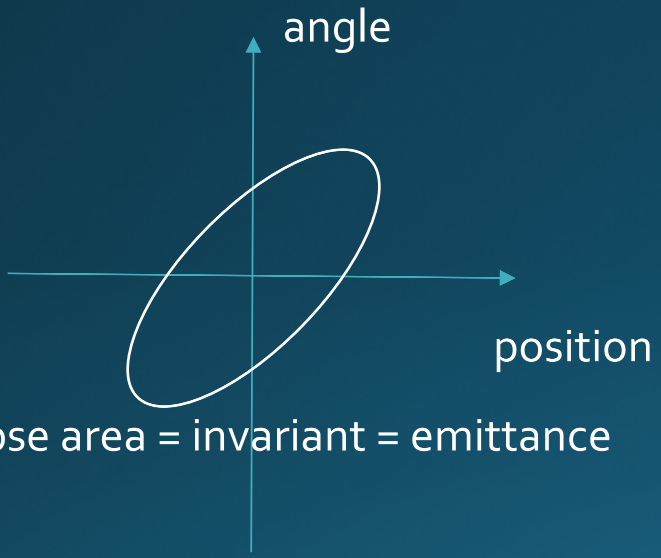
Introduction to longitudinal beam dynamics II

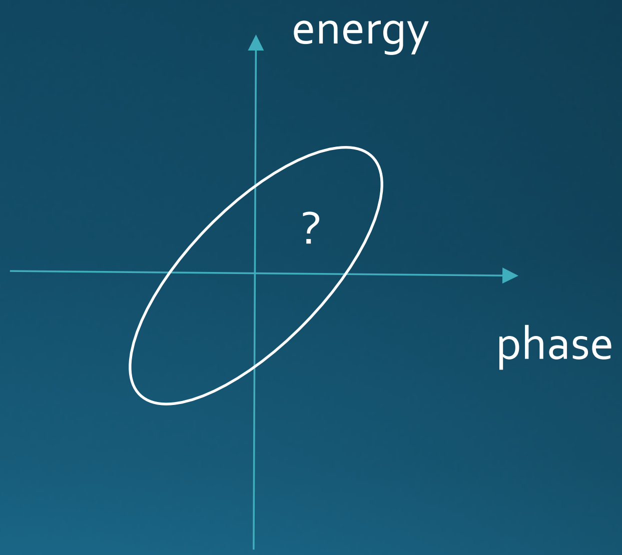
Longitudinal phase space

- As in the case of transverse beam dynamics, where it is more convenient to work in phase space because we can answer in one go what is the position and energy of the system at any moment …
- we also prefer to work in phase space when dealing with the longitudinal motion
- In this case we can answer in one go what is the phase difference and energy difference between the ideal particle and the real particle



Ellipse area = invariant = emittance

TRANSVERSE BEAM DYNAMICS



LONGITUDINAL BEAM DYNAMICS

- In the phase space plane $(w, \Delta\varphi)$ or $(\frac{\Delta p}{p_s}, \Delta\varphi)$ the **synchronous particle** is at the **origin of coordinates** and the **real** particle **describes a given trajectory around it**.
- Expressing the longitudinal phase space in terms of $(\frac{\Delta p}{p_s}, \Delta\varphi)$ is usually more convenient because $\frac{\Delta p}{p_s}$ determines by how much the closed orbit will be different from the ideal one thanks to the Eq. 5: $\frac{\Delta l_{\Delta E}}{l_s} = \alpha \frac{\Delta p}{p_s}$.
- The **stability condition** will determine the **range of initial values** such $(w, \Delta\varphi)$ or $(\frac{\Delta p}{p_s}, \Delta\varphi)$ will be **bounded** during the **movement**.

As we said in Part I, in order to obtain the first equation of motion, Eq. 25, we have assumed that the beam energy can be changed only by the applied RF field, and we have neglected any other energy variation due to interaction with the environment or the synchrotron radiation.

We are dealing with a conservative system and therefore there has to be an invariant and this is usually the energy. Let's calculate the invariant.

From Part I:

$$w = \frac{\Delta W}{W_s} = \frac{W - W_s}{W_s} \qquad \text{Eq. 20}$$

$$\Delta\varphi = \varphi - \varphi_s \qquad \text{Eq. 21}$$

To obtain the invariant we cross multiply the first and second equation of motion, and we integrate:

FIRST EQUATION OF MOTION $\dot{w} = \frac{qV_{max}\Omega_s}{2\pi W_s}\left(\sin(\Delta\varphi + \varphi_s) - sin\varphi_s\right)$ Eq. 25

SECOND EQUATION OF MOTION $\Delta\dot{\varphi} = \frac{\delta(\Delta\varphi)}{\delta t} = h\Gamma_s\Omega_s w$ Eq. 34

$$h\Gamma_s\Omega_s w dw = \frac{qV_{max}\Omega_s}{2\pi W_s}\left(\sin(\Delta\varphi + \varphi_s) - sin\varphi_s\right)d\varphi \quad \text{Eq. 45}$$

Now we integrate Eq. 45 and define as integration constant, C, the one that for $\Delta\varphi$=w=0 ➔ Total Energy = 0:

$$\int wdw - \int_{\varphi_s}^{\varphi} \frac{qV_{max}}{2\pi h\Gamma_s W_s}\left(\sin(\Delta\varphi + \varphi_s) - sin\varphi_s\right)d\varphi = C \quad \text{Eq. 46}$$

$$\boxed{\frac{1}{2}w^2} + \boxed{\frac{qV_{max}}{2\pi h\Gamma_s W_s}\left(\cos(\Delta\varphi + \varphi_s) - cos\varphi_s + \Delta\varphi sin\varphi_s\right)} = Total\ Energy = Hamiltonian \quad \text{Eq. 47}$$

Kinetic energy

Potential energy
➔ for a sinusoidal RF field

The value of the energy of the system is given by the initial conditions

For a general RF field, the Hamiltonian is:

$$\frac{1}{2}w^2 - \int_{\varphi_s}^{\varphi} \frac{q}{2\pi h \Gamma_s W_s}[V(\varphi') - V(\varphi_s)]d\varphi' = Total\ Energy$$
$$= Hamiltonian$$

Eq. 48

Remember: the potential energy is “minus” the integral of the RF voltage

The voltage function can be single harmonic (one RF system):

$$V(\varphi) = V_{max} sin\varphi$$

The voltage function can be double harmonic (two RF systems):

$$V(\varphi) = V^{A}_{max} sin\varphi + V^{B}_{max} \sin(n\varphi + \emptyset)$$

where **n** is the **frequency ratio of both RF systems** and **∅** is the **relative phase between** them.

e.g. SPS: the main RF system is 200 MHz and the second system is 800 MHz

The stability of the particle motion can be better understood from the plot of the RF potential.

Let's plot the potential energy of the Hamiltonian of Eq. 47

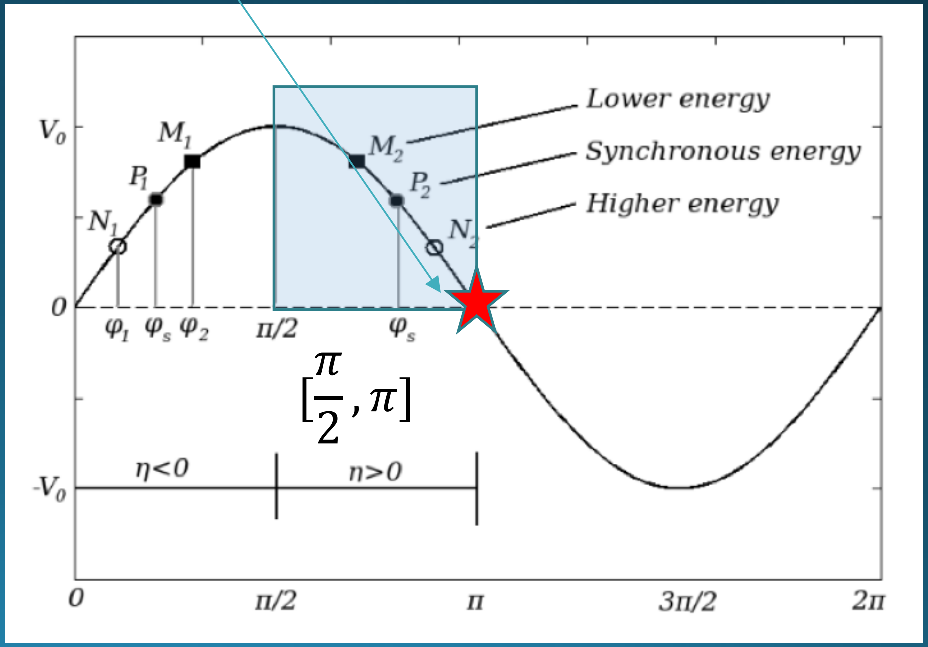
$$U(\Delta\varphi) = \frac{qV_{max}}{2\pi h\Gamma_s W_s}\left(\cos(\Delta\varphi + \varphi_s) - cos\varphi_s + \Delta\varphi sin\varphi_s\right) \qquad \text{Eq. 51}$$

CASE 1: NO ACCELERATION (ABOVE TRANSITION) ➔ STATIONARY BUCKET

No acceleration and we are above transition ➔ $\boldsymbol{\varphi_s = \pi} \quad \boldsymbol{(cos\pi = -1)}$

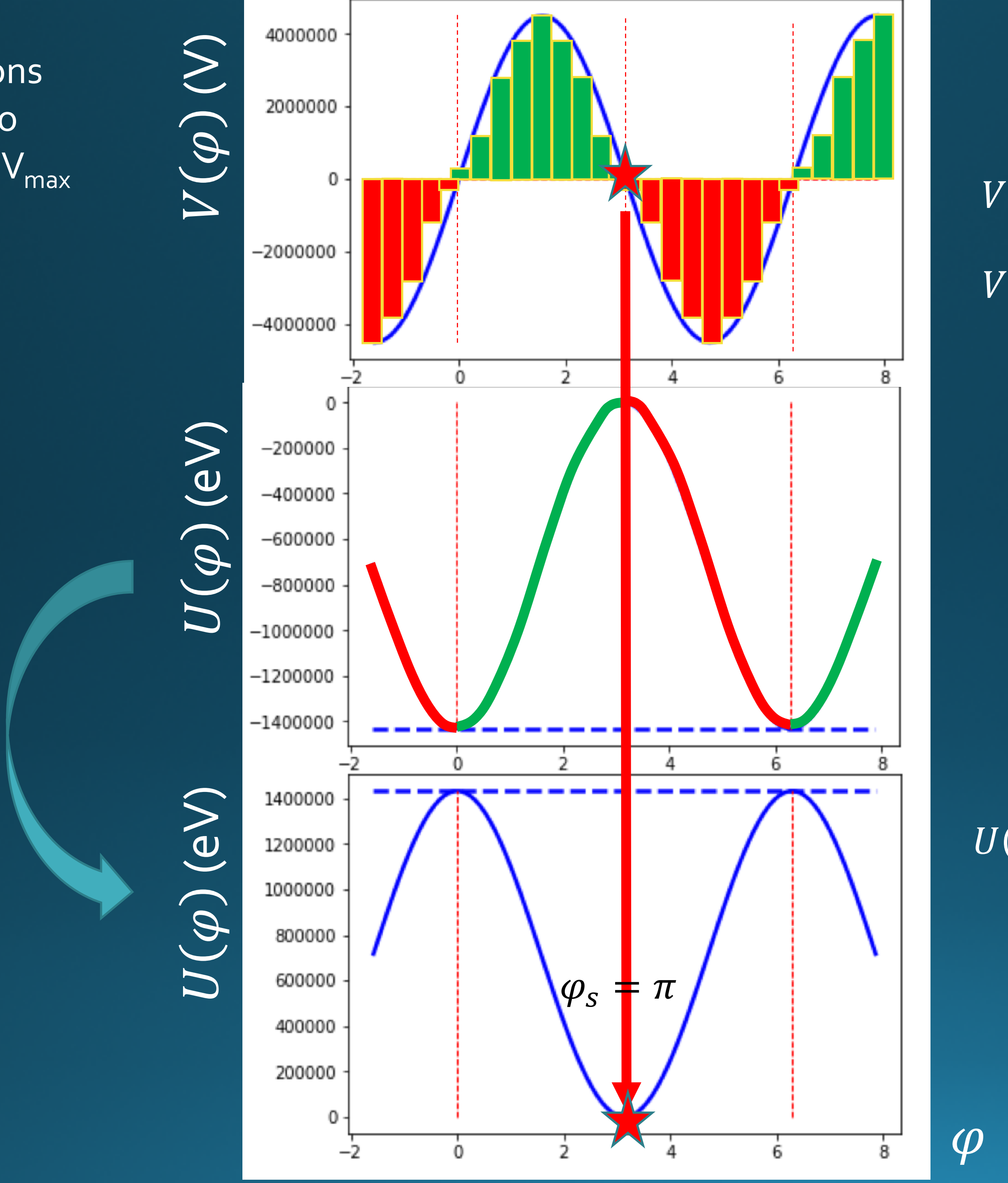
$$U(\Delta\varphi) = \frac{qV_{max}}{2\pi h\Gamma_s W_s}(1 - \cos\Delta\varphi) = \frac{qV_{max}}{2\pi h\Gamma_s W_s}(1 + \cos\varphi)$$

