Formalism for power spectral density estimation for non-identical and correlated noise using the null channel in Einstein Telescope

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Paper accessible at: https://arxiv.org/abs/2205.00416

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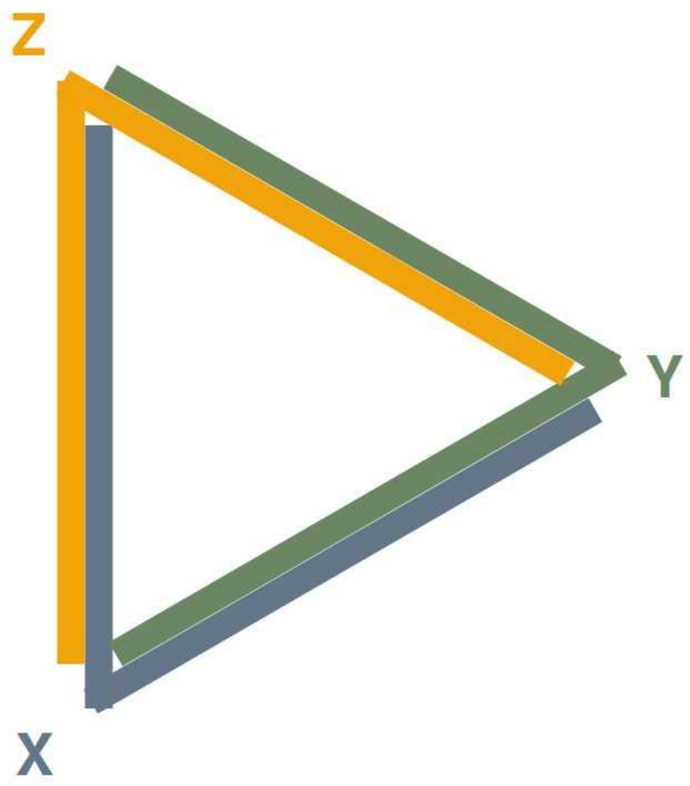
Introduction of the formalism



$$
\begin{aligned}
A &= \frac{1}{\sqrt{2}}(Z - X) \\
E &= \frac{1}{\sqrt{6}}(X - 2Y + Z) \qquad (6) \\
T &= \frac{1}{\sqrt{3}}(X + Y + Z).
\end{aligned}
$$

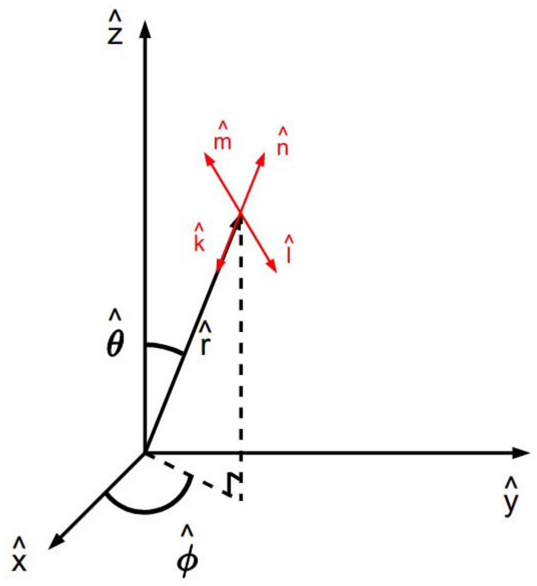
$$
\begin{cases}
\mathbf{e}_X = \frac{1}{2}\left(\sqrt{3}, -1, 0\right) \\
\mathbf{e}_Y = \frac{1}{2}\left(\sqrt{3}, 1, 0\right) \\
\mathbf{e}_Z = (0, 1, 0)
\end{cases}
$$

Null Channel T

$$T = \frac{1}{\sqrt{3}}(X + Y + Z)\text{, with}$$

$$I = n^I(t) + d^I_{ij}h^{ij}(t), (I = [X, Y, Z]) \quad F^I_{+,\times} = d^I_{ij}e^{ij}_{+,\times}$$

$$\begin{cases} F^Y_{+,\times}(\theta, \phi, \psi) = F^X_{+,\times}(\theta, \phi + \frac{2\pi}{3}, \psi) \\ F^Z_{+,\times}(\theta, \phi, \psi) = F^X_{+,\times}(\theta, \phi - \frac{2\pi}{3}, \psi) \end{cases}$$



By combination of the three detector output

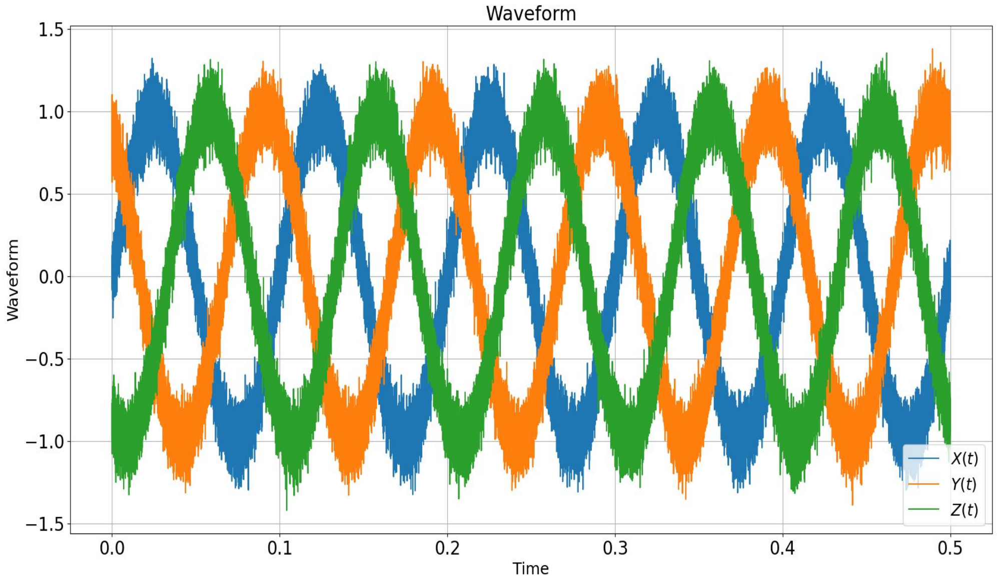
$$T = \frac{1}{\sqrt{3}}\sum_I \left[n^I(t) + d^I_{ij}h^{ij}(t)\right] = \frac{1}{\sqrt{3}}\sum_I n^I(t)$$

