Scattering amplitudes for QCD, gravity and massive particles

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2111.06847 [hep-th]

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CP3 Seminar
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Invitation

Parke-Taylor formula:

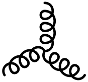
Parke, Taylor (1986)

$$A(1^-, 2^+, 3^-, 4^+, \ldots, n^+) = \frac{i\langle 1\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle \ldots \langle n\,1\rangle}$$

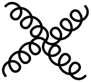
Reminder*

QCD Feynman rules, color-stripped:

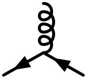
for color ordering see e.g. Dixon's 1995 TASI lectures



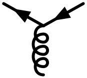
$$= \frac{i}{\sqrt{2}}\left[g^{\lambda\mu}(k-p)^{\nu} + g^{\mu\nu}(p-q)^{\lambda} + g^{\nu\lambda}(q-k)^{\mu}\right],$$



$$= i\, g^{\lambda\nu} g^{\mu\rho} - \frac{i}{2}\left(g^{\lambda\mu} g^{\nu\rho} + g^{\lambda\rho} g^{\mu\nu}\right),$$



$$= \frac{i}{\sqrt{2}}\gamma^{\mu},$$



$$= -\frac{i}{\sqrt{2}}\gamma^{\mu},$$

$$= -\frac{i g_{\mu\nu}}{p^2},$$

$$= \frac{i(\not{p}+m)}{p^2-m^2}.$$

*Disclaimer: all momenta outgoing

Invitation

Parke-Taylor formula:

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$$A(1^-, 2^+, 3^-, 4^+, \dots, n^+) = \frac{i\langle 1\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle \dots \langle n\,1\rangle}$$

Simplification w.r.t. Feynman rules due to

- gauge invariance
- massless spinor-helicity variables

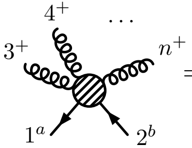
This talk:

- possible for **massive** quarks, higher-spin particles, etc!
- concentrate on 2 tree-level methods in 4d
  (loop methods also available and dimreg-compatible)

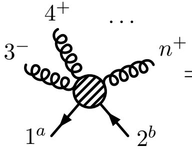
Bern, Dixon, Dunbar, Kosower; Britto, Cachazo, Feng; Forde; Badger;
Frellesvig, Peraro, Zhang; Abreu, Febres Cordero, Ita, Page, etc.

Results with 2 quarks

AO (2018)



$$= \frac{i\,m\langle 1^a 2^b\rangle [3|\prod_{j=3}^{n-2}\{\not{p}_{13\ldots j}\,\not{p}_{j+1} + (s_{13\ldots j}-m^2)\}|n]}{(s_{13}-m^2)(s_{134}-m^2)\ldots(s_{13\ldots(n-1)}-m^2)\,\langle 34\rangle\langle 45\rangle\ldots\langle n-1|n\rangle}$$



$$= -\frac{i\langle 3|1|2|3\rangle\left(\langle 1^a 3\rangle[2^b|1+2|3\rangle + \langle 2^b 3\rangle[1^a|1+2|3\rangle\right)}{s_{12}\langle 34\rangle\ldots\langle n-1|n\rangle\langle 3|1|1+2|n\rangle}$$

$$+\sum_{k=4}^{n-1}\frac{i\,m\langle 3|\not{p}_1\,\not{p}_{3\ldots k}|3\rangle\left(\langle 1^a 2^b\rangle\langle 3|\not{p}_1\,\not{p}_{3\ldots k}|3\rangle + \langle 1^a 3\rangle\langle 2^b 3\rangle s_{3\ldots k}\right)}{s_{3\ldots k}\,(s_{13\ldots k}-m^2)\ldots(s_{13\ldots(n-1)}-m^2)\,\langle 34\rangle\ldots\langle k-1|k\rangle\langle 3|\not{p}_1\,\not{p}_{3\ldots k}|k\rangle}$$

$$\times\frac{\langle 3|\not{p}_{3\ldots k}\prod_{j=k}^{n-2}\{\not{p}_{13\ldots j}\,\not{p}_{j+1} + (s_{13\ldots j}-m^2)\}|n]}{\langle 3|\not{p}_1\,\not{p}_{3\ldots k}|k+1\rangle\langle k+1|k+2\rangle\ldots\langle n-1|n\rangle}$$

KK relations: $$A(1,\beta,2,\alpha) = (-1)^{|\beta|}\sum_{\sigma\in\alpha\sqcup\!\sqcup\beta^T} A(1,2,\sigma)$$

Kleiss, Kuijf (1988)

Outline

1. Massive spinor helicity
2. 4-pt Compton amplitude
3. n-pt amplitudes via BCFW
4. Massive higher spins & BHs

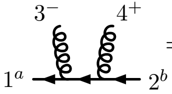
5. Summary & outlook

Massive spinor helicity

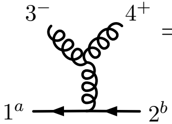
Feynman-rules calculation of $A(\underline{1}^a, 3^-, 4^+, \overline{2}^b)$

Start at 4 pts, textbook way manageable

$A(\underline{1}^a, 3^-, 4^+, \overline{2}^b) \ni$ 2 color-ordered diagrams:



$$= -\frac{i}{2(s_{13}-m^2)}\,(\bar{u}_1^a \not{\varepsilon}_3^- (\not{p}_{13}+m) \not{\varepsilon}_4^+ v_2^b)$$



$$= \frac{i}{2s_{34}} \Big\{ (\varepsilon_3^- \cdot \varepsilon_4^+)(\bar{u}_1^a (\not{p}_3 - \not{p}_4) v_2^b) + 2(p_4 \cdot \varepsilon_3^-)(\bar{u}_1^a \not{\varepsilon}_4^+ v_2^b) - 2(p_3 \cdot \varepsilon_4^+)(\bar{u}_1^a \not{\varepsilon}_3^- v_2^b) \Big\}$$

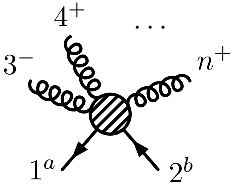
All done?

Why spinor helicity?

Consider color-ordered QCD amplitude $A(\underline{1}^a, 3^-, 4^+, \ldots, n^+, \overline{2}^b)$

Feynman rules give function of

- momenta p_i^μ
- polarization vectors $\varepsilon_\pm^\mu(p_i)$
- external spinors $\bar{u}^a(p_1)$, $v^b(p_2)$



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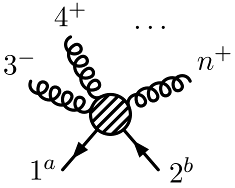

But all vector, spinor indices must be contracted

Remaining indices	$\Leftrightarrow$	physical quantum numbers:
► helicities $\pm$	$\Leftrightarrow$	spins $\{\pm 1/2\}_p$, $\{\pm \mathbf{1}\}_{\mathbf{p}}$, etc.
► SU(2) labels a, b	$\Leftrightarrow$	spins $\{\pm \mathbf{1/2}\}_{\mathbf{q}}$, $\{\pm 1, 0\}_q$, etc.