with $\Delta\varphi = \varphi - \varphi_s$



case 1: SPS protons above transition, no acceleration, q = 1, V_{max} = 4.5e6 V

The potential energy is "minus" the integral of the RF voltage



$$V(\varphi) = V_{max} sin\varphi$$

$$V(\varphi_s) = V_{max} sin\varphi_s = 0$$ ➔ "no acceleration"

$$\int_{\varphi_s}^{\varphi} \frac{q}{2\pi h \Gamma_s W_s} [V(\varphi') - V(\varphi_s)] d\varphi'$$

$$U(\varphi) = -\int_{\varphi_s}^{\varphi} \frac{q}{2\pi h \Gamma_s W_s} [V(\varphi') - V(\varphi_s)] d\varphi'$$

$$U(\varphi) = \frac{qV_{max}}{2\pi h \Gamma_s W_s} (1 + \cos\varphi)$$

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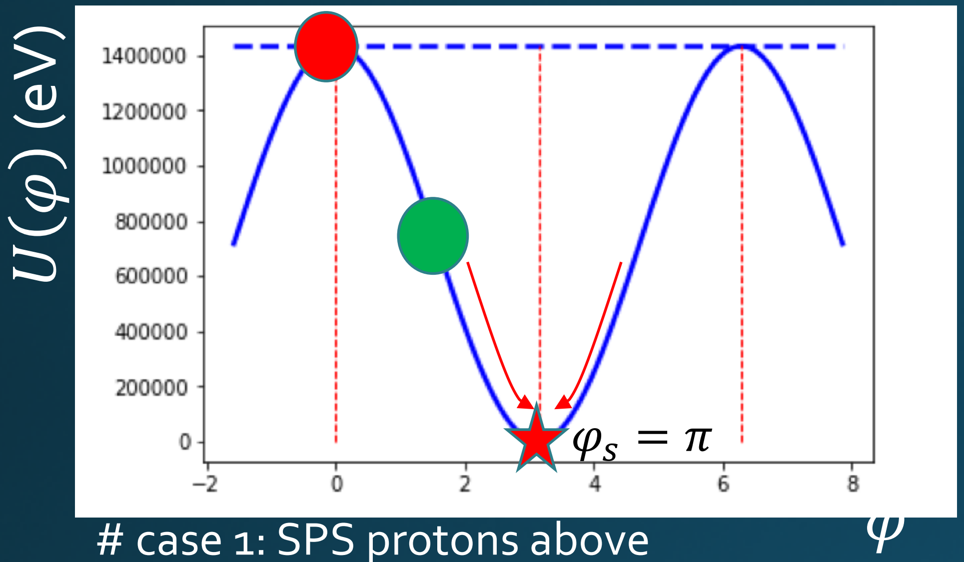


case 1: SPS protons above transition, no acceleration, q = 1, V_{max} = 4.5e6 V, single RF system

Near the synchronous phase the particles feel a restoring force which allows them to execute oscillations around it. In phase space these oscillations translate into closed trajectories which have an angular frequency called the synchrotron frequency.

In the case of **small amplitude oscillations**, i.e. when $\Delta\varphi = \varphi - \varphi_s \ll 1$ the angular synchrotron frequency can be analytically calculated:

$$\Omega_{sy} = \sqrt{-\frac{qh\Gamma_s V_{max} cos\varphi_s}{2\pi W_s}}\,\Omega_s \qquad \text{Eq. 35'}$$

However, quite a number of particles in the beam have **large amplitude** oscillations and then **behave non-linear** due to the shape of the potential produced by the sinusoidal RF voltage.

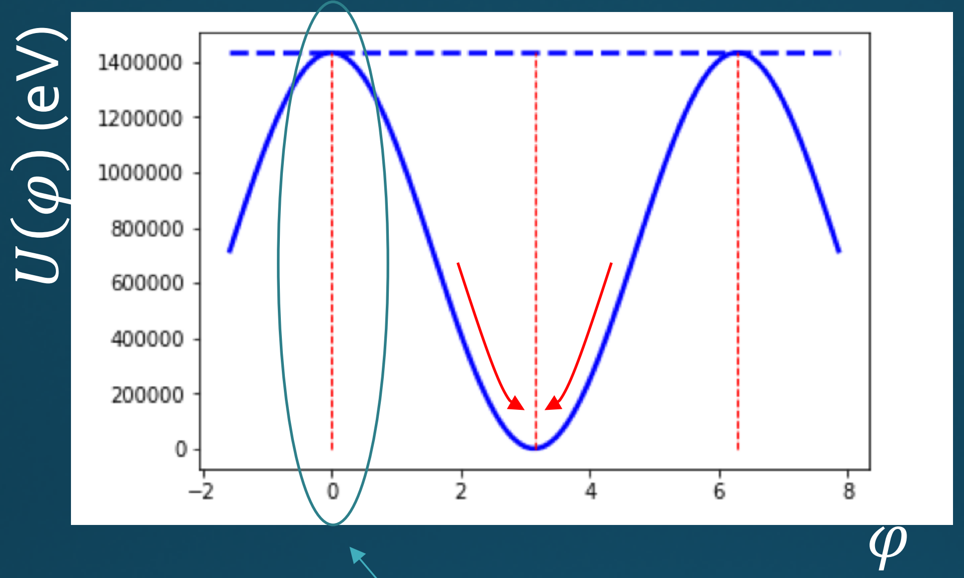
And for **very large amplitudes** they may even **scape from the stable region**, leading to particle losses

The division of the phase space into regions of bounded and unbounded motion in synchrotrons is the reason of grouping the particles into bunches.

The boundary between both regions is called the **SEPARATRIX.**
The phase space area enclosed by the separatrix is called the **BUCKET**.

Let's now **calculate the trajectories in phase space** that correspond to the plotted potential below:

$$U(\varphi) = \frac{qV_{max}}{2\pi h\Gamma_s W_s}(1+\cos\varphi)$$



case 1: SPS protons above transition, no acceleration, q = 1, V_{max} = 4.5e6 V, single RF system

First let's calculate the Hamiltonian or total energy of the system, **which is a constant**:

$$Hamiltonian = Total\ Energy = Kinetic\ energy + U(\varphi) = \frac{1}{2}w^2 + \frac{qV_{max}}{2\pi h\Gamma_s W_s}(1+\cos\varphi) \qquad \text{Eq. 53}$$

The simplest thing to do is to calculate the total energy when the kinetic energy is 0, i.e. w=0, and the potential energy is maximum. The potential energy is maximum when $\varphi = 0$.

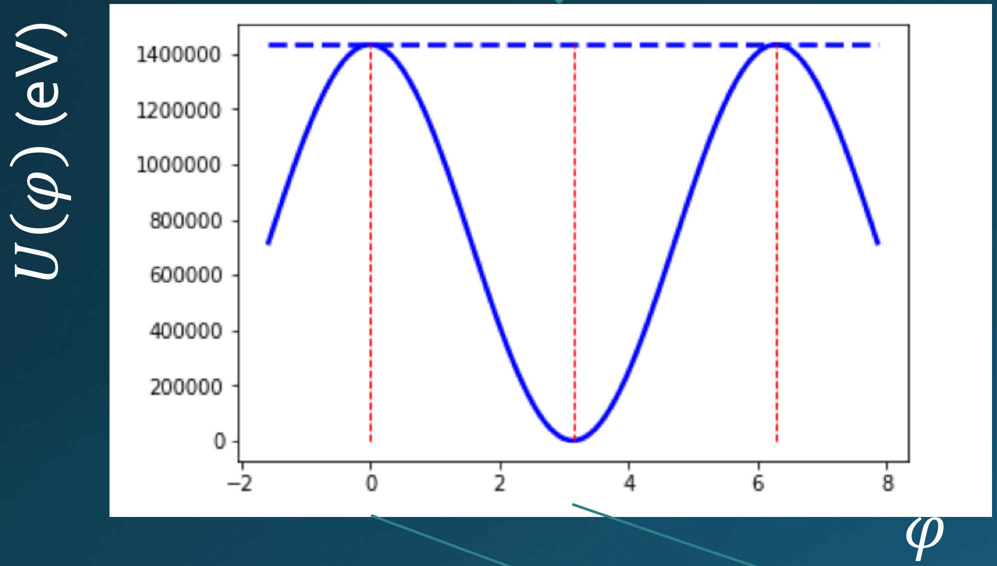
$$Hamiltonian = Total\ Energy = 0 + U(0) = \frac{qV_{max}}{\pi h\Gamma_s W_s} \qquad \text{Eq. 54}$$

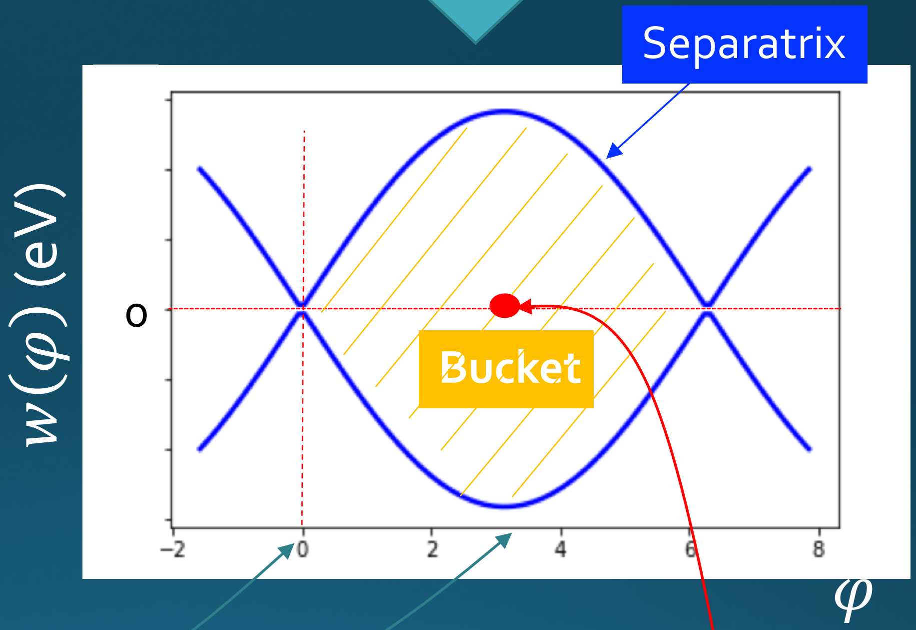
We put Eq. 54 in Eq. 53 and solve for w:

$$Hamiltonian = Total\ Energy = \frac{qV_{max}}{\pi h \Gamma_s W_s} = \frac{1}{2}w^2 \quad + \frac{qV_{max}}{2\pi h \Gamma_s W_s}(1 + \cos\varphi) \qquad \text{Eq. 55}$$

$$U(\varphi) = \frac{qV_{max}}{2\pi h \Gamma_s W_s}(1 + \cos\varphi)$$

$$w = \pm\sqrt{\frac{qV_{max}}{\pi h \Gamma_s W_s}(1 - \cos\varphi)} \qquad \text{Eq. 56}$$





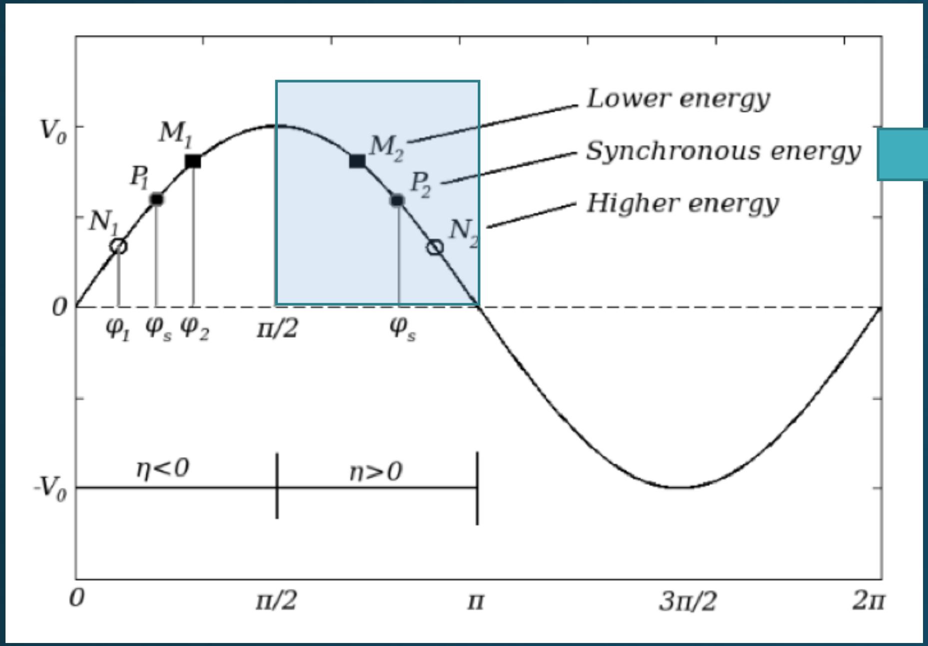
Stationary bucket ➔ particles are not accelerated