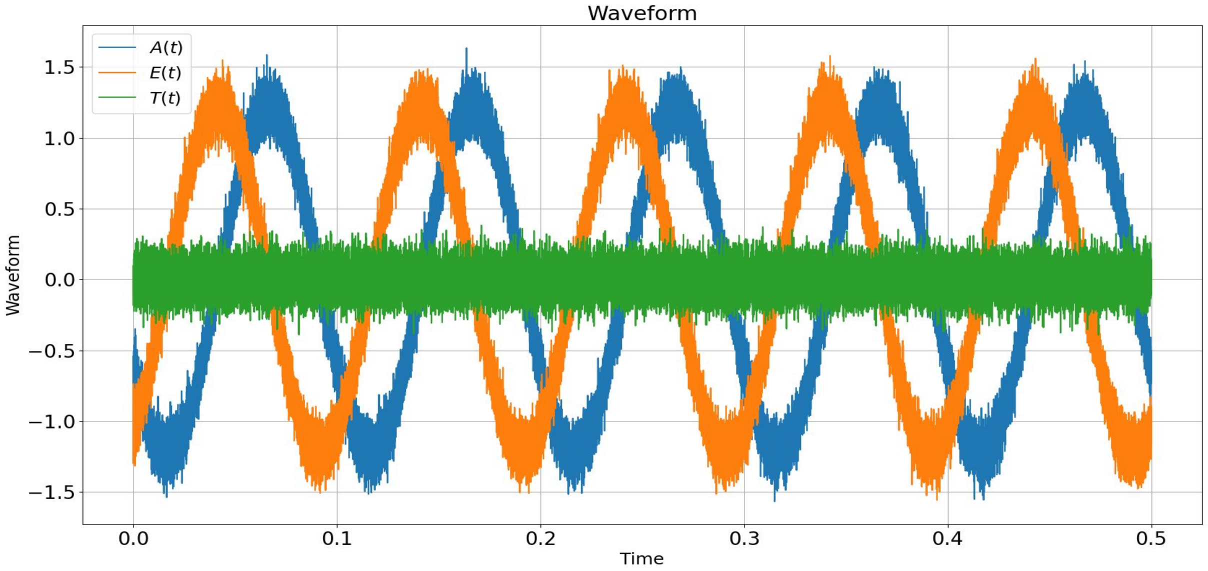
$\Longrightarrow$ No GW signal in the T channel

https://arxiv.org/abs/1201.3563

Toy Model Null channel

Toy model, 3 sinusoidal signals dephased by 2π/3 phase + 3 independent Gaussian noise.





The formalism
Identical noise in X, Y and Z

$$S_n^{XY}(f) = S_n^{XZ}(f) = S_n^{YZ}(f) \equiv S_n^{IJ}(f)$$

$$S_n^{X}(f) = S_n^{Y}(f) = S_n^{Z}(f) \equiv S_n^{I}(f)$$

$$\langle T(f)T^*(f')\rangle = \frac{1}{2}\delta(f-f')\left[S_n^I(f) + 2S_n^{IJ}(f)\right]$$

$$\langle A(f)A^*(f')\rangle = \frac{1}{2}\delta(f-f')\left[S_n^I(f) - S_n^{IJ}(f) + \frac{9}{20}S_h(f)\right]$$

$$\langle E(f)E^*(f')\rangle = \frac{1}{2}\delta(f-f')\left[S_n^I(f) - S_n^{IJ}(f) + \frac{9}{20}S_h(f)\right]$$

$$\langle T(f)A^*(f')\rangle = \langle A(f)T^*(f')\rangle = 0$$
$$\langle T(f)E^*(f')\rangle = \langle E(f)T^*(f')\rangle = 0$$
$$\langle E(f)A^*(f')\rangle = \langle A(f)E^*(f')\rangle = 0.$$

For an isotropic SGWB with equal levels of tensor cross- and plus- polarization,

$$S_h^I(f) = \frac{3}{10}S_h(f)$$
$$S_h^{IJ}(f) = -\frac{3}{20}S_h(f).$$

The formalism
Unique noise in X, Y and Z

$$\langle T(f)T^*(f')\rangle = \frac{1}{2}\delta(f-f')\left[\frac{1}{3}(S_n^X(f)+S_n^Y(f)+S_n^Z(f))+\frac{2}{3}(S_n^{XY}(f)+S_n^{XZ}(f)+S_n^{YZ}(f))\right]$$

$$\langle A(f)A^*(f')\rangle = \frac{1}{2}\delta(f-f')\left[\frac{1}{2}(S_n^X(f)+S_n^Z(f))-S_n^{XZ}(f)+\frac{9}{20}S_h(f)\right]$$

$$\langle E(f)E^*(f')\rangle = \frac{1}{2}\delta(f-f')\left[\frac{1}{6}(S_n^X(f)+4S_n^Y(f)+S_n^Z(f))+\frac{1}{3}S_n^{XZ}(f)-\frac{2}{3}(S_n^{XY}(f)+S_n^{YZ}(f))\right.$$
$$\left.+\frac{9}{20}S_h(f)\right]$$

$$\langle T(f)A^*(f')\rangle = \langle A(f)T^*(f')\rangle = \frac{1}{2}\delta(f-f')\left[\frac{1}{\sqrt{6}}(S_n^Z(f)-S_n^X(f)+S_n^{YZ}(f)-S_n^{XY}(f))\right]$$

$$\langle T(f)E^*(f')\rangle = \langle E(f)T^*(f')\rangle = \frac{1}{2}\delta(f-f')\left[\frac{1}{3\sqrt{2}}(S_n^X(f)-2S_n^Y(f)+S_n^Z(f)-S_n^{XY}(f)+2S_n^{XZ}(f)-S_n^{YZ}(f))\right]$$

$$\langle E(f)A^*(f')\rangle = \langle A(f)E^*(f')\rangle = \frac{1}{2}\delta(f-f')\left[\frac{1}{2\sqrt{3}}(-S_n^X(f)+S_n^Z(f)-2S_n^{YZ}(f)+2S_n^{XY}(f))\right],$$

The formalism
Unique noise in X, Y and Z

$$\langle T(f)A^*(f')\rangle = \langle A(f)T^*(f')\rangle = \frac{1}{2}\delta(f-f')\left[\frac{1}{\sqrt{6}}(S_n^Z(f) - S_n^X(f) + S_n^{YZ}(f) - S_n^{XY}(f))\right]$$

$$\langle T(f)E^*(f')\rangle = \langle E(f)T^*(f')\rangle = \frac{1}{2}\delta(f-f')\left[\frac{1}{3\sqrt{2}}(S_n^X(f) - 2S_n^Y(f) + S_n^Z(f) - S_n^{XY}(f) + 2S_n^{XZ}(f) - S_n^{YZ}(f))\right]$$

$$\langle T(f)T^*(f')\rangle = \frac{1}{2}\delta(f-f')\left[\frac{1}{3}(S_n^X(f) + S_n^Y(f) + S_n^Z(f)) + \frac{2}{3}(S_n^{XY}(f) + S_n^{XZ}(f) + S_n^{YZ}(f))\right]$$

$$\Downarrow$$

$$\langle T(f)X^*(f')\rangle = \langle X(f)T^*(f')\rangle = \frac{1}{2}\delta(f-f')\left[\frac{1}{\sqrt{3}}(S_n^X(f) + S_n^{XY}(f) + S_n^{XZ}(f))\right]$$

$$\langle T(f)Y^*(f')\rangle = \langle Y(f)T^*(f')\rangle = \frac{1}{2}\delta(f-f')\left[\frac{1}{\sqrt{3}}(S_n^Y(f) + S_n^{YX}(f) + S_n^{YZ}(f))\right]$$

$$\langle T(f)Z^*(f')\rangle = \langle Z(f)T^*(f')\rangle = \frac{1}{2}\delta(f-f')\left[\frac{1}{\sqrt{3}}(S_n^Z(f) + S_n^{ZX}(f) + S_n^{ZY}(f))\right]$$