Crucial on-shell notion — LITTLE GROUP

Little groups

- Quantum fields $\Leftarrow$ reps of $\mathrm{SO}(1,3)$
- Quantum states $\Leftarrow$ reps of LITTLE GROUP
  - massless states $\Leftarrow$ $\mathrm{SO}(2)$
  - massive states $\Leftarrow$ $\mathrm{SO}(3)$

Little groups

- Quantum fields $\Leftarrow$ reps of $\mathrm{SO}(1,3) \quad \subset \quad \mathrm{SL}(2,\mathbb{C})$
- Quantum states $\Leftarrow$ reps of LITTLE GROUP's dbl cover
  - massless states $\Leftarrow$ $\mathrm{SO}(2) \quad \subset \quad \mathbf{U(1)}$
  - massive states $\Leftarrow$ $\mathrm{SO}(3) \quad \subset \quad \mathbf{SU(2)}$

Minor complication: spinorial reps use groups' double covers

U(1) and SU(2) arise naturally in spinor helicity

Spinor map

Basis for spinor helicity

- Minkowski space isomorphism:*

$$
\begin{aligned}
\mathrm{M}^{2\times 2,\mathbb{C}}_{\mathrm{Hermitian}} \quad &\leftrightarrow \quad \mathbb{R}^{1,3} \\
p_{\alpha\dot\beta} = p_\mu \sigma^\mu_{\alpha\dot\beta} &= \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix} \\
\det\{p_{\alpha\dot\beta}\} \; &= \; m^2
\end{aligned}
$$

---

*$\sigma^0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\sigma^1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma^2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\sigma^3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

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\det\{p_{\alpha\dot\beta}\} \; &= \; m^2
\end{aligned}
$$

- Lorentz group homomorphism:

$$
\begin{aligned}
\mathrm{SL}(2,\mathbb{C}) \quad &\rightarrow \quad \mathrm{SO}(1,3) \\
p_{\alpha\dot\delta} \to S_\alpha{}^\beta p_{\beta\dot\gamma} \left(S_\delta{}^\gamma\right)^* \quad &\Rightarrow \quad p^\mu \to L^\mu{}_\nu p^\nu, \quad L^\mu{}_\nu = \frac{1}{2}\,\mathrm{tr}\big(\bar\sigma^\mu S \sigma_\nu S^\dagger\big)
\end{aligned}
$$

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Massless vs massive spinor helicity

Arkani-Hamed, Huang, Huang (2017)

MASSLESS	MASSIVE
$\det\{p_{\alpha\dot\beta}\} = 0$	$\det\{p_{\alpha\dot\beta}\} = m^2$
$p_{\alpha\dot\beta} = \lambda_{p\alpha}\tilde\lambda_{p\dot\beta} \equiv \vert p\rangle_\alpha [p\vert_{\dot\beta}$	$p_{\alpha\dot\beta} = \lambda_{p\alpha}^{\ a}\epsilon_{ab}\tilde\lambda_{p\dot\beta}^{\ b} \equiv \vert p^a\rangle_\alpha [p_a\vert_{\dot\beta}$ $\det\{\lambda_{p\alpha}^{\ a}\} = \det\{\tilde\lambda_{p\dot\alpha}^{\ a}\} = m$
$p^\mu = \frac{1}{2}\langle p\vert\sigma^\mu\vert p]$	$p^\mu = \frac{1}{2}\langle p^a\vert\sigma^\mu\vert p_a]$
$p_{\alpha\dot\beta}\tilde\lambda_p^{\dot\beta} = 0$	$p_{\alpha\dot\beta}\tilde\lambda_p^{a\dot\beta} = m\lambda_{p\alpha}^{\ a}$
$\langle p\,q\rangle = -\langle q\,p\rangle \quad\Rightarrow\quad \langle p\,p\rangle = 0$ $[p\,q] = -[q\,p] \quad\Rightarrow\quad [p\,p] = 0$ $\langle p\,q\rangle[q\,p] = 2p\cdot q$	$\langle p^a q^b\rangle = -\langle q^b p^a\rangle$ e.g. $\langle p^a p^b\rangle = -m\epsilon^{ab}$ $[p^a q^b] = -[q^b p^a]$ e.g. $[p^a p^b] = m\epsilon^{ab}$ $\langle p^a q^b\rangle[q_b\,p_a] = 2p\cdot q$

Wavefunctions from helicity spinors

$$\varepsilon^{\mu}_{p+} = \frac{1}{\sqrt{2}} \frac{\langle q|\sigma^{\mu}|p]}{\langle q\, p\rangle} \qquad \qquad \begin{cases} \varepsilon^{\pm}_{p} \cdot p = \varepsilon^{\pm}_{p} \cdot q = 0 \\ \varepsilon^{\mu}_{p+}\varepsilon^{\nu}_{p-} + \varepsilon^{\mu}_{p-}\varepsilon^{\nu}_{p+} = -\eta^{\mu\nu} + \dfrac{p^{\mu}q^{\nu} + q^{\mu}p^{\nu}}{p\cdot q} \\ \varepsilon^{h_1}_{p} \cdot \varepsilon^{h_2}_{p} = -\delta^{h_1(-h_2)} \end{cases}$$

$$\varepsilon^{\mu}_{p-} = \frac{1}{\sqrt{2}} \frac{\langle p|\sigma^{\mu}|q]}{[p\, q]} \qquad \Rightarrow$$

Wavefunctions from helicity spinors

$$\varepsilon^{\mu}_{p+} = \frac{1}{\sqrt{2}} \frac{\langle q|\sigma^{\mu}|p]}{\langle q\,p\rangle} \qquad \varepsilon^{\mu}_{p-} = \frac{1}{\sqrt{2}} \frac{\langle p|\sigma^{\mu}|q]}{[p\,q]} \quad \Rightarrow \quad \begin{cases} \varepsilon^{\pm}_{p}\cdot p = \varepsilon^{\pm}_{p}\cdot q = 0 \\ \varepsilon^{\mu}_{p+}\varepsilon^{\nu}_{p-} + \varepsilon^{\mu}_{p-}\varepsilon^{\nu}_{p+} = -\eta^{\mu\nu} + \dfrac{p^{\mu}q^{\nu} + q^{\mu}p^{\nu}}{p\cdot q} \\ \varepsilon^{h_1}_{p}\cdot\varepsilon^{h_2}_{p} = -\delta^{h_1(-h_2)} \end{cases}$$

$$u^a_p = \begin{pmatrix} |p^a\rangle \\ |p^a] \end{pmatrix} \qquad \bar{u}^a_p = \begin{pmatrix} -\langle p^a| \\ [p^a| \end{pmatrix} \quad \Rightarrow \quad \begin{cases} (\not{p} - m)u^a_p = \bar{u}^a_p(\not{p} - m) = 0 \\ \bar{u}^a_p u^b_p = 2m\epsilon^{ab} \\ \bar{u}^a_p \gamma^{\mu} u^b_p = 2p^{\mu}\epsilon^{ab} \\ u^a_p \bar{u}_{pa} = u^a_p \epsilon_{ab} \bar{u}^b_p = \not{p} + m \end{cases}$$

$$v^a_p = \begin{pmatrix} -|p^a\rangle \\ |p^a] \end{pmatrix} \qquad \bar{v}^a_p = \begin{pmatrix} \langle p^a| \\ [p^a| \end{pmatrix} \quad \Rightarrow \quad \begin{cases} (\not{p} + m)v^a_p = \bar{v}^a_p(\not{p} + m) = 0 \\ \bar{v}^a_p v^b_p = 2m\epsilon^{ab} \\ \bar{v}^a_p \gamma^{\mu} v^b_p = -2p^{\mu}\epsilon^{ab} \\ v^a_p \bar{v}_{pa} = v^a_p \epsilon_{ab} \bar{v}^b_p = -\not{p} + m \end{cases}$$

