifold-valued mappings
    - Optimal design, Γ-convergence
    - Schrödinger operator
    - Sobolev functions
    - Singular integrals
- Prizes and recognitions:
    - Prix Jacques Deruyts (Jean Van Schaftingen, 2020)

Pars pro toto:

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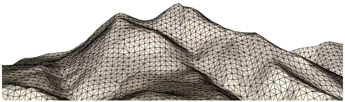
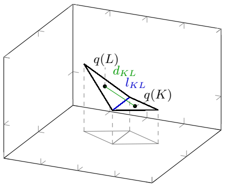
- **Peter Gladbach (Bonn), Heiner Olbermann: Approximation of the Willmore energy by a discrete geometry model**, *Adv. Calc. Var. 16 (2023), 403–424*
- Discrete differential geometry aims at developing discrete analogues to concepts known from (continuous) differential geometry
- Objects of interest: *Polyhedral surfaces*, i.e. finite collections $\mathcal{T} = \{K_i \,:\, i = 1, \ldots, N\}$ of regular triangles $K_i = [x_i, y_i, z_i] \subset \mathbb{R}^3$



- These objects are of interest in applied mathematics and engineering, in particular in *computer graphics*
- For the mathematical statements in the paper, it is assumed that polyhedral surfaces are given as *graphs*

Discrete-to-continuum limit: Main result

Discrete Willmore functional associated to a polyd-edral surface:



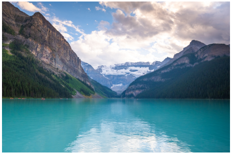
$$E(\mathcal{T}) = \sum_{\substack{\text{neighboring} \\ \text{triangles } K,L}} \frac{l_{KL}}{d_{KL}} |n(K) - n(L)|^2$$

where l_{KL} is the length of the shared side of $K, L \in \mathcal{T}$, d_{KL} is the distance between the centers of circumcircles of K, L, $n(K), n(L)$ denotes the surface normal.

- Note: This is *not* a standard approximation as in the theory of finite elements
- Suppose $\mathcal{T}_\varepsilon$ is a polyhedral surface that minimizes E under some suitable boundary conditions within the class of polyhedral surfaces that consist of (uniformly) regular triangles of diameter $\simeq \varepsilon$, and are graphs
- *Theorem:* Considering a sequence of such minimizers for $\varepsilon \to 0$, this sequence converges to a minimizer of the Willmore functional $\int |Dn|^2 \mathrm{d}\mathcal{H}^2$
- The rigorous claim is a statement about Γ-convergence of the involved functionals

Lake equation

- **Justin Dekeyser and Jean Van Schaftingen, Vortex motion for the lake equations**, *Comm. Math. Phys.* 375 (2020), n. 2, 1459-1501
- This work considers the system of equations



$$\begin{cases} \nabla \cdot (b\mathbf{u}) = 0 & \text{on } \mathbb{R} \times D, \\ \partial_t \mathbf{u} + (u \cdot \nabla)\mathbf{u} = -\nabla h & \text{on } \mathbb{R} \times D, \\ \mathbf{u} \cdot \nu = 0 & \text{on } \mathbb{R} \times \partial D, \end{cases}$$

- ...where $D \subset \mathbb{R}^2$ can be identified with the "lake" of depth $b : D \to (0, \infty)$, $h : \mathbb{R} \times D \to \mathbb{R}$ is the surface height, and $\mathbf{u} : \mathbb{R} \times D \to \mathbb{R}^2$ is the velocity field
- For the case $b = \mathrm{const.}$ this is just the two-dimensional inviscid Euler system
- The *vorticity* of $\mathbf{u}$ is given by $\omega = \nabla \times \mathbf{u}$
- It is well known that there exist distributional solutions of the Euler system where the vorticity consists of δ-type distributions whose position is governed by a dynamical system. These solutions are limits of smooth solutions of the Euler system

Lake equation: Main result

- Technical assumption: b is constant on the connected components of ∂D
- The main result concerns sequences $(\mathbf{u}_k, h_k)$ of classical solutions to the lake equation such that the vorticity at time zero converges (after renormalization) to a δ-type distribution
- It is proved that the vorticity of these solutions converges (again, after renormalization) towards a δ-type distribution whose position q is governed by the equation

$$q'(s) = -(\nabla^{\perp} b^{-1})(q(s))$$

  where $\nabla^{\perp} b^{-1} = (\partial_2 b^{-1}, -\partial_1 b^{-1})$.
- A similar result is proved for the energy density

Schrödinger operators

- **Luigi Orsina (Rome) and Augusto Ponce, On the nonexistence of Green's function and failure of the strong maximum principle**, *J. Math. Pures Appl.* (9) 134 (2020), 72–121

- This paper is a study of the role of the *universal zero-set* Z for the Schrödinger operator $-\Delta + V$, where $V : \Omega \to [0, +\infty]$ is a Borel function, and makes clear how strong maximum principles and existence of Green's functions depend on this set

- Given V, the universal zero-set Z is defined as the set of all points x such that whenever $f \in L^\infty(\Omega)$ and $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ is a distributional solution of

$$-\Delta u + Vu = f$$

  we have that $u(x) = 0$ (where here and in the following we assume that u agrees with its "precise representative")

Ω=open connected subset of $\mathbb{R}^n$ with smooth boundary

Schrödinger operators: Main results

- On every (Sobolev-)connected component D of the (Sobolev-)open set $\Omega \setminus Z$, the strong maximum principle holds in the sense that for any solution u of $-\Delta u + Vu = f$ with non-negative $f \in L^\infty(\Omega)$, one has the dichotomy

$$\text{either } u > 0 \text{ on } D \quad \text{or} \quad u = 0 \text{ on } D\,.$$

- Given $x \in \Omega$, there exists a Green's function $G_x \in W_0^{1,1} \cap L^1(\Omega; V\mathrm{d}x)$ satisfying $-\Delta G_x + VG_x = \delta_x$ if and only if $x \in \Omega \setminus Z$. For such x, and for every solution u of $-\Delta u + Vu = f$ with $u \in W_0^{1,2} \cap L^\infty(\Omega)$, one has $u(x) = \int_\Omega G_x f$.
- The Dirichlet problem

$$\begin{cases} -\Delta u + Vu = \mu \text{ in } \Omega \\ \qquad\qquad u = 0 \text{ on } \partial\Omega \end{cases}$$

with a non-negative finite Borel measure μ possesses a distributional solution if and only if $\mu(Z) = 0$.