ET Correlation noise : Newtonian noise and Schumann resonance

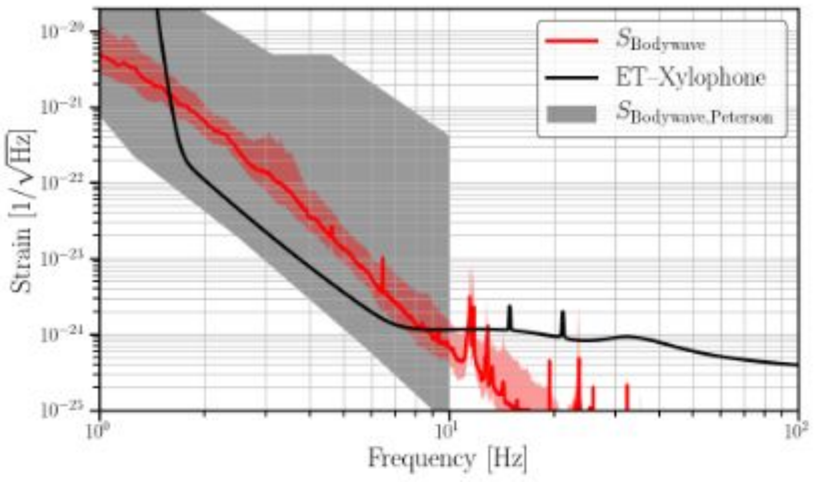


FIG. 13: Strain of the NN with CSD of the Homestakes underground seismometers D2000 and E2000 vertical displacement measurement (see Fig. 8) with a horizontal distance of 405m at a depth of $\sim 610\,\mathrm{m}$ (red curve). The solid line is the body waves NN strain from the 50 % percentile and the surface associated is delimited by the 10^{th} and 90^{th} percentiles CSD. The gray surface, delimited by the low and high limits of Peterson measurement [36], are the body wave NN strain at 610 m depth. The black line is the ET-Xylophone design sensitivity

https://arxiv.org/abs/2206.06809

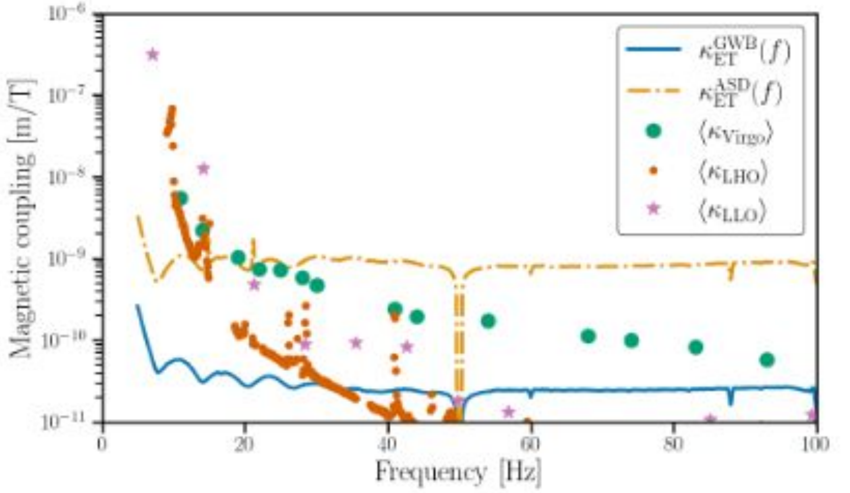


FIG. 3: "ASD" and "GWB" magnetic coupling function upper limits for ET – X design sensitivity. Also included are the average of the measurements of the coupling functions at LIGO Hanford, LIGO Livingston and Virgo during the O3 run for comparison.

https://arxiv.org/abs/2110.14730

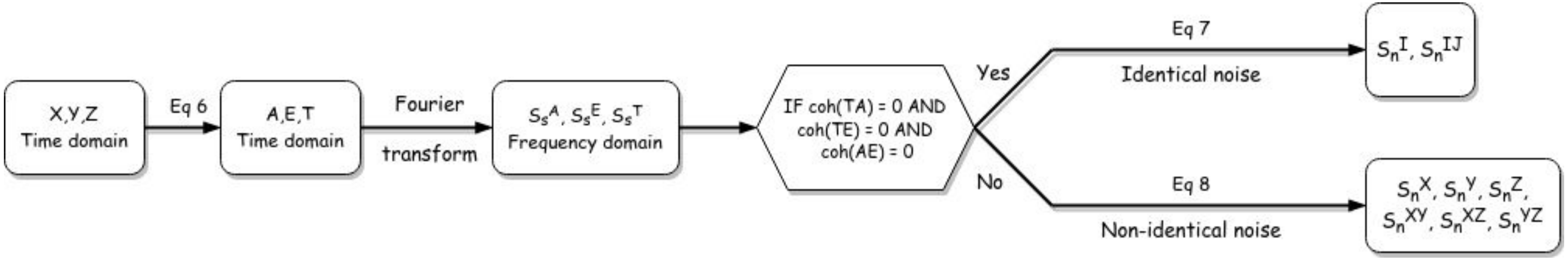


FIG. 1: Diagram summarizing the formalism one can use for the estimation of noise spectral densities. In the presence of non-identical noise more information can be gained by using the CSD between TX, TY and TZ compared to just using the PSD TT, as shown in Eq. 8

Toy example for the Einstein Telescope

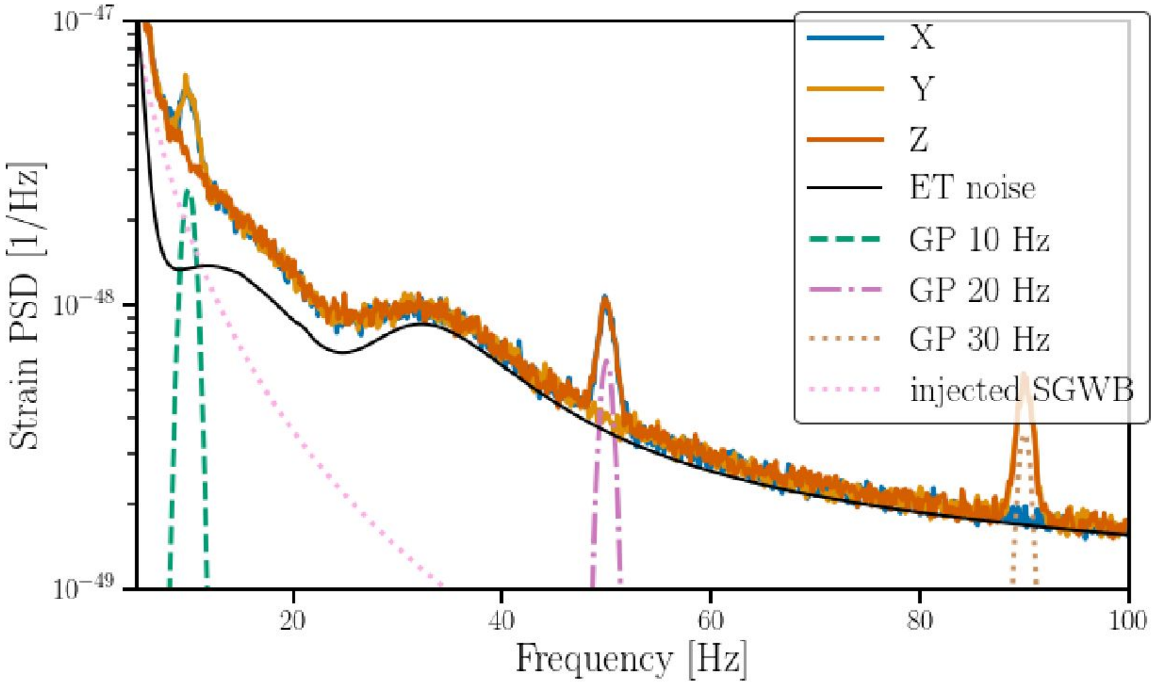


FIG. 1: The PSDs of the X, Y, Z channels, the ET noise and the injected correlated noise $S_n^{\mathrm{GP},10}$, $S_n^{\mathrm{GP},50}$ and $S_n^{\mathrm{GP},90}$(see text for a more detailed description). The contribution of the injected SGWB S_h^{SGWB} is also shown. We point out that at high frequencies the X, Y and Z PSDs seem to not match the ET noise curve. This is due to the small but non-negligble contribution of the GW signal, as can be seen by the perfect match for the T PSD in Fig. 5.