Little group transformations

Consider Lorentz transformation $p^\mu \to L^\mu{}_\nu p^\nu$

MASSLESS:

$$|p\rangle \to S|p\rangle = e^{i\phi/2}|Lp\rangle \qquad \langle p| \to \langle p|S^{-1} = e^{i\phi/2}\langle Lp|$$

$$|p] \to S^{\dagger\,-1}|p] = e^{-i\phi/2}|Lp] \qquad [p| \to [p|S^{\dagger} = e^{-i\phi/2}[Lp|$$

$\Rightarrow\ \varepsilon_p^\pm \to L\varepsilon_p^\pm \sim e^{\mp i\phi}\varepsilon_{Lp}^\pm$

$e^{ih\phi} \in \mathrm{U}(1)$ encode $2d$ rotations in frame where $p = (E, 0, 0, E)$

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$e^{ih\phi} \in \mathrm{U}(1)$ encode $2d$ rotations in frame where $p = (E, 0, 0, E)$

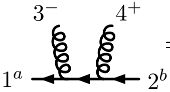
Massive:

$$|p^a\rangle \to S|p^a\rangle = \omega^a{}_b|Lp^b\rangle \qquad |p^a\rangle \to |p^a\rangle S^{-1} = \omega^a{}_b|Lp^a\rangle$$
$$|p^a] \to S^{\dagger\,-1}|p^a] = \omega^a{}_b[Lp^b| \qquad [p^a| \to [p^a|S^\dagger = \omega^a{}_b[Lp^b|$$

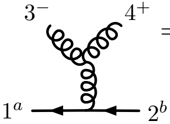
$\omega \in \mathrm{SU}(2)$ encode $3d$ rotations in rest frame where $p = (m, 0, 0, 0)$

4-pt Compton amplitude

Feynman-rules calculation of $A(\underline{1}^a, 3^-, 4^+, \overline{2}^b)$



$$= -\frac{i}{2(s_{13}-m^2)}\,(\bar{u}_1^a \not{\varepsilon}_3^- (\not{p}_{13}+m) \not{\varepsilon}_4^+ v_2^b)$$



$$= \frac{i}{2s_{34}}\Big\{(\varepsilon_3^- \cdot \varepsilon_4^+)(\bar{u}_1^a (\not{p}_3 - \not{p}_4) v_2^b) + 2(p_4 \cdot \varepsilon_3^-)(\bar{u}_1^a \not{\varepsilon}_4^+ v_2^b) - 2(p_3 \cdot \varepsilon_4^+)(\bar{u}_1^a \not{\varepsilon}_3^- v_2^b)\Big\}$$

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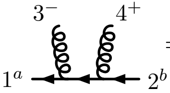
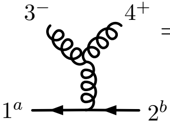
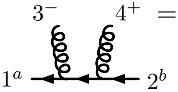


$$= -\frac{i}{2(s_{13}-m^2)}\,(\bar{u}_1^a\,\not{\varepsilon}_3^-\,(\not{p}_{13}+m)\,\not{\varepsilon}_4^+\,v_2^b)$$

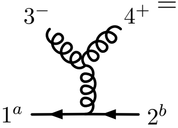


$$= \frac{i}{2s_{34}}\Big\{(\varepsilon_3^-\cdot\varepsilon_4^+)(\bar{u}_1^a(\not{p}_3-\not{p}_4)v_2^b) + 2(p_4\cdot\varepsilon_3^-)(\bar{u}_1^a\,\not{\varepsilon}_4^+\,v_2^b) - 2(p_3\cdot\varepsilon_4^+)(\bar{u}_1^a\,\not{\varepsilon}_3^-\,v_2^b)\Big\}$$

- plug in external wavefunctions:

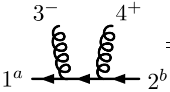


$$= \frac{-i}{(s_{13}-m^2)[3\,q_3]\langle 4\,q_4\rangle}\Big\{\langle 1^a 3\rangle[q_3|p_{13}|q_4\rangle[4\,2^b] + [1^a q_3]\langle 3|p_{13}|4]\langle q_4 2^b\rangle - m\langle 1^a 3\rangle[q_3 4]\langle q_4 2^b\rangle - m[1^a q_3]\langle 3\,q_4\rangle[4\,2^b]\Big\}$$

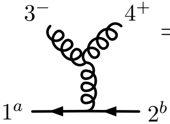


$$= \frac{-i}{s_{34}[3\,q_3]\langle 4\,q_4\rangle}\Big\{-\frac{1}{2}\langle 3\,q_4\rangle[4\,q_3]\big(\langle 1^a|p_3-p_4|2^b] + [1^a|p_3-p_4|2^b\rangle\big) - \langle 3|4|q_3]\big(\langle 1^a q_4\rangle[4\,2^b] + [1^a 4]\langle q_4 2^b\rangle\big) + \langle q_4|3|4]\big(\langle 1^a 3\rangle[q_3 2^b] + [1^a q_3]\langle 3\,2^b\rangle\big)\Big\}$$

Feynman-rules calculation of $A(\underline{1}^a, 3^-, 4^+, \overline{2}^b)$

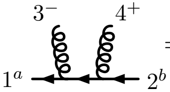


$$= -\frac{i}{2(s_{13}-m^2)}\,(\bar{u}_1^a\,\not{\varepsilon}_3^-\,(\not{p}_{13}+m)\,\not{\varepsilon}_4^+\,v_2^b)$$

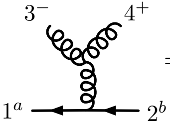


$$= \frac{i}{2s_{34}}\Big\{(\varepsilon_3^-\cdot\varepsilon_4^+)(\bar{u}_1^a(\not{p}_3-\not{p}_4)v_2^b) + 2(p_4\cdot\varepsilon_3^-)(\bar{u}_1^a\,\not{\varepsilon}_4^+\,v_2^b) - 2(p_3\cdot\varepsilon_4^+)(\bar{u}_1^a\,\not{\varepsilon}_3^-\,v_2^b)\Big\}$$

- plug in external wavefunctions with $q_3 = p_4$, $q_4 = p_3$:



$$= \frac{i\langle 3|1|4]}{(s_{13}-m^2)s_{34}}\big(\langle 1^a 3\rangle[2^b 4] + [1^a 4]\langle 2^b 3\rangle\big)$$