$\varphi = 0$ ➔ Maximum potential, zero kinetic energy

$\varphi = \pi$ ➔ $U(\varphi) = 0,\ K_{max} = \sqrt{\frac{2qV_{max}}{\pi h \Gamma_s W_s}}$

In a stationary bucket, the synchronous particle is always at $w(\varphi) = 0, \varphi = \varphi_s = 0\ or\ \pi\ (below\ or\ above\ transition)\ because\ V = qV_{max}sin\varphi_s = 0!$

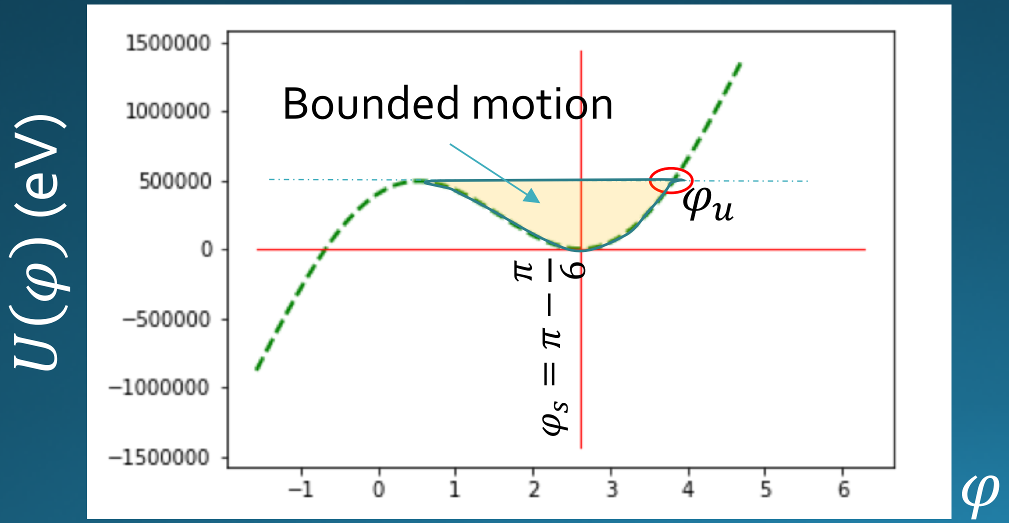
CASE 2: ACCELERATION (ABOVE TRANSITION)

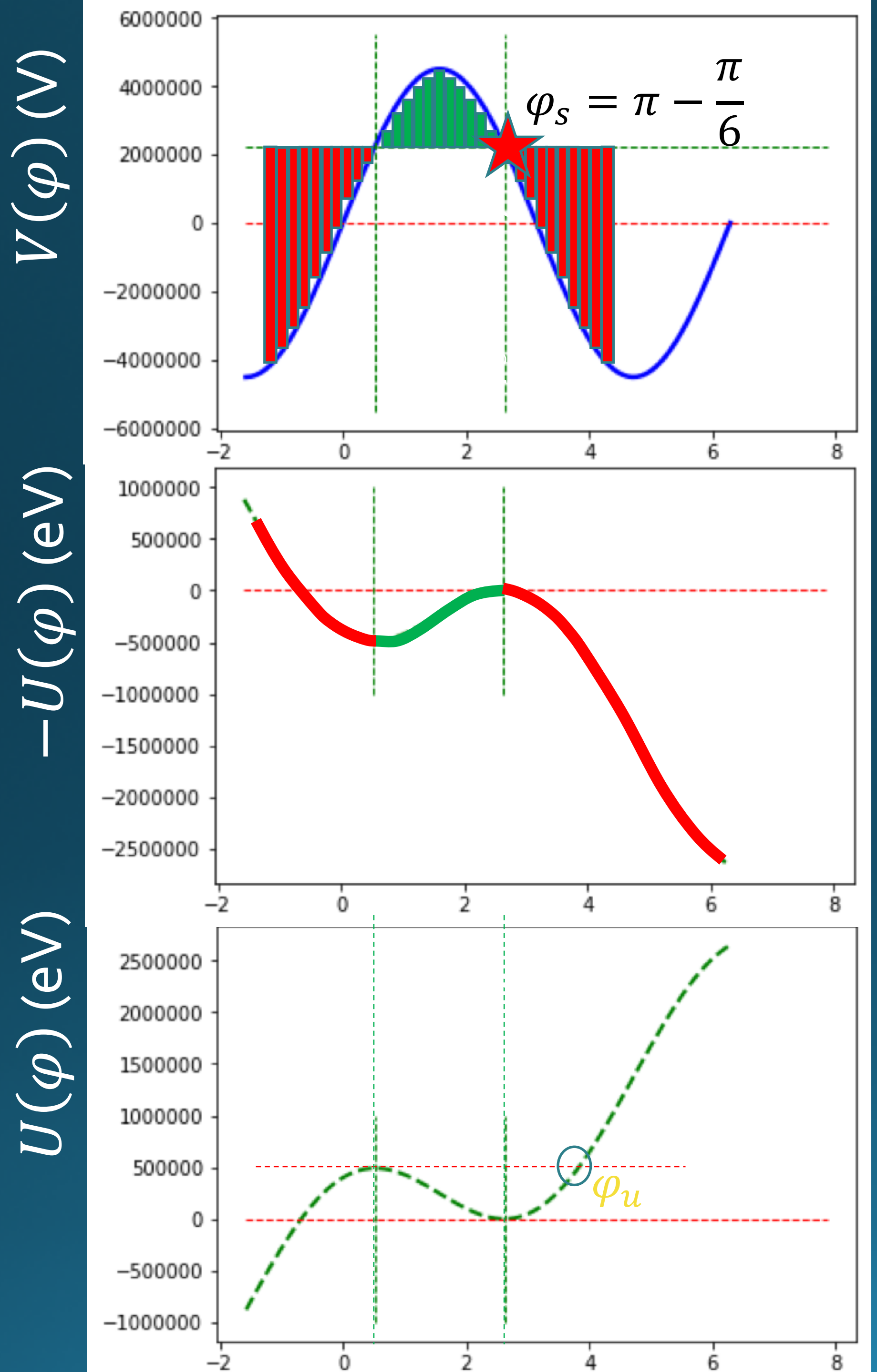


$$\varphi_s = \pi - \frac{\pi}{6}$$

$$U(\Delta\varphi) = \frac{qV_{max}}{2\pi h \Gamma_s W_s}\left(\cos(\Delta\varphi + \varphi_s) - cos\varphi_s + \Delta\varphi sin\varphi_s\right) \qquad \text{Eq. 51}$$

case 2: SPS protons above transition, acceleration, q = 1, V_{max} = 4.5e6 V, $\varphi_s = \pi - \frac{\pi}{6}$





$$V(\varphi_s) = V_{max} sin\varphi_s = V_{max}\, sin\left(\pi - \frac{\pi}{6}\right)$$
$$= 4.5e6\, sin\left(\pi - \frac{\pi}{6}\right)$$

$$V(\varphi) = V_{max} sin\varphi$$

$$\int_{\varphi_s}^{\varphi} \frac{q}{2\pi h \Gamma_s W_s}[V(\varphi') - V(\varphi_s)]d\varphi'$$

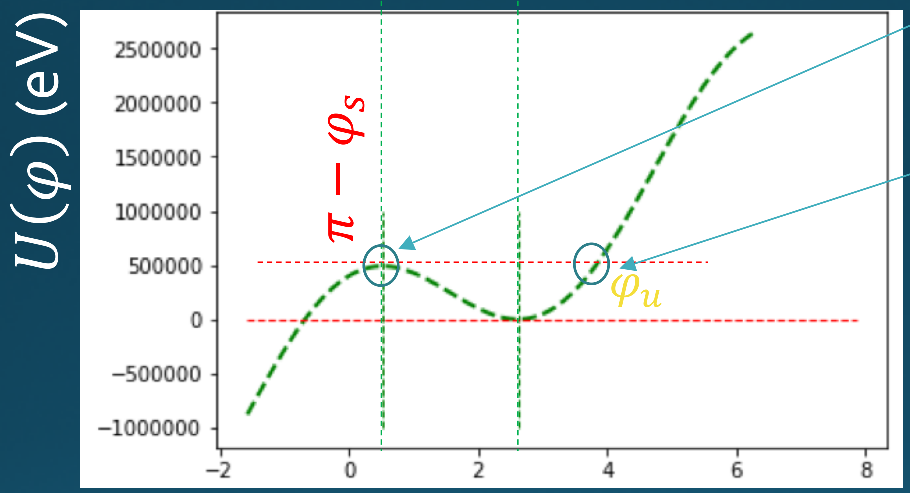
$$U(\varphi) = -\int_{\varphi_s}^{\varphi} \frac{q}{2\pi h \Gamma_s W_s}[V(\varphi') - V(\varphi_s)]d\varphi'$$

$$U(\Delta\varphi) = \frac{qV_{max}}{2\pi h \Gamma_s W_s}(\cos(\Delta\varphi + \varphi_s) - cos\varphi_s + \Delta\varphi sin\varphi_s) \qquad \text{Eq. 51}$$

Let's now **calculate the trajectories in phase space** that correspond to the plotted potential before

First let's calculate the Hamiltonian or total energy of the system, which is a constant.

The simplest thing to do is to calculate the total energy when the kinetic energy is 0, i.e. w=0, and the potential energy is maximum. The **potential energy is maximum when $\varphi = \pi - \varphi_s$**.



First point where particles are still bounded within the separatrix

The second point where particles are still bounded is φ_u

$$U(\pi - \varphi_s) = U(\varphi_u)$$

For single RF systems the total energy or separatrix can be calculated analytically by replacing $\varphi = \pi - \varphi_s$ in Eq. 51:

$$U(\Delta\varphi) = \frac{qV_{max}}{2\pi h \Gamma_s W_s}\left(\cos(\Delta\varphi + \varphi_s) - cos\varphi_s + \Delta\varphi sin\varphi_s\right) \qquad \text{Eq. 51}$$

First replace $\Delta\varphi$ by $\varphi - \varphi_s$, and then $\varphi = \pi - \varphi_s$ to obtain:

$$H_{sep} = U(\pi - \varphi_s) = \frac{qV_{max}}{2\pi h \Gamma_s W_s}(-2\cos\varphi_s + (\pi - 2\varphi_s)sin\varphi_s) \qquad \text{Eq. 57}$$

The phase space trajectory is then:

$$H_{sep} = \frac{1}{2}w^2 + U(\varphi) = \frac{1}{2}w^2 + \frac{qV_{max}}{2\pi h \Gamma_s W_s}(\cos\varphi - \cos\varphi_s + (\varphi - \varphi_s)sin\varphi_s) \qquad \text{Eq. 58}$$

Solving for $w(\varphi)$ we get:

$$\boldsymbol{w(\varphi) = \pm\sqrt{2\left(H_{sep} - U(\varphi)\right)}} \qquad \text{Eq. 59}$$

Coming back to the second point where particles are still bounded within the separatrix, denoted φ_u in the previous figure, we know the energy deviation there should be zero, i.e. $\omega = 0$. In this case:

$$H_{sep} = \frac{1}{2}\underbrace{w^2}_{=0} + U(\varphi_u) = \frac{qV_{max}}{2\pi h \Gamma_s W_s}(\cos\varphi_u - \cos\varphi_s + (\varphi_u - \varphi_s)sin\varphi_s) = U(\pi - \varphi_s) \qquad \text{Eq. 60}$$

⇓

$$(\cos\varphi_u - \cos\varphi_s + (\varphi_u - \varphi_s)sin\varphi_s) = (-2\cos\varphi_s + (\pi - 2\varphi_s)sin\varphi_s) \qquad \text{Eq. 61}$$

⇓

$$\cos\varphi_u + \varphi_u sin\varphi_s = \cos(\pi - \varphi_s) + (\pi - \varphi_s)sin\varphi_s \qquad \text{Eq. 62}$$

⇓

$$\boldsymbol{\left(\varphi_u - (\pi - \varphi_s)\right)}sin\varphi_s = \cos(\pi - \varphi_s) - \cos\varphi_u \qquad \text{Eq. 63}$$

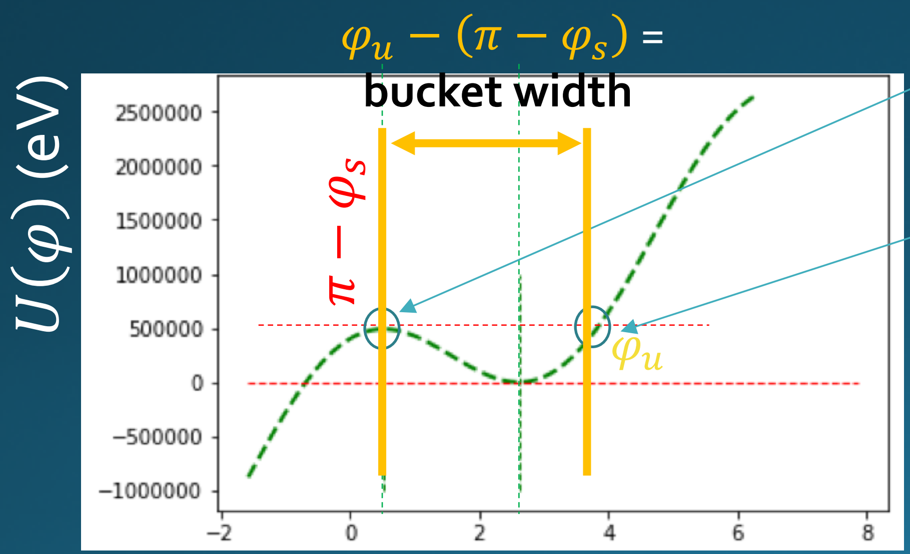
The term $\varphi_u - (\pi - \varphi_s)$ is called the **bucket width**