Toy example for the Einstein Telescope

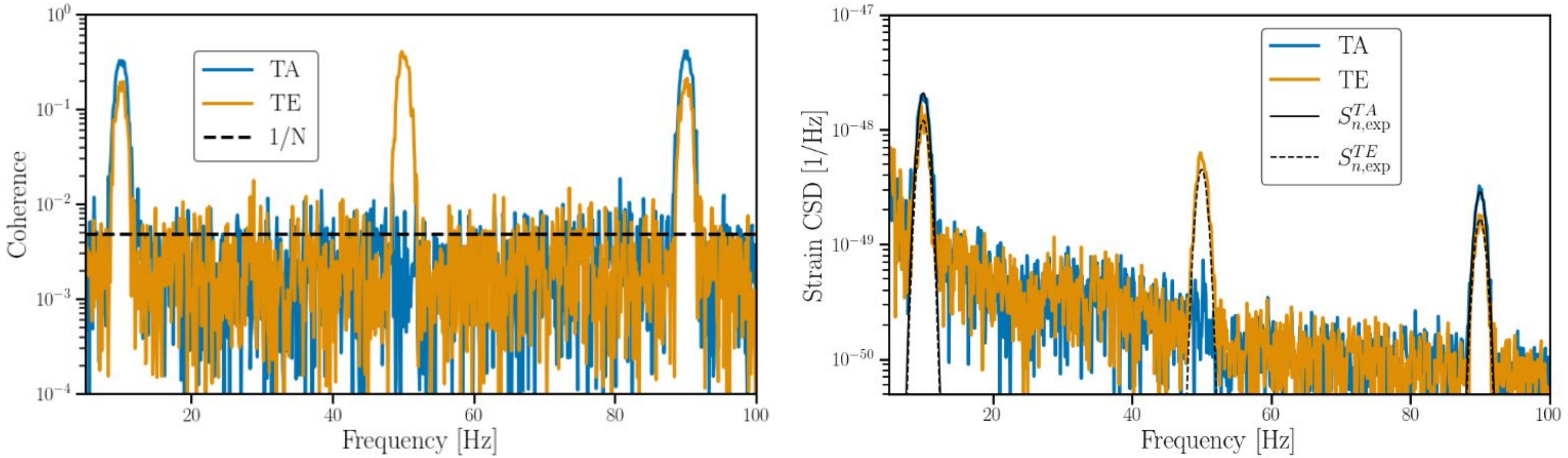


FIG. 3: Left: the coherence between the T and A, E channels. The black dashed line represents the level of coherence expected from independent Gaussian data, which goes approximately as $1/N$, where N is the number of time segments over which was averaged. Right: the modulus of the CSD between the T and A, E channels. The expected cross spectral densities associated with the T and A channels, and T and E channels given by Eq. 17 for the toy model example are shown in black. The expected CSD is in agreement with the observed CSD.