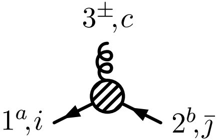


$$= 0$$

- spinor helicity helps no matter method

3-pt amplitudes

Modern methods require on-shell 3-pt input only

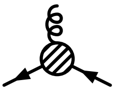


$$\mathcal{A}(1_i^a, 2_{\bar{\jmath}}^b, 3_c^+) = -\frac{iT^c_{i\bar{\jmath}}}{\langle 3\, q\rangle}\big(\langle 1^a q\rangle[2^b 3] + [1^a 3]\langle 2^b q\rangle\big) = -iT^c_{i\bar{\jmath}}\frac{\langle 1^a 2^b\rangle[3|1|q\rangle}{m\langle 3\, q\rangle}$$

$$\mathcal{A}(1_i^a, 2_{\bar{\jmath}}^b, 3_c^-) = \frac{iT^c_{i\bar{\jmath}}}{[3\, q]}\big(\langle 1^a 3\rangle[2^b q] + [1^a q]\langle 2^b 3\rangle\big) = iT^c_{i\bar{\jmath}}\frac{[1^a 2^b]\langle 3|1|q]}{m[3\, q]}$$

3-pt amplitudes

Modern methods require on-shell 3-pt input only



$$\mathcal{A}(1^a_i, 2^b_{\bar{\jmath}}, 3^+_c) = -\frac{iT^c_{i\bar{\jmath}}}{\langle 3\, q\rangle}\big(\langle 1^a q\rangle[2^b 3] + [1^a 3]\langle 2^b q\rangle\big) = -iT^c_{i\bar{\jmath}}\frac{\langle 1^a 2^b\rangle[3|1|q\rangle}{m\langle 3\, q\rangle}$$

$$\mathcal{A}(1^a_i, 2^b_{\bar{\jmath}}, 3^-_c) = \frac{iT^c_{i\bar{\jmath}}}{[3\, q]}\big(\langle 1^a 3\rangle[2^b q] + [1^a q]\langle 2^b 3\rangle\big) = iT^c_{i\bar{\jmath}}\frac{[1^a 2^b]\langle 3|1|q]}{m[3\, q]}$$

NB! Independent of ref. momentum q

$$p_2^2 - m^2 = \langle 3|1|3] = 0 \qquad \Rightarrow \qquad \exists\, x_3 \in \mathbb{C} : |1|3] = -m x_3 |3\rangle$$

$$\Rightarrow \qquad x_3 = \frac{[3|1|q\rangle}{m\langle 3\, q\rangle} \quad \text{indep. of } q$$

BCFW calculation of $A(\underline{1}^a, 3^-, 4^+, \overline{2}^b)$

BCFW shift: $\begin{cases} |3] \to |\hat{3}] = |3] - z|4] \\ |4\rangle \to |\hat{4}\rangle = |4\rangle + z|3\rangle \end{cases} \quad \Rightarrow \quad \mathcal{A} \to \mathcal{A}(z)$

Britto, Cachazo, Feng, Witten (2005)

Residue thm: $0 = \oint \frac{\mathrm{d}z}{2\pi i} \frac{\mathcal{A}(z)}{z} = \mathcal{A}(0) + \sum_{\text{poles of } \mathcal{A}(z)} \frac{1}{z_{\rm p}} \operatorname*{Res}_{z=z_{\rm p}} \mathcal{A}(z)$

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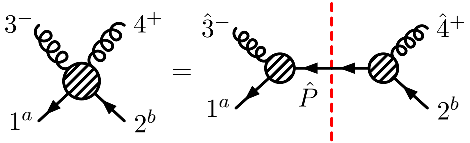


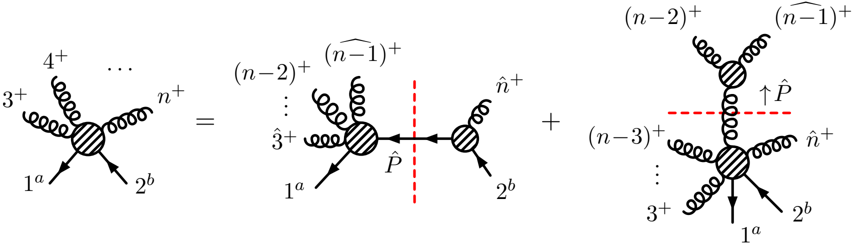
$$= \operatorname*{Res}_{z=z_{13}} A(\underline{1}^a, \hat{3}^-, \hat{4}^+, \overline{2}^b) = A(\underline{1}^a, \hat{3}^-, -\hat{P}^c) \frac{i}{s_{13} - m^2} A(\hat{P}_c, \hat{4}^+, \overline{2}^b)$$

$$= \frac{-i}{(s_{13} - m^2)[3\,4]\langle 4\,3\rangle} \big(\langle 1^a 3\rangle [4\hat{P}^c] - [1^a 4]\langle 3\hat{P}^c\rangle\big)\big(\langle \hat{P}_c\, 3\rangle [2^b 4] + [\hat{P}_c\, 4]\langle 2^b 3\rangle\big)$$

$$= \frac{i\langle 3|1|4]}{(s_{13} - m^2)s_{34}} \big(\langle 1^a 3\rangle [2^b 4] + [1^a 4]\langle 2^b 3\rangle\big)$$

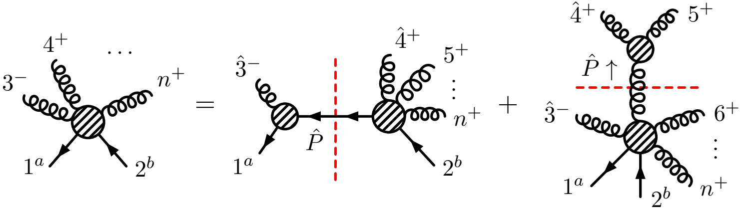
n-pt amplitudes via BCFW

BCFW recursion for $A(\underline{1}^a, 3^+, \dots, n^+, \overline{2}^b)$



$$= \dots = \frac{i\, m \langle 1^a 2^b \rangle [3| \prod_{j=3}^{n-2} \left\{ \not{p}_{13\dots j}\, \not{p}_{j+1} + (s_{13\dots j} - m^2) \right\} |n]}{(s_{13} - m^2)(s_{134} - m^2) \dots (s_{13\dots(n-1)} - m^2)\, \langle 34 \rangle \langle 45 \rangle \dots \langle n-1|n \rangle}$$

BCFW recursion for $A(\underline{1}^a, 3^-, 4^+, \ldots, n^+, \overline{2}^b)$



$$= \ldots = -\frac{i\langle 3|1|2|3\rangle\left(\langle 1^a 3\rangle[2^b|1+2|3\rangle + \langle 2^b 3\rangle[1^a|1+2|3\rangle\right)}{s_{12}\langle 3\,4\rangle\ldots\langle n-1|n\rangle\langle 3|1|1+2|n\rangle}$$

$$+\sum_{k=4}^{n-1}\frac{i\,m\langle 3|\not{p}_1\,\not{p}_{3\ldots k}|3\rangle\left(\langle 1^a 2^b\rangle\langle 3|\not{p}_1\,\not{p}_{3\ldots k}|3\rangle + \langle 1^a 3\rangle\langle 2^b 3\rangle s_{3\ldots k}\right)}{s_{3\ldots k}\,(s_{13\ldots k}-m^2)\ldots(s_{13\ldots(n-1)}-m^2)\,\langle 3\,4\rangle\ldots\langle k-1|k\rangle\langle 3|\not{p}_1\,\not{p}_{3\ldots k}|k\rangle}$$