The **bucket height** at φ_s can be evaluated from Eq. 59:

$$w(\varphi) = \pm\sqrt{2\frac{qV_{max}}{2\pi h\Gamma_s W_s}\left((\pi-\varphi_s-\varphi)sin\varphi_s-(cos\varphi+cos\varphi_s)\right)} \qquad \text{Eq. 64}$$

bucket height at φ_s

$$w(\varphi=\varphi_s) = +\sqrt{2\frac{qV_{max}}{2\pi h\Gamma_s W_s}\left((\pi-2\varphi_s)sin\varphi_s-2cos\varphi_s\right)} \qquad \text{Eq. 65}$$

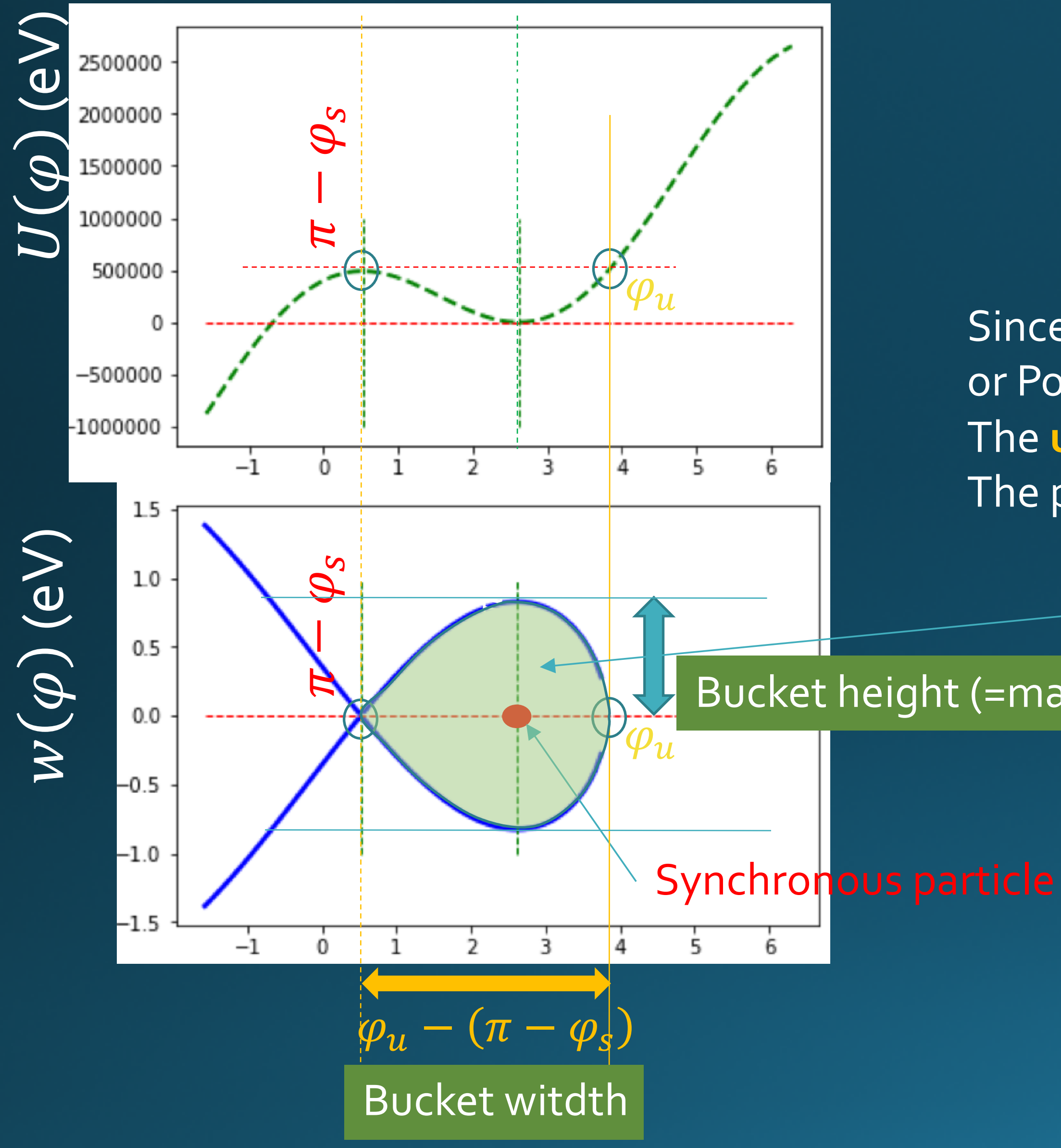


First point where particles are still bounded within the separatrix

The second point where particles are still bounded is φ_u

$$U(\pi-\varphi_s) = U(\varphi_u)$$

RF bucket parameters



Phase space area enclosed by the particle trajectory is:

$$A = \oint w d\varphi \qquad \text{Eq. 65}$$

Since (φ, w) are canonical conjugate variables, the **integral** is the action or Poincare invariant, therefore a **constant of motion**.
The **units** of this area are (**energy x time**) ➔ (**eVs**)
The phase space area enclosed by the separatrix is the

bucket area

Bucket height (=maximum energy deviation of the separatrix) ➔ Eq. 65

The local maximum of the potential at $\pi - \varphi_s$ is an unstable fixed point in the longitudinal phase space, while the local minimum gives a stable fixed point, φ_s, which corresponds to the centre of the bucket. At the stable and unstable fixed point the energy deviation is zero.

Using Eq. 59 and the symmetry around the φ axis, one can write for the bucket area:

bucket area
or longitudinal acceptance

$$A = 2\int_{\varphi_1}^{\varphi_u} w(\varphi)d\varphi = 2\int_{\varphi_1}^{\varphi_u} \sqrt{2\big(H_{sep} - U(\varphi)\big)}d\varphi \qquad \text{Eq. 66}$$

where $\varphi_1 = \pi - \varphi_s$, and φ_u can be found from Eq. 63

In the special case of a stationary bucket ($\varphi_s = 0$ or π), the bucket area and height can be calculated analytically
➔ exercise

LONGITUDINAL EMITTANCE AND BUNCH CHARACTERISTICS

All calculated variables in the previous slides, where calculated to the full extend of the stable area.
In practice, in order to avoid particle losses only a fraction of the stable area is usually occupied by the beam, enclosed by a single particle trajectory in phase space.
This area is called single particle emittance.

single particle longitudinal emittance

The trajectory of this particle can be derived from Eq. 59, but now we replace H_{sep} by the new value of the Hamiltonian at a phase where the trajectory crosses the horizontal axis. We call this phase φ_1 and the Hamiltonian $H_c = U(\varphi_1)$. The second point at φ_2 also satisfies that the energy deviation is 0, therefore:

$$U(\varphi_1) = U(\varphi_2) \qquad \text{Eq. 67}$$

For a single RF system this means:

$$cos\varphi_1 + \varphi_1 sin\varphi_s = cos\varphi_2 + \varphi_2 sin\varphi_s \qquad \text{Eq. 68}$$

After identifying the two turning points, the area under a given trajectory can be calculated from the integral:

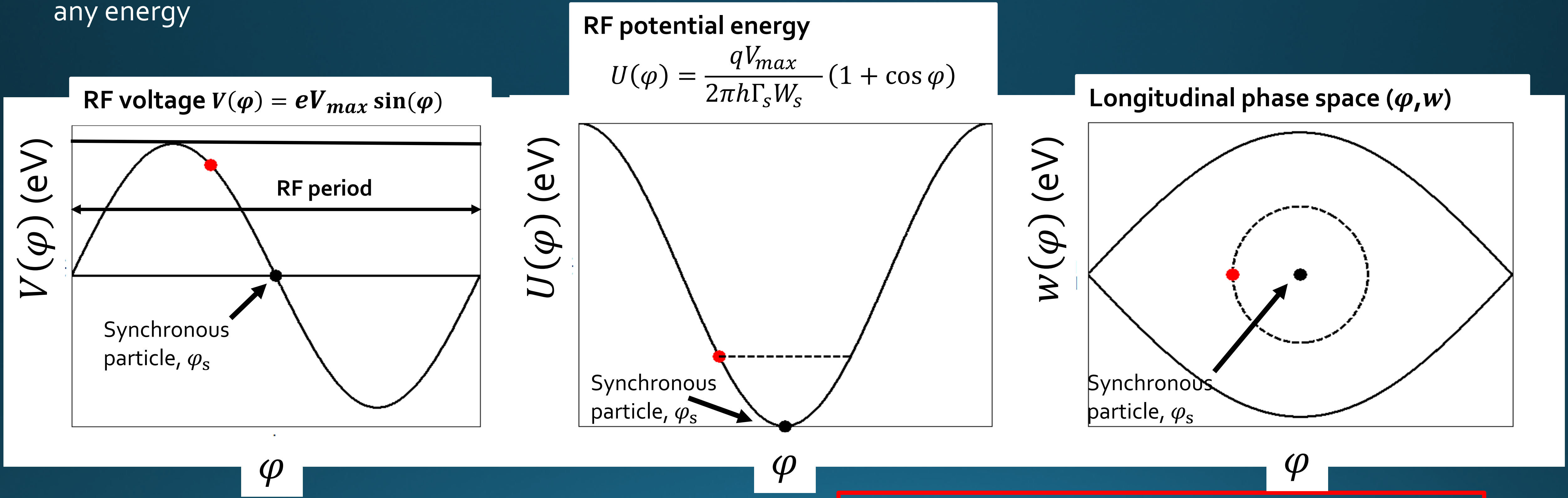
single particle longitudinal emittance

$$\varepsilon_l = 2\int_{\varphi_1}^{\varphi_2} \sqrt{2(H_c - U(\varphi))}d\varphi \qquad \text{Eq. 69}$$

Summary

Single particle synchrotron motion – no acceleration

- All particles are **oscillating around the synchronous one** with frequencies called **synchrotron frequencies $f_s(\varphi)$**.
- **Stationary case**: In the LHC $\varphi_s = \pi$ (above transition) → $V(\pi) = eV_{max}\sin(\pi) = 0$ → synchronous particle does not gain any energy



The state of any particle, at every moment is represented by a single point **(φ,w)** in the **Longitudinal phase-space.**

Large Amplitude Oscillations

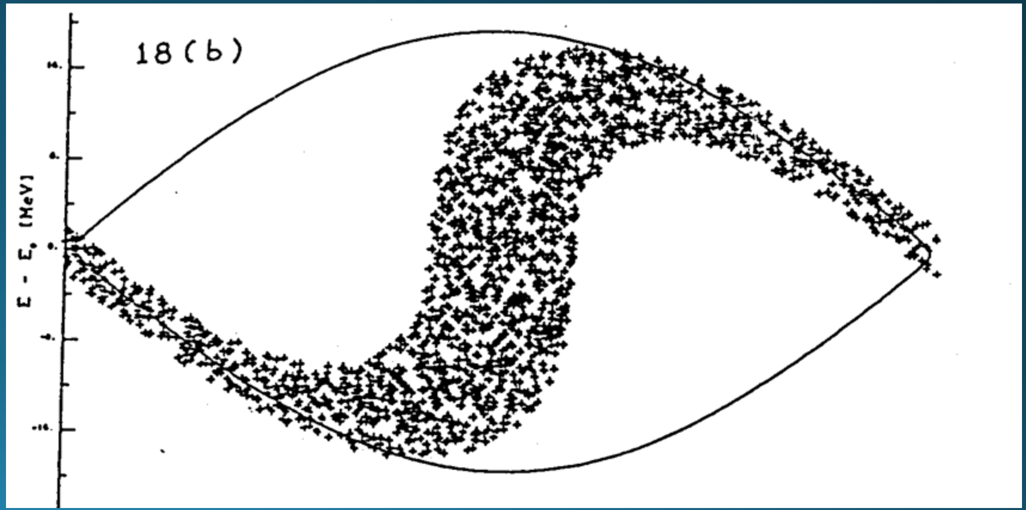
For larger phase (or energy) deviations from the reference (i.e. synchronous) the second order differential equation is non-linear:

FIRST EQUATION OF MOTION $\dot{w} = \frac{qV_{max}\Omega_s}{2\pi W_s}(\sin(\Delta\varphi + \varphi_s) - sin\varphi_s)$ Eq. 25

SECOND EQUATION OF MOTION $\Delta\dot{\varphi} = \frac{\delta(\Delta\varphi)}{\delta t} = h\Gamma_s\Omega_s w$ Eq. 34

$\Delta\ddot{\varphi} = h\Gamma_s\Omega_s\dot{w}$ Eq. 36

The restoring force cannot be linearised anymore and the differential equation has no analytic solution