Toy example for the Einstein Telescope

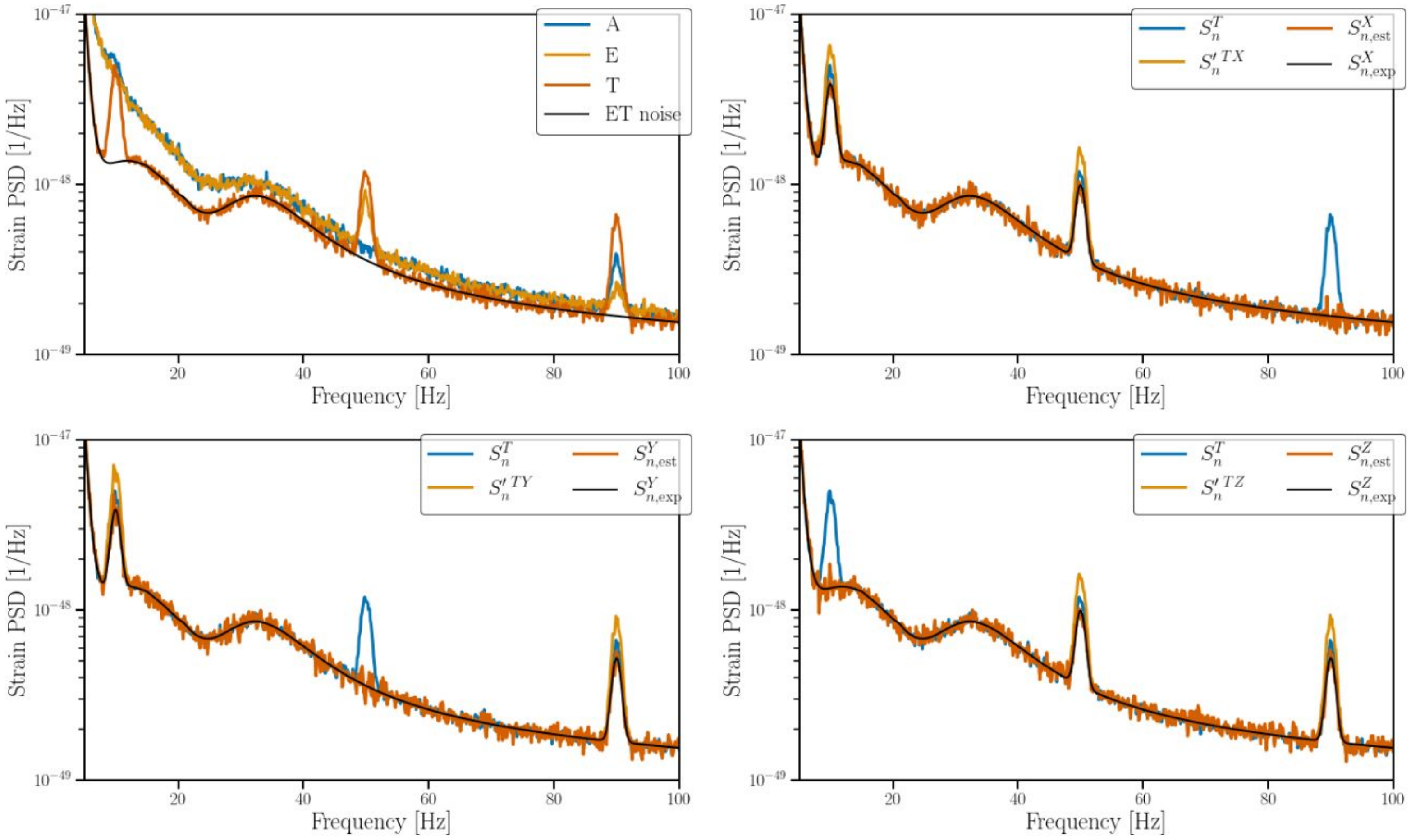
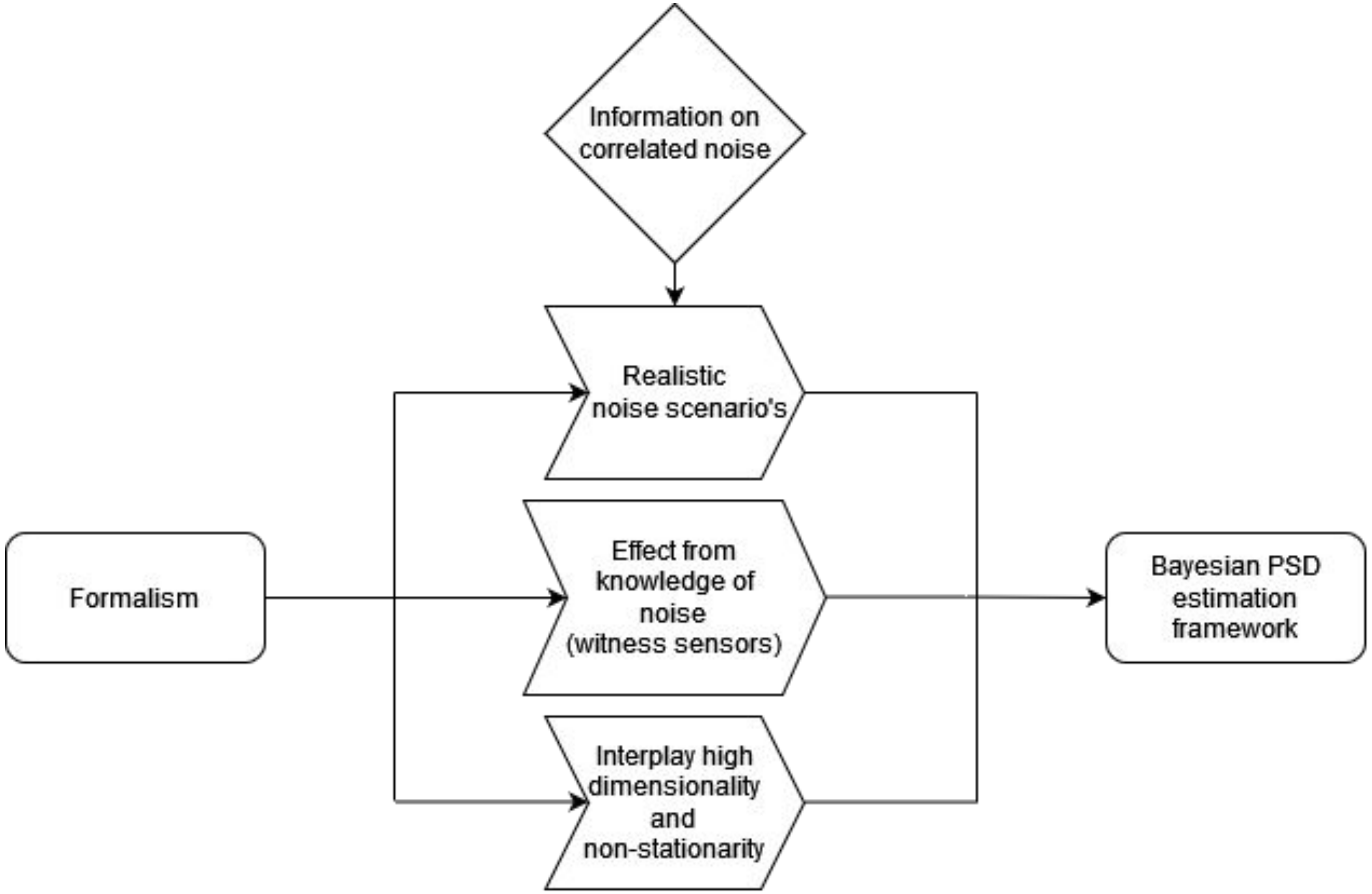


FIG. 2: Top left: The PSDs of the A, E, T channels and the ET noise. Top right/bottom left/bottom right: The PSD of the null channel T, the CSD of the T and $X/Y/Z$ channels, normalised such that it can serve as an estimate of $S_n^X/S_n^Y/S_n^Z$. The expected PSD of $X/Y/Z$, as shown in Eq. 13 and the estimated PSD as calculated in Eq. 15 and Eq. 16.

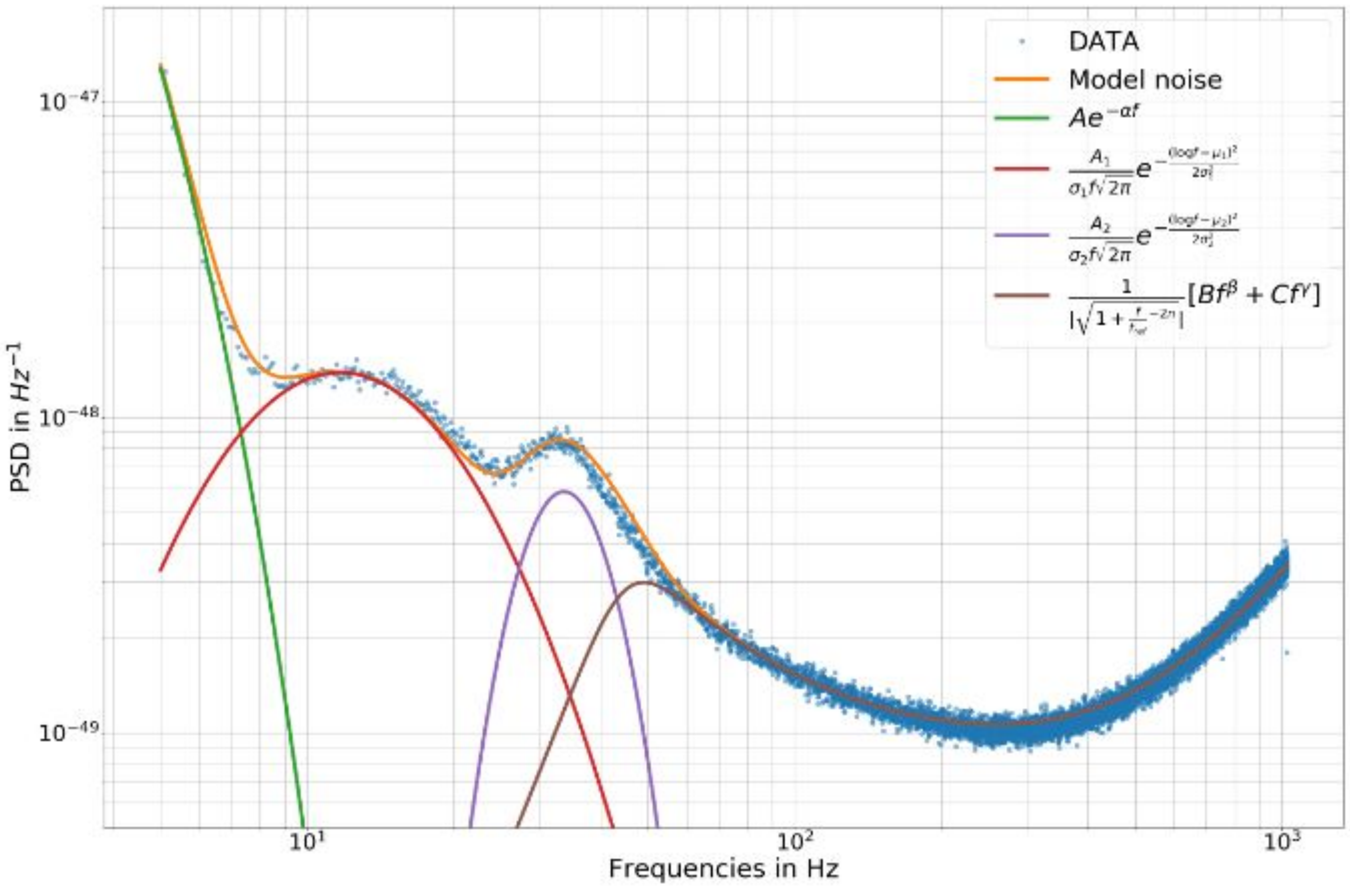
Recipe to transform the extended null channel formalism into a PSD estimation framework