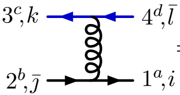
$$\times\frac{\langle 3|\not{p}_{3\ldots k}\prod_{j=k}^{n-2}\{\not{p}_{13\ldots j}\,\not{p}_{j+1}+(s_{13\ldots j}-m^2)\}|n]}{\langle 3|\not{p}_1\,\not{p}_{3\ldots k}|k+1\rangle\langle k+1|k+2\rangle\ldots\langle n-1|n\rangle}$$

Four-quark amplitudes

Lazopoulos, AO, Shi (2021)

$$\mathcal{A}(\underline{1},\overline{2},\underline{3},\overline{4},5,\dots,n) = \mathcal{A}(\underline{1},\overline{2},\underline{3},\overline{4},5,\dots,n) - \mathcal{A}(\underline{1},\overline{2},\underline{3},\overline{4},5,\dots,n)$$

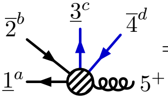
identical flavors from distinct flavors



$$= -\frac{iT^a_{i\bar{\jmath}}T^a_{k\bar{l}}}{s_{12}}\left(\langle 1^a 4^d\rangle[2^b 3^c] + [1^a 4^d]\langle 2^b 3^c\rangle + \langle 1^a 3^c\rangle[2^b 4^d] + [1^a 3^c]\langle 2^b 4^d\rangle\right)$$

Color ordering from adjoint rep.: $T^a_{i\bar{\jmath}} \to \tilde{f}^{ia\bar{\jmath}}$

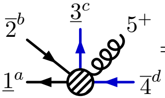
Adding gluons to four-quark amplitudes



$$= i\Bigg\{ \frac{[1^a5]\langle 2^b3^c\rangle[4^d5] + [1^a5]\langle 2^b4^d\rangle[3^c5]}{(s_{15}-m_1^2)s_{34}} + \frac{[1^a5]\langle 2^b3^c\rangle[4^d5] + \langle 1^a3^c\rangle[2^b5][4^d5]}{s_{12}(s_{45}-m_3^2)}$$

$$+ \frac{\langle 1^a4^d\rangle[2^b5][3^c5] + [1^a5]\langle 2^b3^c\rangle[4^d5] + \langle 1^a3^c\rangle[2^b5][4^d5] + [1^a5]\langle 2^b4^d\rangle[3^c5]}{s_{12}s_{34}}$$

$$+ \frac{s_{12}[5|1|4|5] - (s_{15}-m_1^2)[5|3|4|5]}{s_{12}s_{34}(s_{15}-m_1^2)(s_{45}-m_3^2)} \big(\langle 1^a4^d\rangle[2^b3^c] + [1^a4^d]\langle 2^b3^c\rangle + \langle 1^a3^c\rangle[2^b4^d] + [1^a3^c]\langle 2^b4^d\rangle\big)\Bigg\}$$



$$= i\Bigg\{ \frac{\langle 1^a4^d\rangle[2^b5][3^c5] + [1^a5]\langle 2^b4^d\rangle[3^c5]}{s_{12}(s_{35}-m_3^2)} - \frac{\langle 1^a3^c\rangle[2^b5][4^d5] + [1^a5]\langle 2^b3^c\rangle[4^d5]}{s_{12}(s_{45}-m_3^2)}$$

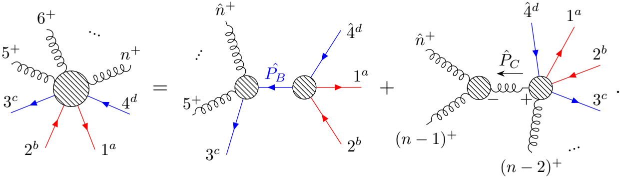
$$- \frac{[5|3|4|5]}{s_{12}(s_{35}-m_3^2)(s_{45}-m_3^2)} \big(\langle 1^a4^d\rangle[2^b3^c] + [1^a4^d]\langle 2^b3^c\rangle + \langle 1^a3^c\rangle[2^b4^d] + [1^a3^c]\langle 2^b4^d\rangle\big)\Bigg\}$$

Recall: fixed quarks 1 and 2 together by KK relations
Moreover, quark 3 maybe be locked in using (gen.) BCJ relations

Bern, Carrasco, Johansson (2008); Johansson, AO (2015)

Four-quark amplitudes with plus-helicity gluons

Lazopoulos, AO, Shi (2021)



$$A(1^a,2^b,3^c,5^+,\ldots,n^+,4^d) = \frac{-i}{s_{12}\prod_{j=5}^{n-1}\langle j|j{+}1\rangle}\Bigg\{\frac{1}{\prod_{j=5}^{n} D_{4j\ldots n}\langle 5|3|d_5^n]}$$

$$\times\Big[\langle 14\rangle[23][d_5^n|3|P_{45\ldots n}|d_5^n] - \langle 14\rangle[2|d_5^n][3|d_5^n]D_{45\ldots n} + m\langle 13\rangle[2|d_5^n]\langle 4|3|d_5^n] \tag{3.10a}$$

$$+\langle 13\rangle[2|P_{45\ldots n}|3|d_5^n]\Big([4n]\prod_{j=5}^{n-1} D_{4j\ldots n} + m\sum_{i=5}^{n-1}\langle 4|i|d_{i+1}^n]\prod_{j=5}^{i-1} D_{4j\ldots n}\Big)\Big]$$

$$+\sum_{i=6}^{n}\frac{m\langle i{-}1|i\rangle[c_{i-1}^5|d_i^n]}{\prod_{j=5}^{i-1} D_{35\ldots j}\prod_{j=i+1}^{n} D_{4j\ldots n}\big(D_{35\ldots(i-1)}\langle i{-}1|P_{4i\ldots n}|d_i^n] + D_{4i\ldots n}\langle i{-}1|P_{35\ldots(i-1)}|d_i^n]\big)}$$

$$\times\Bigg[\frac{\langle 1|P_{4i\ldots n}|d_i^n][2|d_i^n]\langle 34\rangle}{\langle i|P_{4i\ldots n}|d_{i+1}^n]} + \frac{[d_i^n|P_{12}|P_{4i\ldots n}|d_i^n]}{D_{35\ldots i}\langle i|P_{4(i+1)\ldots n}|d_{i+1}^n] + D_{4(i+1)\ldots n}\langle i|P_{35\ldots i}|d_{i+1}^n]} \tag{3.10b}$$