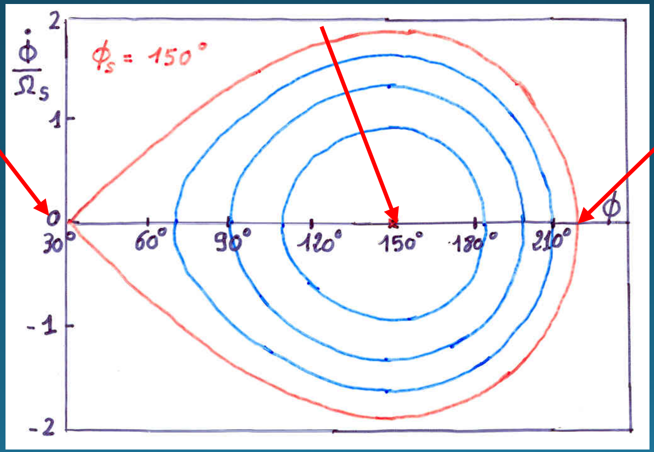
FIRST EQUATION OF MOTION $\dot{w} = \frac{qV_{max}\Omega_s}{2\pi W_s}(\sin(\Delta\varphi + \varphi_s) - sin\varphi_s)$ Eq. 25

For $\varphi = \pi - \varphi_s$: unstable fixed point

$$(\sin\varphi - sin\varphi_s) => \sin(\pi - \varphi_s) - sin\varphi_s$$
$$\underbrace{sin\pi cos\varphi_s}_{0} - \underbrace{cos\pi sin\varphi_s}_{-1\, sin\, \varphi_s} - sin\varphi_s = 0$$

$$\Delta\ddot{\varphi} = h\Gamma_s\Omega_s\dot{w} = 0$$ Eq. 36

No restoring force!! for $\varphi = \pi - \varphi_s$ and for $\varphi = \varphi_s$ (trivial case) ... and for φ_u



The synchrotron frequency → 0

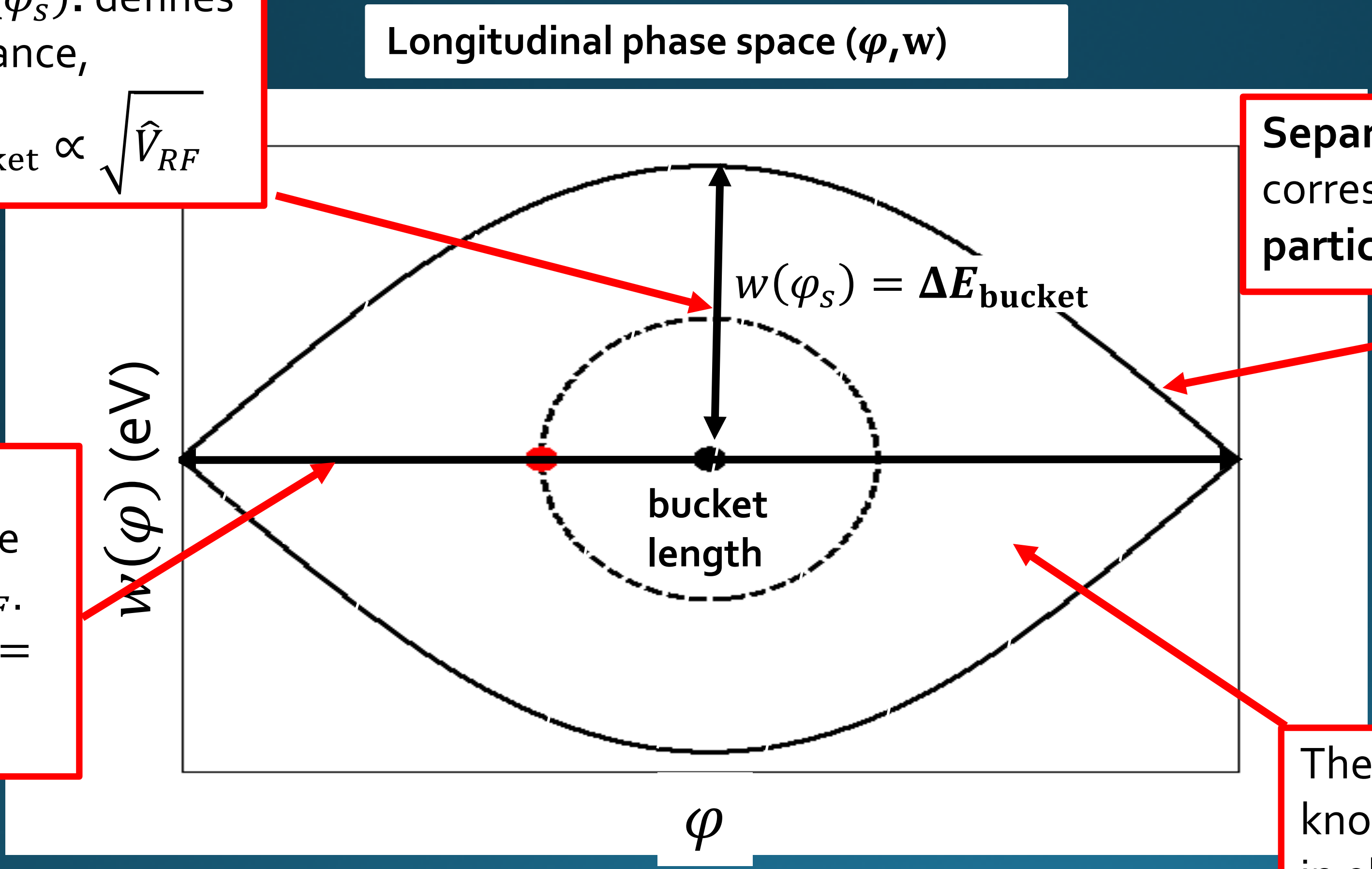
Single particle synchrotron motion – no acceleration

- **Stationary case**: In the LHC $\varphi_0 = \varphi_s = \pi$ (above transition) → $\boldsymbol{V(\pi) = eV_{max}\sin(\pi) = 0}$ → synchronous particle does not gain any energy

Bucket Height, $w(\varphi_s)$**:** defines the bucket acceptance,

$$w(\varphi_s) = \Delta E_{\mathrm{bucket}} \propto \sqrt{\hat{V}_{RF}}$$

Bucket Length: determined by the RF frequency $\boldsymbol{\omega_{RF}}$. In time units: $\Delta\tau = \frac{2\pi}{\omega_{RF}} = T_{RF}$



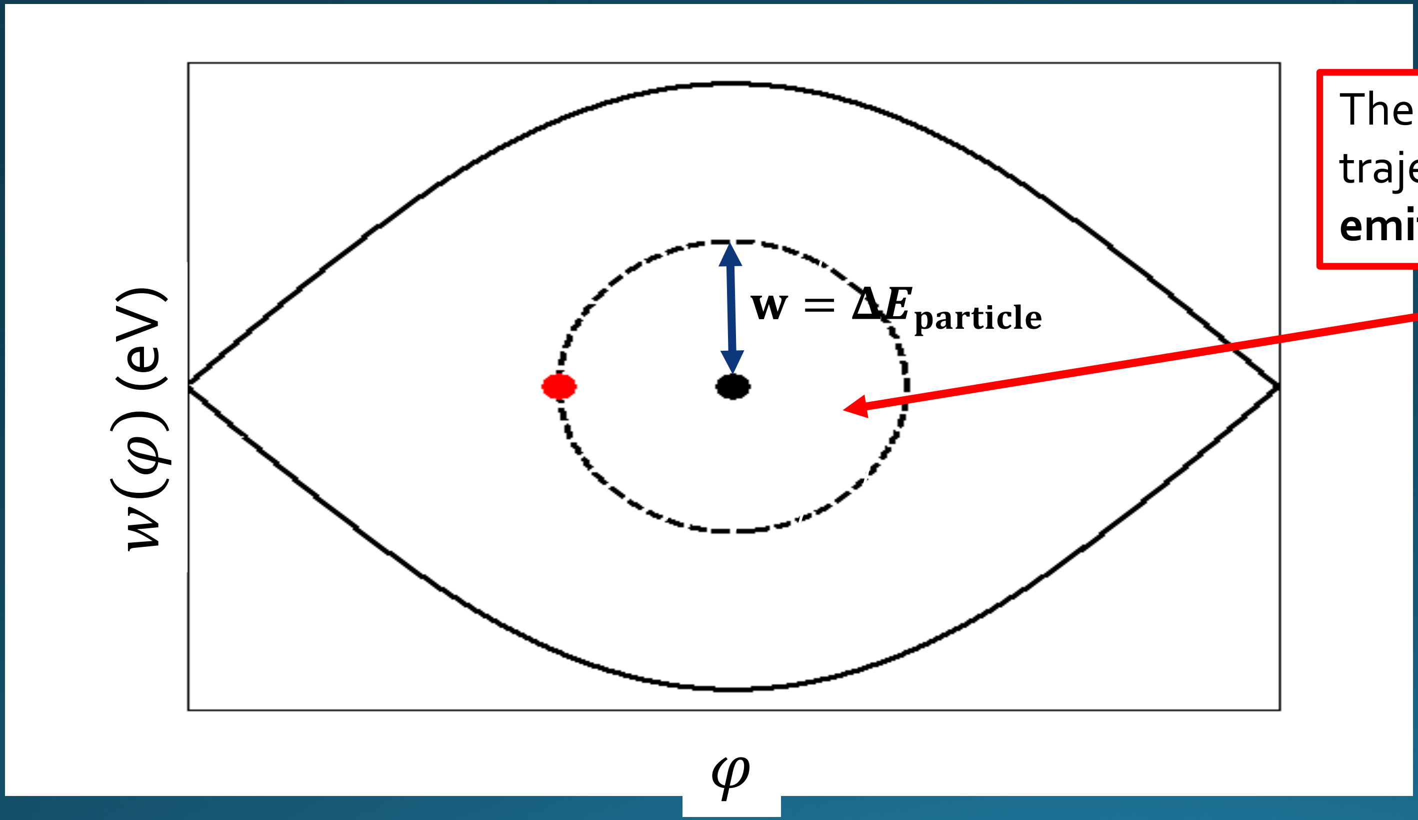
Separatrix: defines the **Bucket** and corresponds to the **trajectory of the particle with maximum offset from** $\boldsymbol{\varphi_s}$

The area enclosed by the separatrix is known as **Bucket Area.** It is measured in electronvolt-seconds (**eVs**)

Single particle synchrotron motion – no acceleration

- **Stationary case**: In the LHC $\varphi_s = \pi$ (above transition) → $\boldsymbol{V(\pi) = eV_{max}\sin(\pi) = 0}$ → synchronous particle does not gain any energy

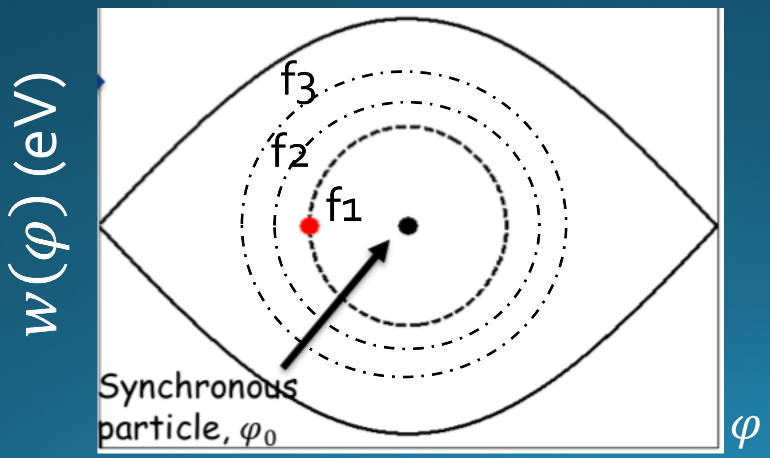
Longitudinal phase space (φ, w)



The area enclosed by the single particle trajectory gives the **single particle emittance.** It is measured in **eVs**.