MCMC for Einstein telescope in non-identical noise and correlation

- MCMC software to test different configuration of noise/correlation/SGWB
- Testing the different channel ‘XYZ’, ‘AET’, ‘AET+TXTYTZ’,...
- For now, we are testing ETD + correlate Gaussian peaks + SGWB (Toy-model)
- We have a large number of parameters (Toy model = 26)
- Statistical comparison use Deviance information criterion (DIC), Bayes Factor.

Analytic model for ET-D *Ad Hoc*



A	4×10^{-45} Hz^{-1}
A_1	2.3×10^{-46} Hz^{-1}
μ_1	15 Hz
σ_1	0.5
f_{ref}	45 Hz
B	4×10^{-45} Hz^{-1}
C	6×10^{-49} Hz^{-1}
D	2.7×10^{-55} Hz^{-1}

α	1.15
A_2	1×10^{-47} Hz^{-1}
μ_2	35 Hz
σ_2	0.2
n	6
β	-2.55
γ	-0.35
δ	2

$$S_{\mathrm{ETD}}(f) = A\exp-\alpha f + \frac{A_1}{\sigma_1 f\sqrt{2\pi}}e^{-\frac{(\log f-\log\mu_1)^2}{2\sigma_1^2}} + \frac{A_2}{\sigma_2 f\sqrt{2\pi}}e^{-\frac{(\log f-\log\mu_2)^2}{2\sigma_2^2}} + \frac{1}{\left|\sqrt{1+\frac{f}{f_{ref}}}^{-2n}\right|}\left[Bf^{\beta}+Cf^{\gamma}+Df^{\delta}\right]$$

MCMC (Markov chain Monte Carlo)

Whittle likelihood $p(\mathcal{D}|\theta)$ with data $\mathcal{D} = (\tilde{d}(f_k))_M$

$$\tilde{d}(f_k) = \frac{1}{\sqrt{T_{Obs}}} \sum_{i=1}^{T} d(t) e^{-jtf_k}, \quad f_k = 2\pi k/T$$

$$p(\mathcal{D}|\theta) = \prod_{k=0}^{N} \frac{1}{\sqrt{\det\left(2\pi \mathcal{C}(\theta, f_k)\right)}} e^{-\frac{1}{2}\mathcal{D}_k^{*T}\mathcal{C}^{-1}(\theta, f_k)\mathcal{D}_k}$$

cross power spectral covariance matrix $\mathcal{C}$

$$\mathcal{L}(\mathcal{D}|\theta) = -\frac{1}{2}\sum_{k=0}^{N}\left[\mathcal{D}_k^{*T}\mathcal{C}^{-1}(\theta, f_k)\mathcal{D}_k + \det\left(2\pi\mathcal{C}(\theta, f_k)\right)\right]$$

Case of XYZ channels :

$$\mathcal{C}(\theta, f) = \begin{pmatrix} S_{XX}(\theta, f) & S_{XY}(\theta, f) & S_{XZ}(\theta, f) \\ S_{YX}(\theta, f) & S_{YY}(\theta, f) & S_{YZ}(\theta, f) \\ S_{ZX}(\theta, f) & S_{ZY}(\theta, f) & S_{ZZ}(\theta, f) \end{pmatrix}$$

- Posterior distribution $p(\theta|d) \propto p(\theta)p(\mathcal{D}|\theta)$
- Using uniform prior $p(\theta) = \prod_i U(\theta_i, a_i, b_i)$
- Estimate parameters :

$$\theta_{ET} + \theta_{10Hz} + \theta_{50Hz} + \theta_{90Hz} + \theta_{SGWB}$$

$$S_h(f) = A_{2/3}\left(\frac{f}{f_*}\right)^{\alpha_{2/3}} = 5.35 \times 10^{-50}\left(\frac{f}{25\text{Hz}}\right)^{-7/3}$$

$$S_\mu^{\text{GP}}(f) = \frac{A^2}{2\pi} e^{\frac{-(f-\mu)^2}{\sigma^2}}$$

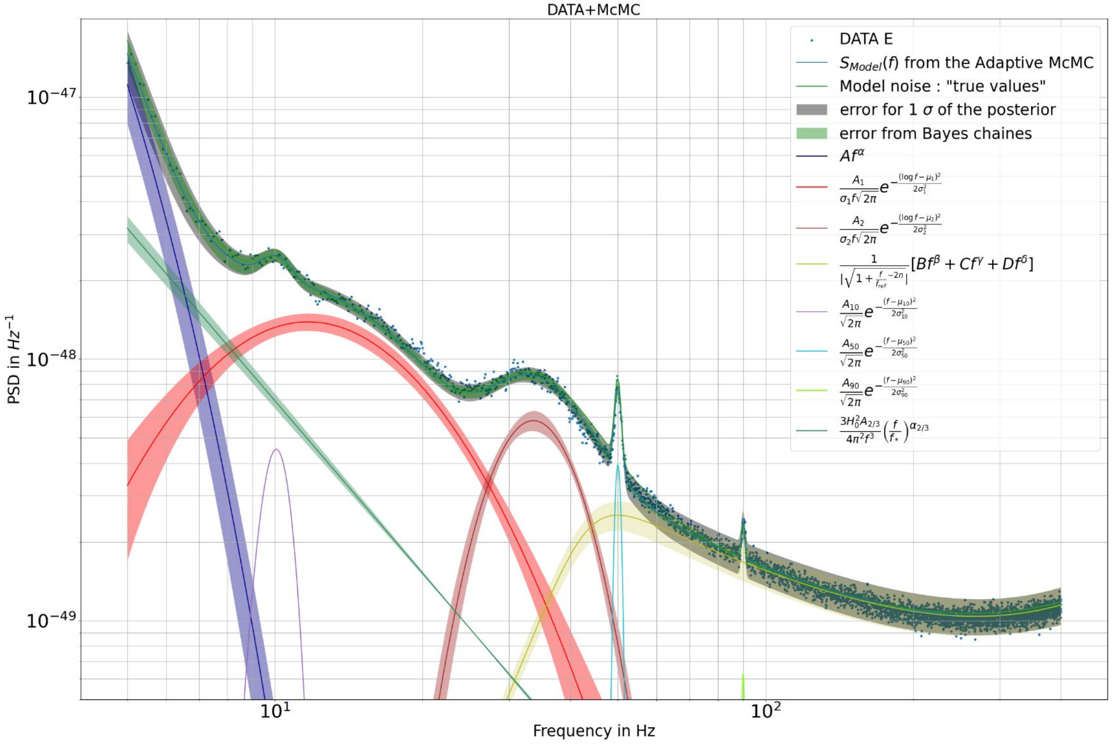
Case Study: ET Channel/sources and noise