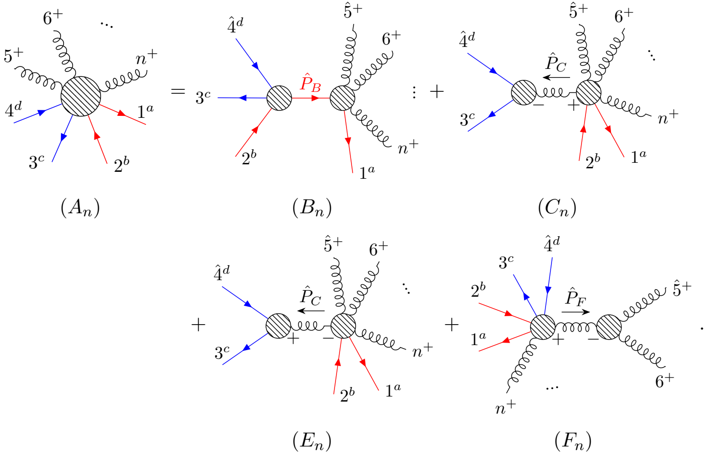
$$\times\Big(\langle 14\rangle[2|P_{12}|3\rangle + \langle 1|P_{4i\ldots n}|2]\langle 34\rangle + \frac{\langle 1|i\rangle[2|d_i^n]\langle 34\rangle}{\langle i|P_{4i\ldots n}|d_{i+1}^n]}\Big)\Bigg]\Bigg\}$$

$$+(1\leftrightarrow 2).$$

Four-quark amplitudes with plus-helicity gluons

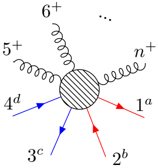
Lazopoulos, AO, Shi (2021)

- ↑ gluon insertions between like-flavored quarks ✓
- ↓ gluon insertions between distinctly flavored quarks ✓



Four-quark amplitudes with plus-helicity gluons

Lazopoulos, AO, Shi (2021)



$$A(1^a,2^b,3^c,4^d,5^+,\dots,n^+) = \frac{-i}{\prod_{j=5}^{n-1}\langle j|j+1\rangle}\Bigg\{\frac{1}{\prod_{j=5}^{n-1} D_{4\dots j}[d_n^5|P_{12}|3|P_{12}|1|d_n^5]}$$

$$\times\Bigg[\frac{[e_n|5][d_n^5|1|P_{123}|d_n^5]}{D_{123}\langle n|P_{123}|d_{n-1}^5]}\Big(\langle 13\rangle[2|d_n^5] + \langle 23\rangle[1|d_n^5]\Big) + \frac{1}{s_{12}\langle n|P_{12}|3|P_{12n}|d_{n-1}^5]}$$

$$\times\Bigg[M\langle 12\rangle[d_n^5|P_{12}|3|d_n^5]\Big([3|d_n^5]\langle 4|P_{12}|d_n^5] + [e_n|5]\langle 3|P_{12}|d_n^5]\Big)$$

$$+ m\langle 34\rangle[d_n^5|2|1|d_n^5]\Big([1|d_n^5]\langle 2|P_{12}|d_n^5] + [2|d_n^5]\langle 1|P_{12}|d_n^5]\Big)\Bigg] \qquad (3.33a)$$

$$+ \frac{[d_n^5|1|P_{123}|d_n^5]}{D_{1n}\langle n|P_{123n}|d_{n-1}^5] - D_{123n}\langle n|P_{1n}|d_{n-1}^5]}\Bigg[\langle 24\rangle[1|d_n^5][3|d_n^5] + \langle 23\rangle[1|d_n^5][e_n|5]$$

$$- \frac{[d_n^5|1|P_{123}|d_n^5]}{D_{123}}\Big([13]\langle 24\rangle + \langle 14\rangle[23]\Big) + \frac{[d_n^5|1|P_{123}|d_n^5]}{D_{123}^2\langle n|P_{123}|d_{n-1}^5]}$$

$$\times\Big(\langle 13\rangle[2|P_{123}|n\rangle[e_n|5] + \langle 23\rangle[1|P_{123}|n\rangle[e_n|5] - m\langle 13\rangle\langle 4n\rangle[2|d_n^5] - m\langle 23\rangle\langle 4n\rangle[1|d_n^5]\Big)\Bigg]\Bigg]$$

$$+ \sum_{i=5}^{n-1}\frac{M\langle i|i+1\rangle[a_{i+1}^n|d_i^5][d_i^5|P_{23}|P_{4\dots i}|d_i^5]}{[d_i^5|P_{3\dots i}|2|3|P_{3\dots i}|d_i^5]\prod_{l=i}^{n-1} D_{2\dots l}\prod_{k=5}^{i-1} D_{4\dots k}\big(D_{4\dots i}\langle i+1|P_{2\dots i}|d_i^5] - D_{2\dots i}\langle i+1|P_{4\dots i}|d_i^5]\big)}$$

$$\times\Bigg[\frac{[d_i^5|P_{23}|P_{4\dots i}|d_i^5]}{D_{4\dots(i-1)}\langle i|P_{2\dots(i-1)}|d_{i-1}^5] - D_{2\dots(i-1)}\langle i|P_{4\dots(i-1)}|d_{i-1}^5]}\Bigg[\langle 1|P_{2\dots i}|3]\langle 24\rangle + M\langle 14\rangle[23]$$

$$- \frac{1}{D_{4\dots i}\langle i|P_{4\dots(i-1)}|d_{i-1}^5]}\Big(\langle 1|P_{23}|P_{4\dots i}|i\rangle\langle 23\rangle[e_i|5] - D_{4\dots i}\langle 1i\rangle\langle 24\rangle[3|d_i^5] \qquad (3.33b)$$

$$+ m\langle 1|P_{2\dots i}|d_i^5]\langle 23\rangle\langle 4i\rangle + M\langle 13\rangle\langle i|P_{4\dots i}|2][e_i|5] + mM\langle 13\rangle[2|d_i^5]\langle 4i\rangle + m^2\langle 1i\rangle\langle 23\rangle[e_i|5]\Big)\Bigg]$$

$$+ \frac{1}{\langle i|P_{4\dots(i-1)}|d_{i-1}^5]}\Bigg[\langle 1|P_{23}|d_i^5]\langle 23\rangle[e_i|5] - \langle 1|P_{4\dots i}|d_i^5][3|d_i^5]\langle 24\rangle + M\langle 13\rangle[2|d_i^5][e_i|5]\Bigg]\Bigg]$$

$$+ \sum_{i=5}^{n-1}\frac{M\langle 12\rangle\langle i|i+1\rangle\big([3|d_i^5]\langle 4|P_{3\dots i}|d_i^5] + \langle 3|P_{3\dots i}|d_i^5][e_i|5]\big)[d_i^5|P_{3\dots i}|3|d_i^5]}{s_{3\dots i}\prod_{l=i}^{n-1} D_{2\dots l}\prod_{k=5}^{i-1} D_{4\dots k}\langle i+1|P_{3\dots i}|3|P_{3\dots i}|d_i^5]\langle i|P_{3\dots(i-1)}|3|P_{3\dots(i-1)}|d_{i-1}^5]}$$

$$\times\frac{[a_{i+1}^n|P_{3\dots i}|2|3|P_{3\dots i}|d_i^5]}{[d_i^5|P_{3\dots i}|2|3|P_{3\dots i}|d_i^5]}$$

$$+ \sum_{i=5}^{n-1}\frac{m\langle 34\rangle\langle i|i+1\rangle}{s_{3\dots i}\prod_{k=5}^{i-1} D_{4\dots k}\langle i|P_{3\dots(i-1)}|3|P_{3\dots(i-1)}|d_{i-1}^5]}$$

$$\times\Bigg[\frac{[d_i^5|P_{3\dots i}|2|1|P_{3\dots i}|d_i^5]\big(\langle 1|P_{3\dots i}|d_i^5][2|P_{12}|P_{3\dots i}|d_i^5] + \langle 2|P_{3\dots i}|d_i^5][1|P_{12}|P_{3\dots i}|d_i^5]\big)}{s_{12}\langle n|P_{12}|2|P_{3\dots i}|d_i^5]\langle i+1|P_{3\dots i}|3|P_{3\dots i}|d_i^5]}$$