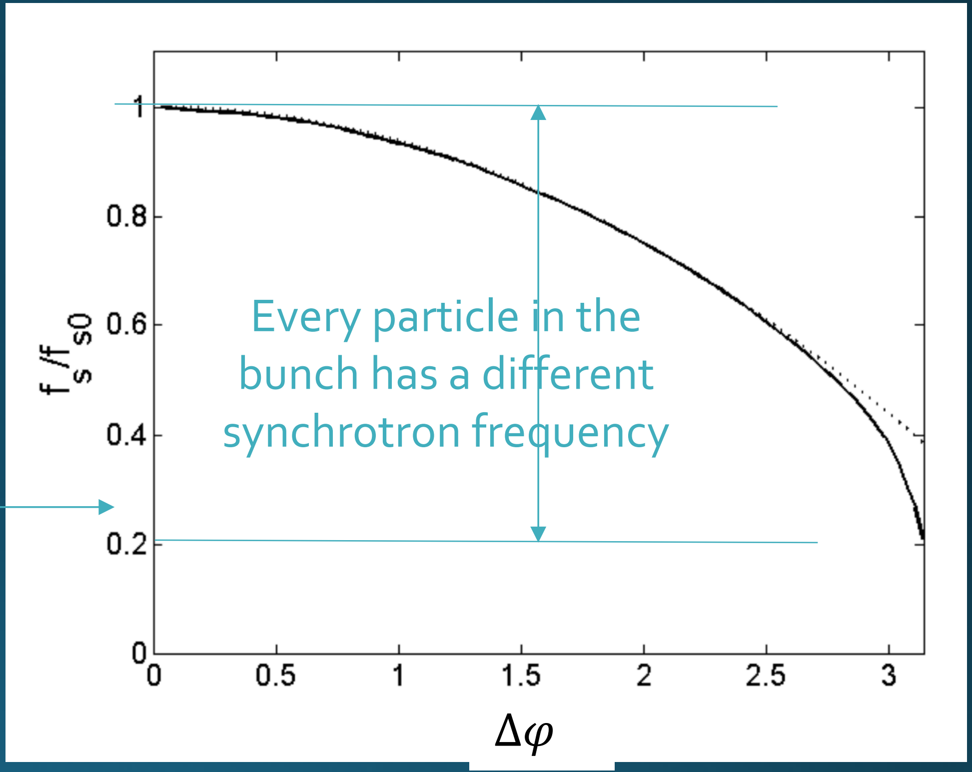
SYNCHROTRON FREQUENCY

- As discussed before, particles bounded within the bucket are performing oscillations around the stable phase.
- The synchrotron oscillation period is the time that takes to travel all around the closed trajectory (defined by the initial conditions) in phase space.
- Particles at different distances from the synchronous particle (synchronous phase) will oscillate with different synchrotron frequencies in the longitudinal phase space.
- The longer the bunch the larger the spread of synchrotron frequencies.
- Large synchrotron frequency spreads within the bunch (i.e. large bunches) provide a self-stabilizing mechanism against coherent beam instabilities, called Landau damping.



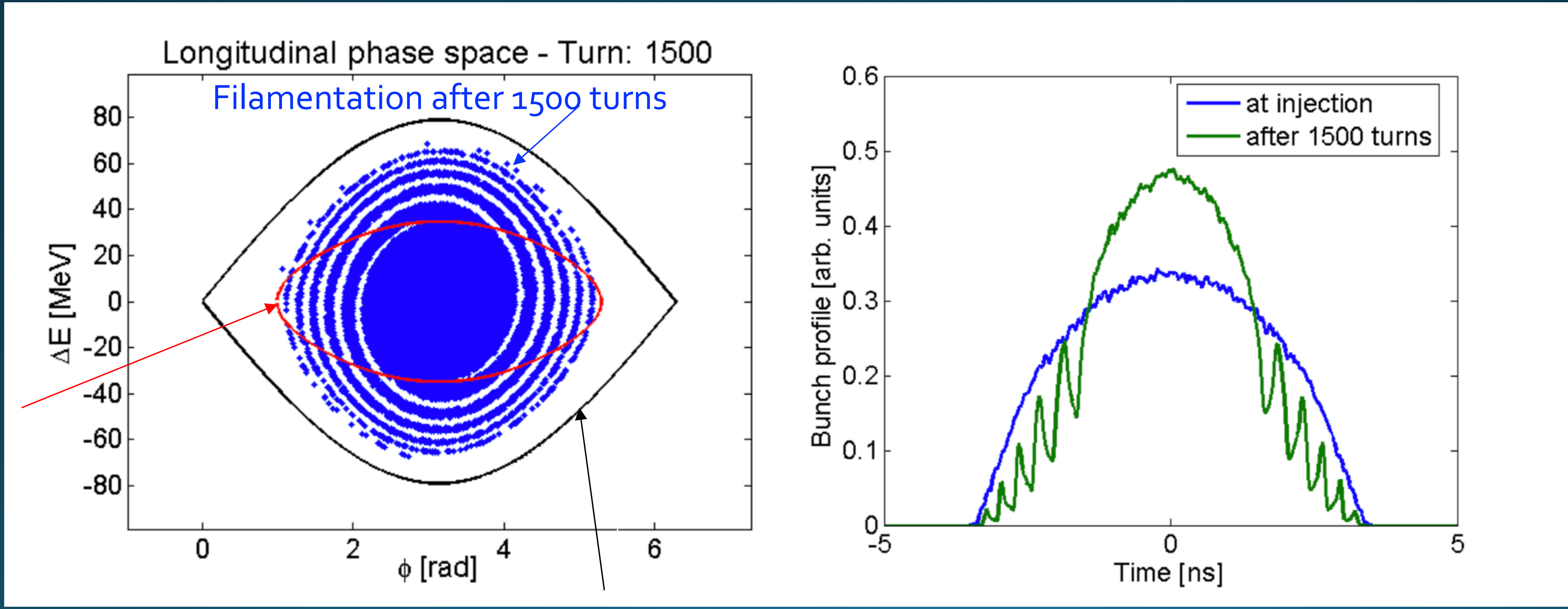
Synchrotron frequency spread

The synchrotron frequency versus the phase in the case of a single RF system and a stationary bucket (The exact curve is the solid line, the approximate solution is the dashed line)



FILAMENTATION

- The spread in synchrotron frequency is the responsible for the beam filamentation.
- E.g. in case of a large bunch (occupies substantial fraction of the bucket area) that it is not matched to the bucket, for example the bunch is injected into another machine, the difference in frequencies leads to beam filamentation.
- This process causes the mismatched bunch distribution to evolve into spirals, diluting the phase space density of the beam:

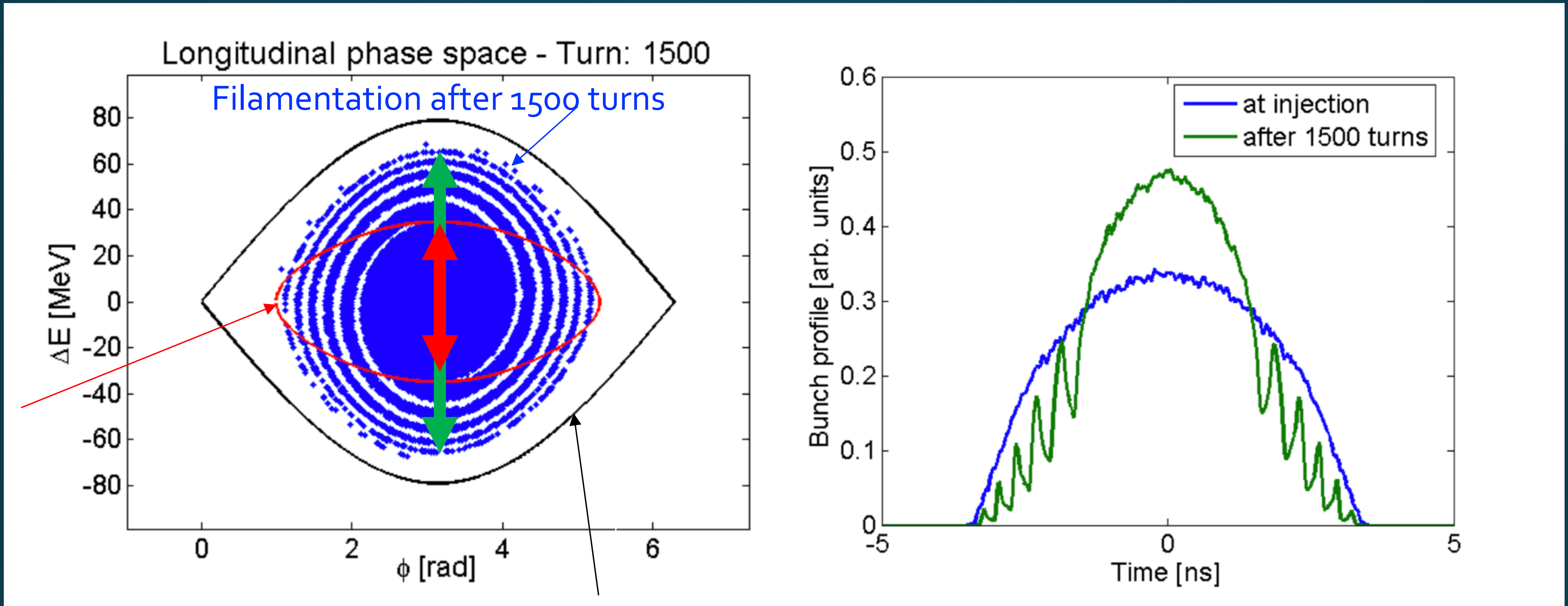


Injected beam from a previous machine

Bucket from the receiving machine

For this machine 1500 turns = 11 synchrotron periods

Injected beam from a previous machine

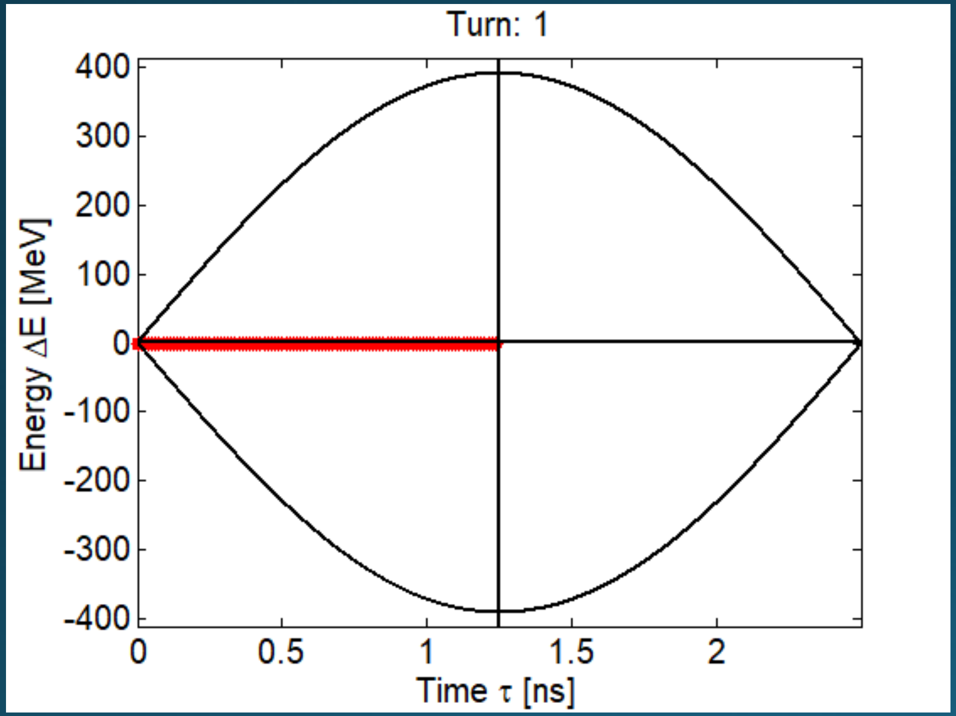


Bucket from the receiving machine

For this machine 1500 turns = 11 synchrotron periods

- The w≡$(\Delta p/p_s)_{\text{max}}$ that we need to take into account when calculating the horizontal displacement of the reference closed orbit ($x_{\Delta E}(s) = D(s)\frac{\Delta p}{p_s}$) is not the initial energy deviation (red arrow) but the energy deviation after filamentation (green arrow)

Single particle synchrotron motion – Synchrotron frequency distribution



- Particles closer to the synchronous one oscillate with higher frequency.
- The zero (from φ_0) amplitude synchrotron frequency is given by:

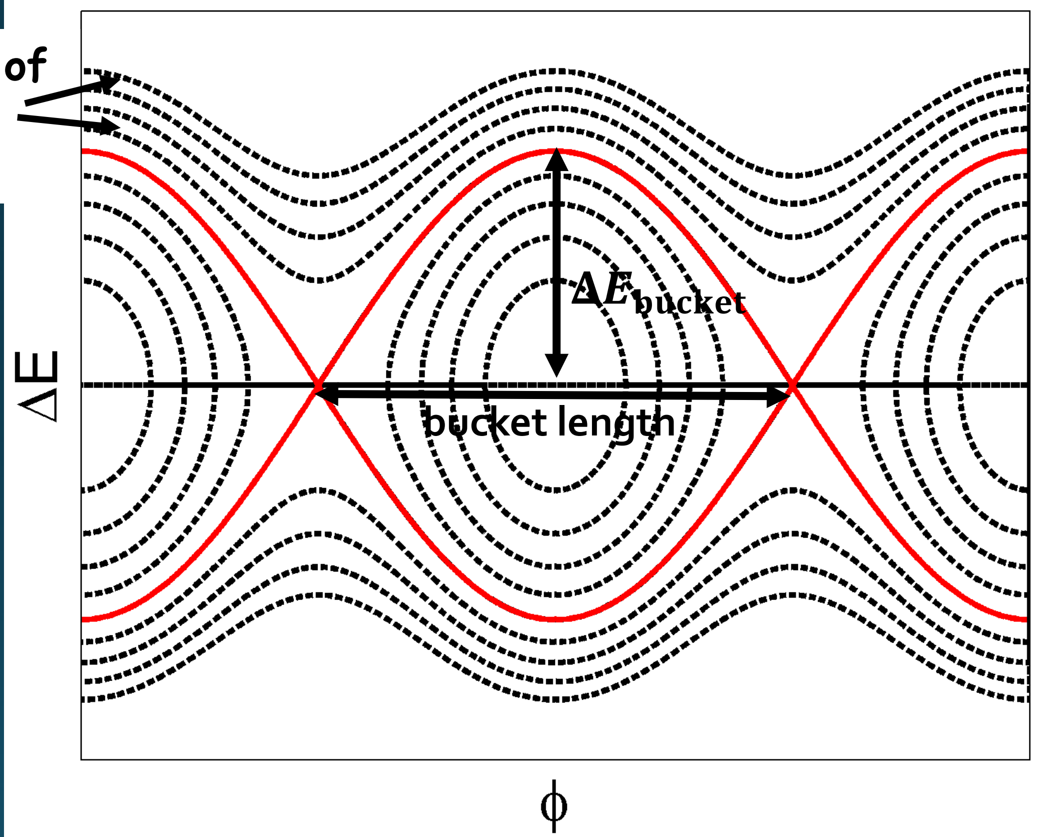
$$\omega_{sy}(0) = 2\pi f_{sy}(0)$$
$$= \sqrt{-\frac{h\omega_s^2 \eta \cos\varphi_s q V_{max}}{2\pi\beta_s^2 \quad W_s}}$$

Single particle synchrotron motion – bucket losses

- Particles with phases outside the **bucket length** will be captured by the adjacent buckets.