ETD	ETD+GPs	ETD+SGWB	ETD+SGWB+GPs
• AET • AET+TX+TY+TZ • AE+TX+TY+TZ • XYZ • T • TX+TY+TZ • T+TX+TY+TZ • T+TA+TE	• AET • AET+TX+TY+TZ • AE+TX+TY+TZ • XYZ • T • TX+TY+TZ • T+TX+TY+TZ • T+TA+TE	• AET • AET+TX+TY+TZ • AE+TX+TY+TZ • XYZ	• AET • AET+TX+TY+TZ • AE+TX+TY+TZ • XYZ

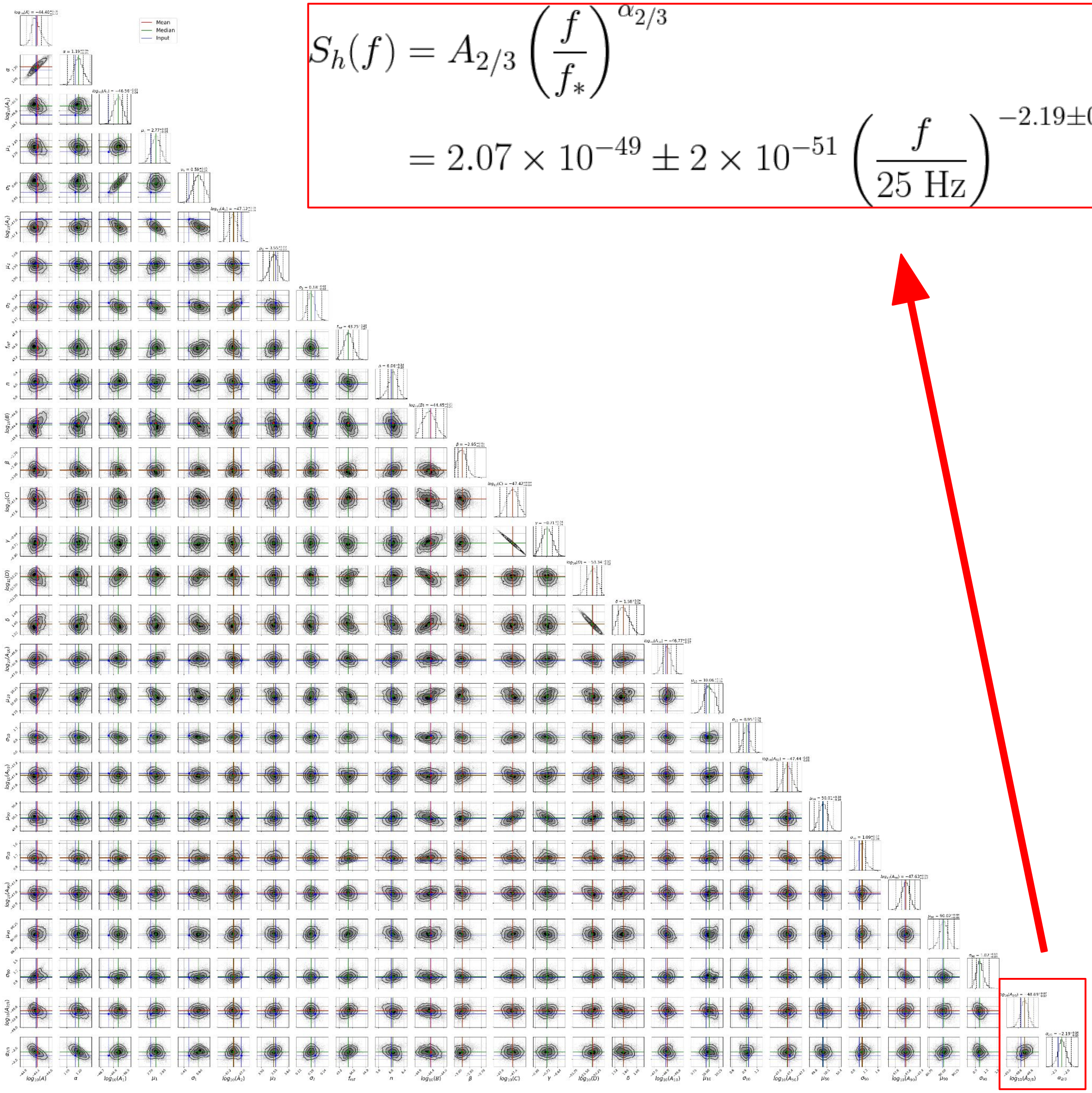
- Possibility to update the MCMC soft with different scenarios ET, Gaussian peaks (GP), SGWB (isotropic ⅔ slope from CBC population)

- Comparison of the different channels to investigate the “best” configuration to separate the different components

Example : AET+TXYZ channel ETD+GPs+SGWB



$$S_h(f) = A_{2/3}\left(\frac{f}{f_*}\right)^{\alpha_{2/3}}$$
$$= 2.07\times 10^{-49} \pm 2\times 10^{-51}\left(\frac{f}{25\text{ Hz}}\right)^{-2.19\pm 0.16}$$



Deviance Information Criterion (DIC)

- Analogous to Akaike information criterion and Bayesian information criterion : criterion for model comparison (BF not sensible for improper prior).
- It combines a measure of model fit with a penalty for the number of independent parameters.
- Easy to compute based on MCMC samples.