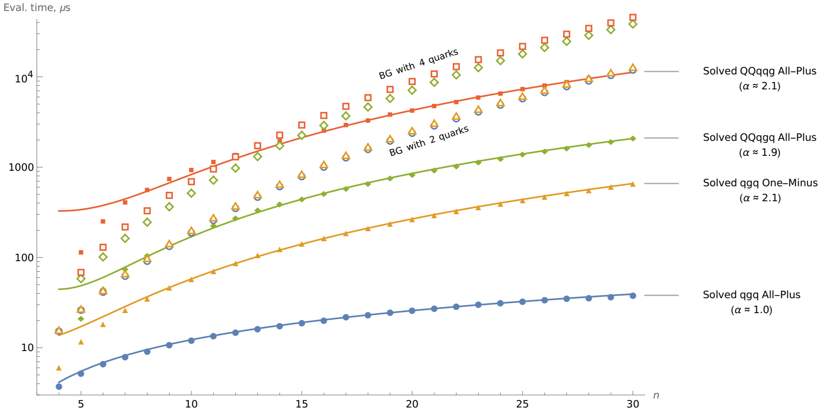
$$+ \frac{Ms_{3\dots i}[d_i^5|2|P_{3\dots i}|d_i^5][a_{i+1}^n|d_i^5]\big(\langle 12\rangle[d_i^5|P_{3\dots i}|2|d_i^5] + \langle 1|P_{3\dots i}|d_i^5]\langle 2|P_{3\dots i}|d_i^5]\big)}{\prod_{j=i}^{n-1} D_{2\dots j}\langle i+1|P_{3\dots i}|2|P_{3\dots i}|d_i^5][d_i^5|P_{3\dots i}|2|3|P_{3\dots i}|d_i^5]}$$

$$+ \sum_{k=i+1}^{n-1}\frac{M\langle k|k+1\rangle[d_i^5|P_{3\dots i}|2|P_{3\dots k}|P_{3\dots i}|d_i^5][a_{k+1}^n|P_{3\dots k}|P_{3\dots i}|d_i^5]}{s_{3\dots k}\prod_{j=k}^{n-1} D_{2\dots j}\langle k|P_{3\dots k}|2|P_{3\dots i}|d_i^5]\langle k+1|P_{3\dots k}|2|P_{3\dots i}|d_i^5]\langle i+1|P_{3\dots i}|3|P_{3\dots i}|d_i^5]}$$

$$\times\Big(\langle 12\rangle[d_i^5|P_{3\dots i}|2|P_{3\dots k}|P_{3\dots i}|d_i^5] + s_{3\dots k}\langle 1|P_{3\dots i}|d_i^5]\langle 2|P_{3\dots i}|d_i^5]\Big)\Bigg]\Bigg\}.$$

Formulae against off-shell recursion

Lazopoulos, AO, Shi (2021)



Num. eval. times against in-house impl. of **off-shell** BG recursion

Berends, Giele (1987)

Massive higher spins & BHs

QCD (hel. ± 1, spin 1/2) vs GR (hel. ± 2, spin s)

QCD	GR
$A(1^a, 2^b, 3^+) = -ig\frac{\langle 1^a 2^b \rangle}{m}x$ $A(1^a, 3^-, 4^+, 2^b) = \frac{ig^2 \langle 3\|1\|4](\langle 1^a 3 \rangle [2^b 4] + [1^a 4] \langle 2^b 3 \rangle)}{(s_{13} - m^2)s_{34}}$	$\mathcal{M}_3^{(s,+)} = -\frac{\kappa}{2}\frac{\langle 12 \rangle^{\odot 2s}}{m^{2s-2}}x^2$ $\mathcal{M}(1^{\{a\}}, 3^-, 4^+, 2^{\{b\}})$

$$\text{3-pt helicity factor } x = -\frac{\sqrt{2}}{m}(p_1 \cdot \varepsilon^+) = \left[\frac{\sqrt{2}}{m}(p_1 \cdot \varepsilon^-)\right]^{-1}$$

$$\mathcal{M}(1^{\{a\}}, 3^-, 4^+, 2^{\{b\}}) = \left(\frac{\kappa}{2}\right)^2 \frac{i\langle 3|1|4]^4}{(s_{13} - m^2)(s_{14} - m^2)s_{34}} \left(\frac{\langle 13 \rangle [24] + [14] \langle 23 \rangle}{\langle 3|1|4]}\right)^{\odot 2s}$$

Together with cl. limit applicable to scattering of black holes!

Summary

- SU(2) covariance $\Leftrightarrow$ arbitrary spin projections
- Elegant form for two-quark amplitudes with
  - all gluons of same helicity (e.g. all plus)
  - one gluon of different helicity (e.g. one minus)

  AO (2018)
- New analytic results for four-quark amplitudes

  Lazopoulos, AO, Shi (2021)
- Applicable to any massive particles with spin, black holes

  Guevara, AO, Vines (2018,2019)
  Aoude, AO (2021)

Thank you!

Backup slides

Solution to BCJ relations

Bern, Carrasco, Johansson (2008)
Johansson, AO (2015)

BCJ relations:

$$A(\underline{1},\overline{2},\alpha,3,\beta) = \sum_{\sigma\in S(\alpha)\sqcup\beta} A(\underline{1},\overline{2},3,\sigma)\prod_{i=1}^{|\alpha|}\frac{\mathcal{F}(q,\sigma,1|i)}{s_{2,\alpha_1,\ldots,\alpha_i}-m_2^2}$$

Kleiss-Kuijf basis of $(n-2)!$ primitives $\left\{A(\underline{1},\overline{2},\sigma)\right\}$

$\Rightarrow$ BCJ basis of $(n-3)!$ primitives $\left\{A(\underline{1},\overline{2},3,\sigma)\right\}$

Solution to BCJ relations for QCD

Johansson, AO (2015)