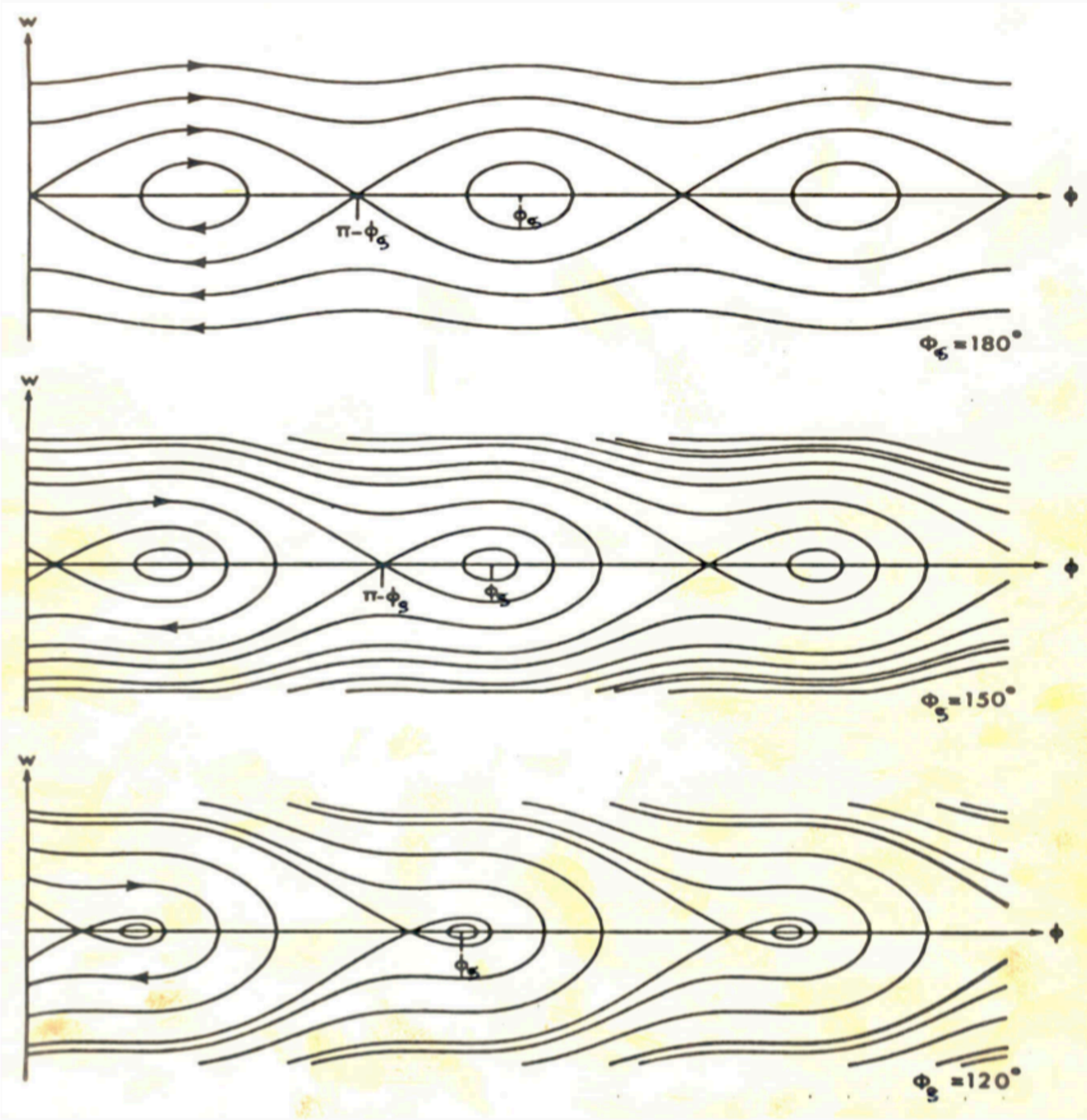
- Particles with $\mathbf{\Delta E_{particle}} > \mathbf{\Delta E_{bucket}}$ will not be captured in the bucket. They will follow the open dotted lines (shown in the plot) and finally **will be lost** when the acceleration starts.

Trajectories of uncaptured particles



The separatrix separates the open and the closed trajectories. Particles that are inside the bucket are grouped into **Bunches.**

RF cavities are used to **accelerate** and to **bunch** the particle beams.

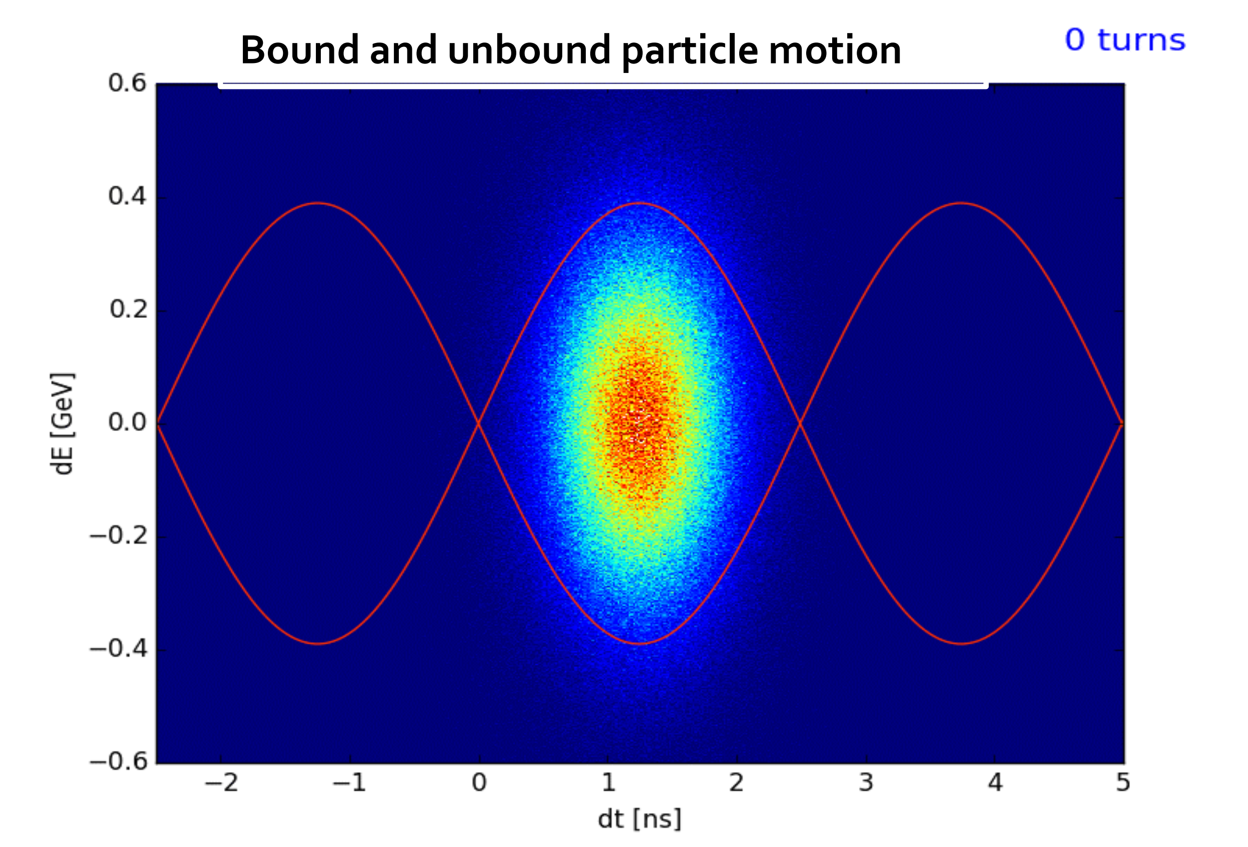


w
ϕ
π-ϕs
ϕs
ϕs =180°
w
ϕ
π-ϕs
ϕs
ϕs =150°
w
ϕ
ϕs
ϕs =120°

Single particle synchrotron motion – bucket losses

- Example of bound and unbound motion of the particles in the longitudinal phase space.

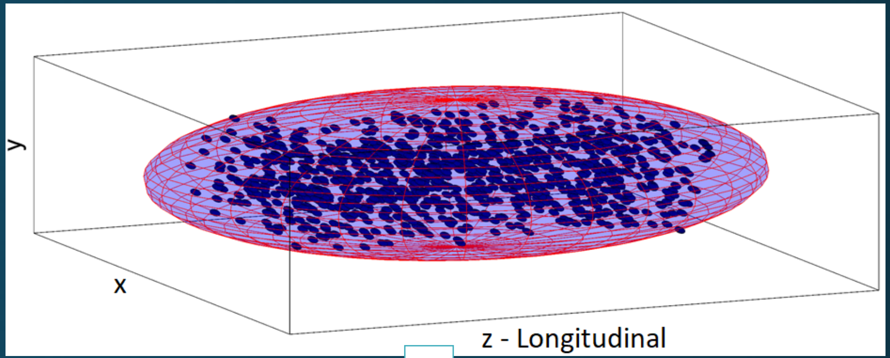
- Note that particles with $\Delta E_{particle} > \Delta E_{bucket}$ do not stay within the bunch but are moving along the separatrices and thus along the ring, forming the **un-bunched circulating beam**.



H. Timko, Simulations with BLonD

Bunch parameters

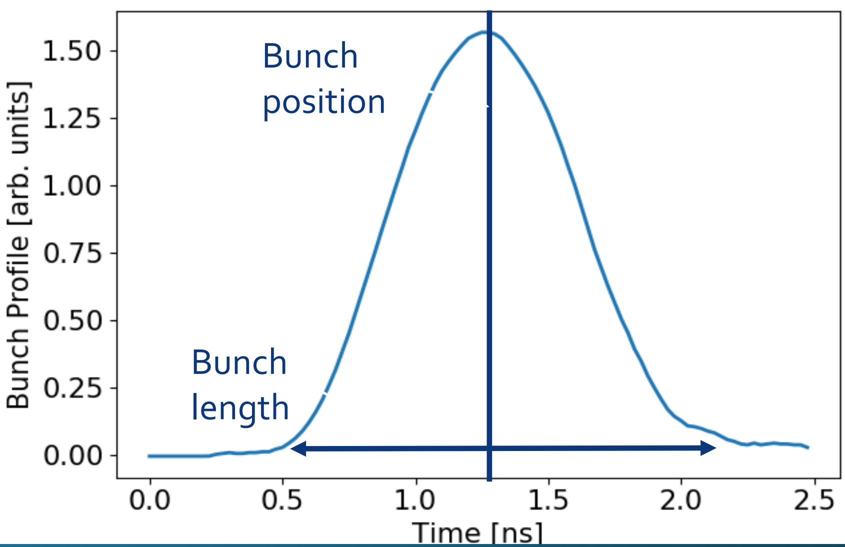
- **Bunches are formed from many particles** ($10^9 - 10^{15}$ particles) → the description of the bunch and its motion is represented by **statistical quantities**.
  - **Bunch position** (usually in seconds): position of the centroid of the bunch.
  - **Bunch length** (usually in seconds): size of the bunch.
  - **Bunch emittance** (in eVs): area in the longitudinal phase-space enclosed by a limiting single particle trajectory.



Projection to the z-axis

- In the **longitudinal plane** we use the **wall-current-monitors** and a **fast sampling oscilloscope** to **measure** the **bunch profile** which is the **projection of the bunch to the z-axis**.

No direct measurement on the energy distribution (ΔE=w) of the particles within the bunch.



- The statistical nature of the bunch parameters imply a **strong dependence on the particle distributions** → **different conventions** of defining the bunch parameters are used in the accelerators.

Bunch parameters

- Usually the **bunch parameters** (position, length or σ) are obtained **after fitting the bunch profiles** → different fitting functions can be used (Gaussian, parabolic, binomial, q-Gaussian etc.)