$$DIC = D(\overline{\theta}) + 2p_d$$

$$D(\theta) = -2\log\mathcal{L}(\mathbf{d}|\theta)$$

$$p_d = \overline{D}(\theta) - D(\overline{\theta})$$

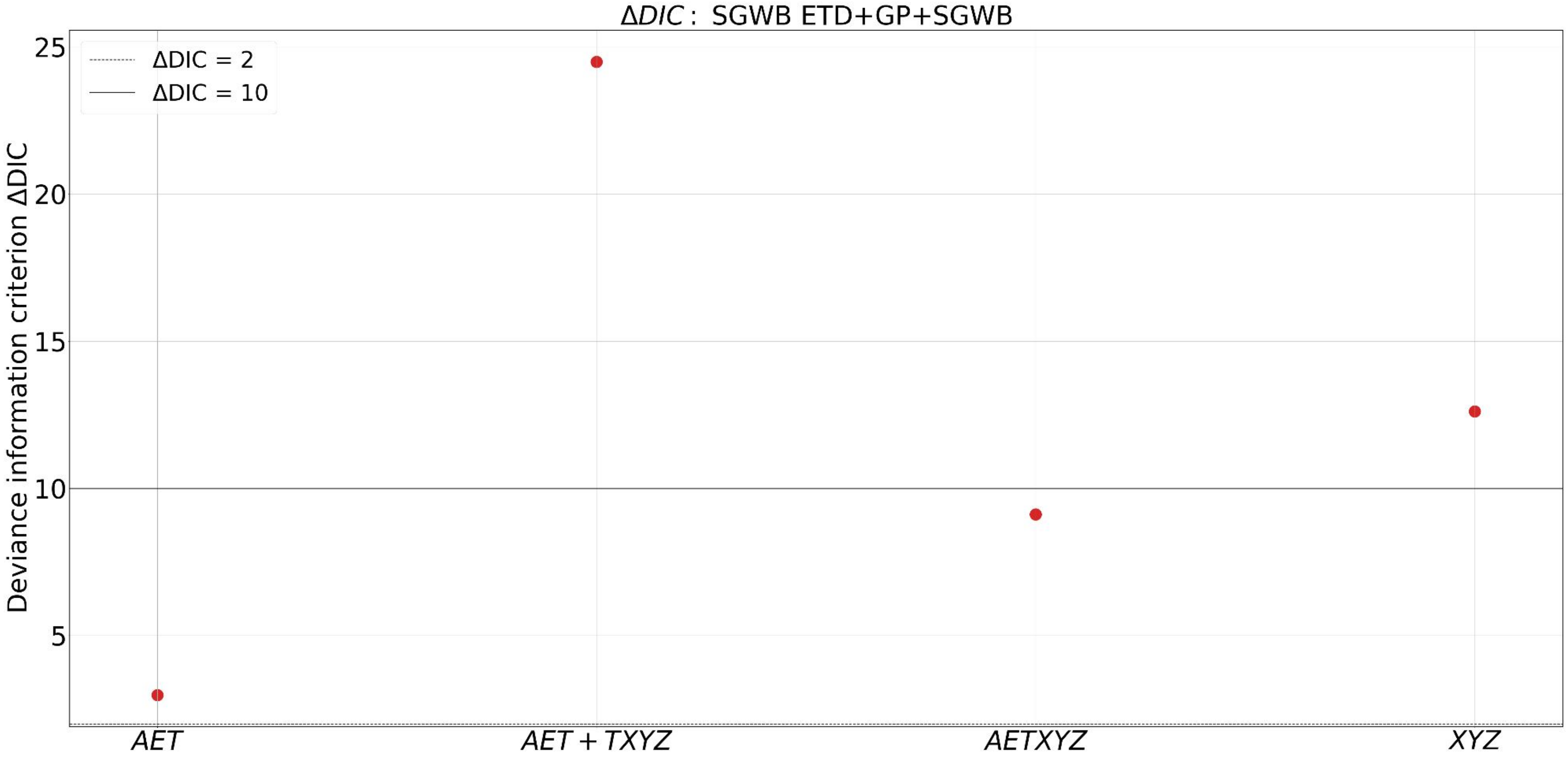
$$\Delta DIC = DIC_{\mathcal{M}_i} - DIC_{\varnothing\mathcal{M}_i}$$

$$\Delta DIC < 2 : \text{Not worth more than a bare mention}$$

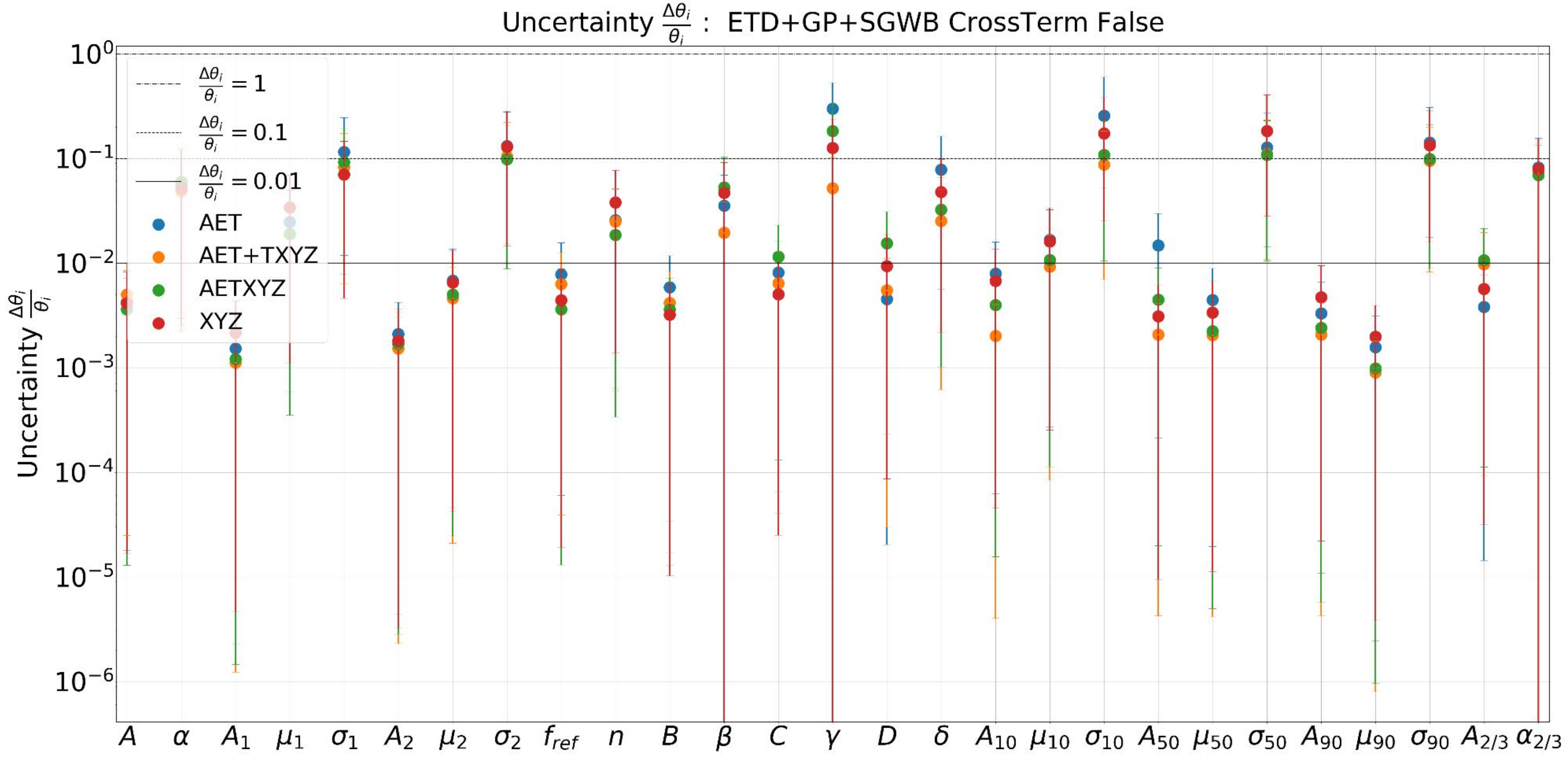
$$\Delta DIC \in [2, 10] : \text{positive}$$

$$\Delta DIC > 10 : \text{very strong}$$

Example of DIC result for SGWB in the context of ETD noise and Gaussian peaks



Uncertainty of the MCMC fitting



Conclusion

- We introduced an extension to the AET formalism where we take into account both non-identical as well as correlated noise.
- The correlation between <TX>, <TY> and <TZ> are valuable channels, free of signal, that could help to fit all the noise parameters and understanding the differences in PSD between the X, Y and Z channels.
- The coherence between the T and the A and E channels is an indicator of the non-identical behavior of the X, Y and Z channels.
- We have demonstrated the formalism in the case of the Einstein telescope for a simple toy example.
- We have developed a software to fit different scenarios ET, Gaussian peaks (GP), SGWB (isotropic ⅔ slope from CBC population) on different scenarios
- <u>Next step:</u> Update the code with NN and magnetic correlation
  - Magnetic : Use different coupling functions, Phys. Rev. D, 102:102005, Nov 2020.
  - NN : Consider different level of correlation in X, Y and Z (10%, 50% and 90%)