General BCJ relations:

$$A(\underline{1},\overline{2},\alpha,\underline{q},\beta)=\sum_{\sigma\in S(\alpha)\sqcup\beta}A(\underline{1},\overline{2},\underline{q},\sigma)\prod_{i=1}^{|\alpha|}\frac{\mathcal{F}(q,\sigma,1|i)}{s_{2,\alpha_1,\ldots,\alpha_i}-m_2^2},$$

where α is purely gluonic

Melia basis of $(n-2)!/k!$ primitives

$$\left\{A(\underline{1},\overline{2},\sigma)\;\middle|\;\sigma\in\mathrm{Dyck}_{k-1}\times\{\text{gluon insertions}\}_{n-2k}\right\}$$

$\Rightarrow$ new BCJ basis of $(n-3)!(2k-2)/k!$ primitives

$$\left\{A(\underline{1},\overline{2},\underline{q},\sigma)\;\middle|\;\{\underline{q},\sigma\}\in\mathrm{Dyck}_{k-1}\times\{\text{gluon insertions in }\sigma\}_{n-2k}\right\}$$

Helicity basis

Arkani-Hamed, Huang, Huang (2017)

Take $p^\mu = (E, P\cos\varphi\sin\theta, P\sin\varphi\sin\theta, P\cos\theta)$

$$|p^a\rangle = \lambda_{p\alpha}^{\ a} = \begin{pmatrix} \sqrt{E-P}\cos\frac{\theta}{2} & -\sqrt{E+P}e^{-i\varphi}\sin\frac{\theta}{2} \\ \sqrt{E-P}e^{i\varphi}\sin\frac{\theta}{2} & \sqrt{E+P}\cos\frac{\theta}{2} \end{pmatrix}$$

$$[p^a| = \tilde{\lambda}_{p\dot{\alpha}}^{\ a} = \begin{pmatrix} -\sqrt{E+P}e^{i\varphi}\sin\frac{\theta}{2} & -\sqrt{E-P}\cos\frac{\theta}{2} \\ \sqrt{E+P}\cos\frac{\theta}{2} & -\sqrt{E-P}e^{-i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$

Then

$$s^\mu(u_p^a) = \frac{1}{2m}\bar{u}_{pa}\gamma^\mu\gamma^5 u_p^a = (-1)^{a-1}s_p^\mu$$

$$s_p^\mu = \frac{1}{m}(P, E\cos\varphi\sin\theta, E\sin\varphi\sin\theta, E\cos\theta)$$

Comparison with earlier results

Older reference-momentum-dep. spinors:

$$\bar{u}_p^{a=1} = \begin{pmatrix} -\langle p^1| \equiv \frac{m\langle q|}{\langle q\, p^\flat\rangle} \\ [p^1| \equiv [p^\flat| \end{pmatrix} = \bar{u}_p^-(q) \qquad v_p^{a=1} = \begin{pmatrix} -|p^1\rangle \equiv -\frac{m|q\rangle}{\langle p^\flat\, q\rangle} \\ |p^1] \equiv |p^\flat] \end{pmatrix} = v_p^-(q)$$

$$\bar{u}_p^{a=2} = \begin{pmatrix} -\langle p^2| \equiv -\langle p^\flat| \\ [p^2| \equiv -\frac{m[q|}{[q\, p^\flat]} \end{pmatrix} = -\bar{u}_p^+(q) \qquad v_p^{a=2} = \begin{pmatrix} -|p^2\rangle \equiv -|p^\flat\rangle \\ |p^2] \equiv \frac{m|q]}{[p^\flat\, q]} \end{pmatrix} = -v_p^+(q)$$

Kleiss, Stirling (1986), Dittmaier (1998), Schwinn, Weinzierl (2005)

⇒ Analytically retrieve older non-SU(2)-covariant formulae

Schwinn, Weinzierl (2007)

$$A(\underline{1}^1, 3^-, 4^+, \ldots, n^+, \bar{\underline{2}}^1) = 0$$

$$A(\underline{1}^1, 3^-, 4^+, \ldots, n^+, \bar{\underline{2}}^2) = \frac{-i\langle 2^\flat 3\rangle}{\langle 1^\flat 3\rangle\langle 34\rangle \ldots \langle n{-}1|n\rangle} \sum_{k=4}^{n} \frac{\langle 3|\not{p}_1\, \not{p}_{3\ldots k}|3\rangle^2}{s_{3\ldots k}\langle 3|\not{p}_1\, \not{p}_{3\ldots k}|k\rangle}$$
$$\times \left\{ \delta_{k=n} + \delta_{k\neq n} \frac{m^2\langle k|k{+}1\rangle\langle 3|\not{p}_{3\ldots k} \prod_{j=k+1}^{n-1}\{(s_{13\ldots j} - m^2) - \not{p}_j\, \not{p}_{13\ldots j}\}|n]}{(s_{13\ldots k} - m^2)\ldots(s_{13\ldots(n-1)} - m^2)\langle 3|\not{p}_1\, \not{p}_{3\ldots k}|k{+}1\rangle} \right\}$$

$$A(\underline{1}^2, 3^-, 4^+, \ldots, n^+, \bar{\underline{2}}^1) = \frac{i\langle 1^\flat 3\rangle}{\langle 2^\flat 3\rangle\langle 34\rangle \ldots \langle n{-}1|n\rangle} \sum_{k=4}^{n} \frac{\langle 3|\not{p}_1\, \not{p}_{3\ldots k}|3\rangle^2}{s_{3\ldots k}\langle 3|\not{p}_1\, \not{p}_{3\ldots k}|k\rangle}$$
$$\times \left\{ \delta_{k=n} + \delta_{k\neq n} \frac{m^2\langle k|k{+}1\rangle\langle 3|\not{p}_{3\ldots k} \prod_{j=k+1}^{n-1}\{(s_{13\ldots j} - m^2) - \not{p}_j\, \not{p}_{13\ldots j}\}|n]}{(s_{13\ldots k} - m^2)\ldots(s_{13\ldots(n-1)} - m^2)\langle 3|\not{p}_1\, \not{p}_{3\ldots k}|k{+}1\rangle} \right\}$$

$$A(\underline{1}^2, 3^-, 4^+, \ldots, n^+, \bar{\underline{2}}^2) = \frac{i\langle 1^\flat 2^\flat\rangle}{m\langle 34\rangle \ldots \langle n{-}1|n\rangle} \sum_{k=4}^{n} \frac{\langle 3|\not{p}_1\, \not{p}_{3\ldots k}|3\rangle^2}{s_{3\ldots k}\langle 3|\not{p}_1\, \not{p}_{3\ldots k}|k\rangle} \left[1 + \frac{s_{3\ldots k}\langle 3\, 2^\flat\rangle}{\langle 3|\not{p}_{3\ldots k}\, \not{p}_1^\flat|2^\flat\rangle} \right]$$
$$\times \left\{ \delta_{k=n} + \delta_{k\neq n} \frac{m^2\langle k|k{+}1\rangle\langle 3|\not{p}_{3\ldots k} \prod_{j=k+1}^{n-1}\{(s_{13\ldots j} - m^2) - \not{p}_j\, \not{p}_{13\ldots j}\}|n]}{(s_{13\ldots k} - m^2)\ldots(s_{13\ldots(n-1)} - m^2)\langle 3|\not{p}_1\, \not{p}_{3\ldots k}|k{+}1\rangle} \right\}$$