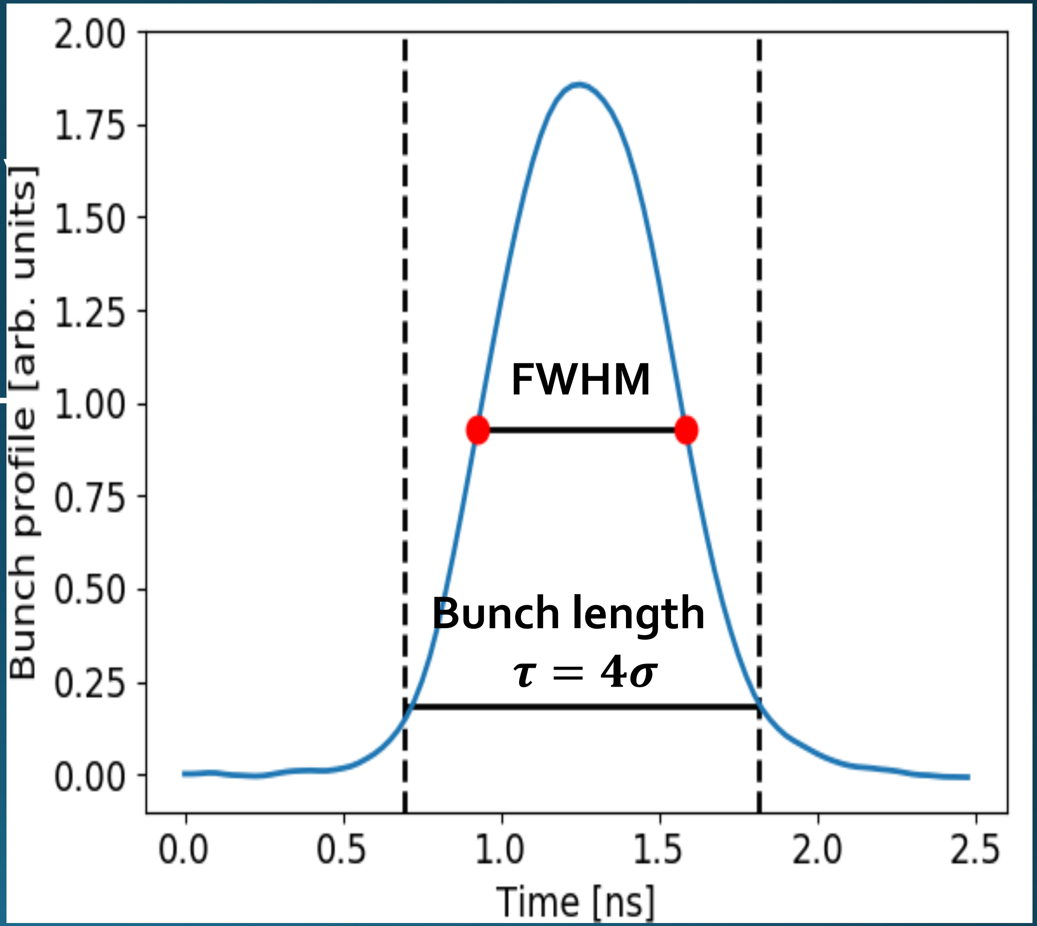
- However, **fitting algorithms are time consuming** and cannot be efficiently applied in the operational tools. Especially in the case of LHC with large number of bunches and long duration of the fills.

- Alternative algorithm based on the **measurement of the Full-Width-Half-Maximum (FWHM)** is applied in the LHC (and the SPS).

- The FWHM of each bunch is quickly measured from the acquired beam profiles and then the **standard deviation σ** of the bunch is obtained **assuming a Gaussian distribution**, according to the equation:

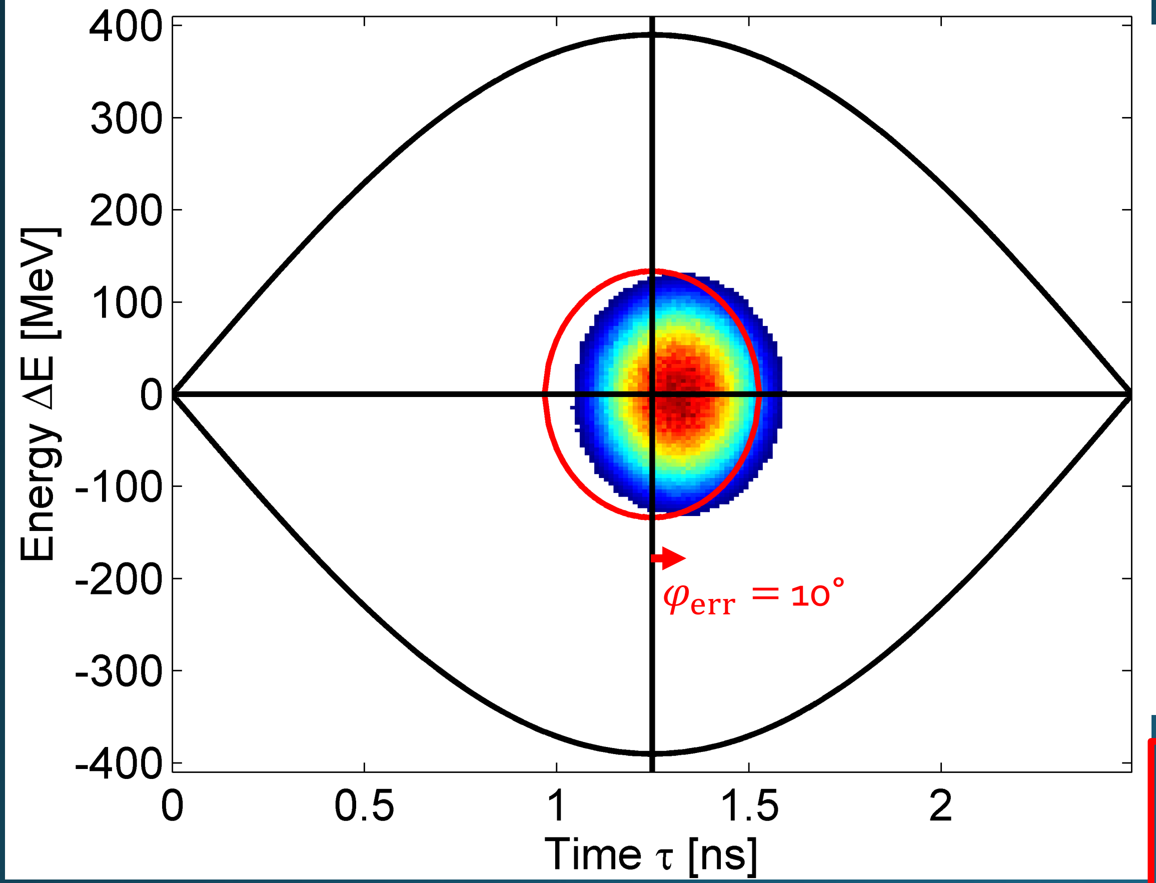
$$\boldsymbol{\sigma = \frac{FWHM}{2\sqrt{2\ln 2}}}$$

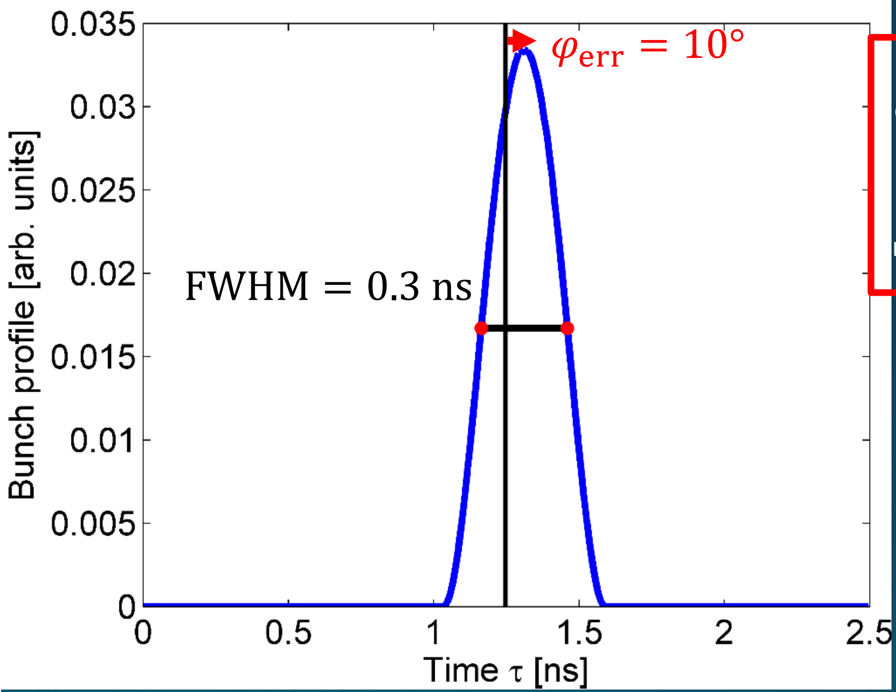
- The **bunch length** is then defined as: $\boldsymbol{\tau = 4\sigma}$



Phase error at injection

- Injection of a bunch into the RF bucket with a phase error $\varphi_{\mathrm{err}} = 10°$ (or $\Delta\tau_{\mathrm{err}} = \varphi_{\mathrm{err}}/\omega_{\mathrm{RF}}$ = 0.07 ns in time)





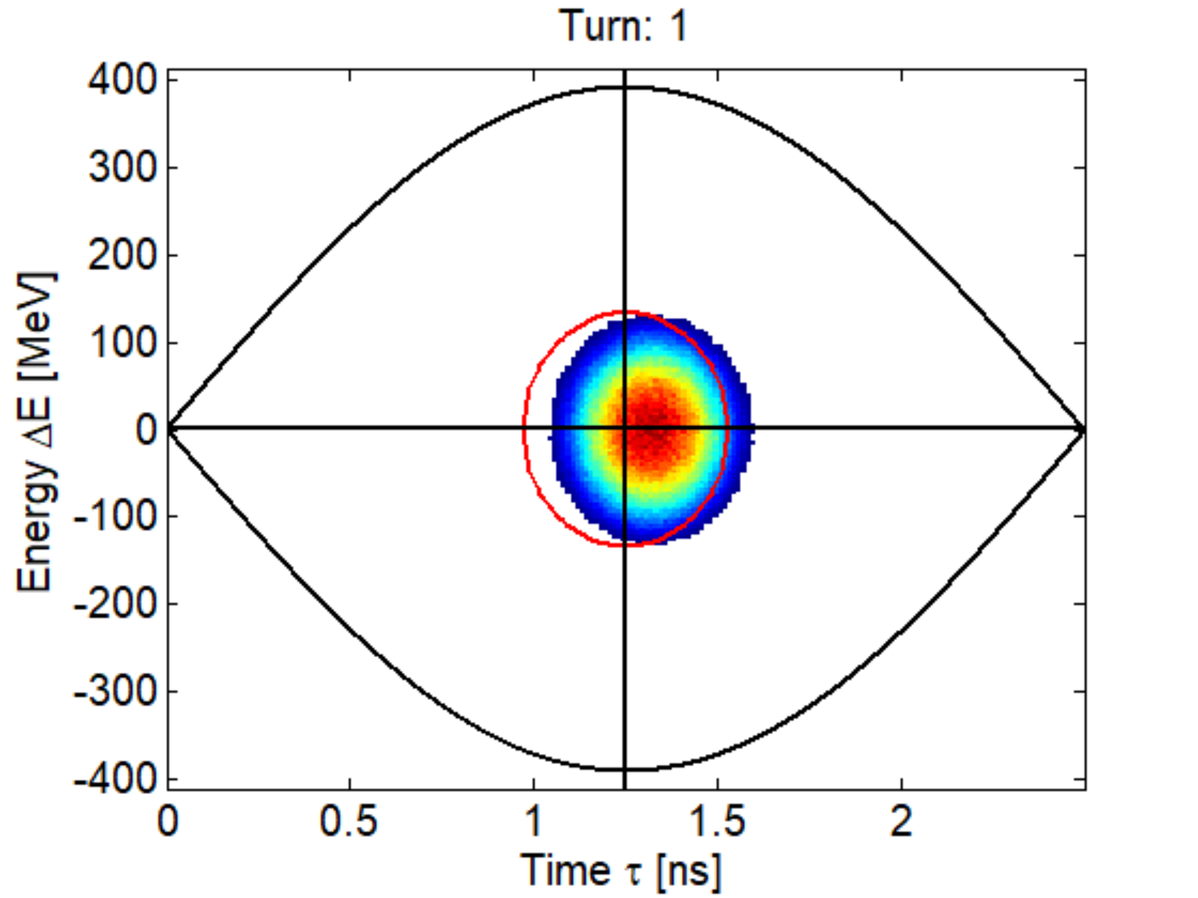
$$\sigma = \frac{\mathrm{FWHM}}{2\sqrt{2\ln 2}} = 0.126\ \mathrm{ns}$$

$\Rightarrow \tau = 4\sigma = 0.5\ \mathrm{ns}$

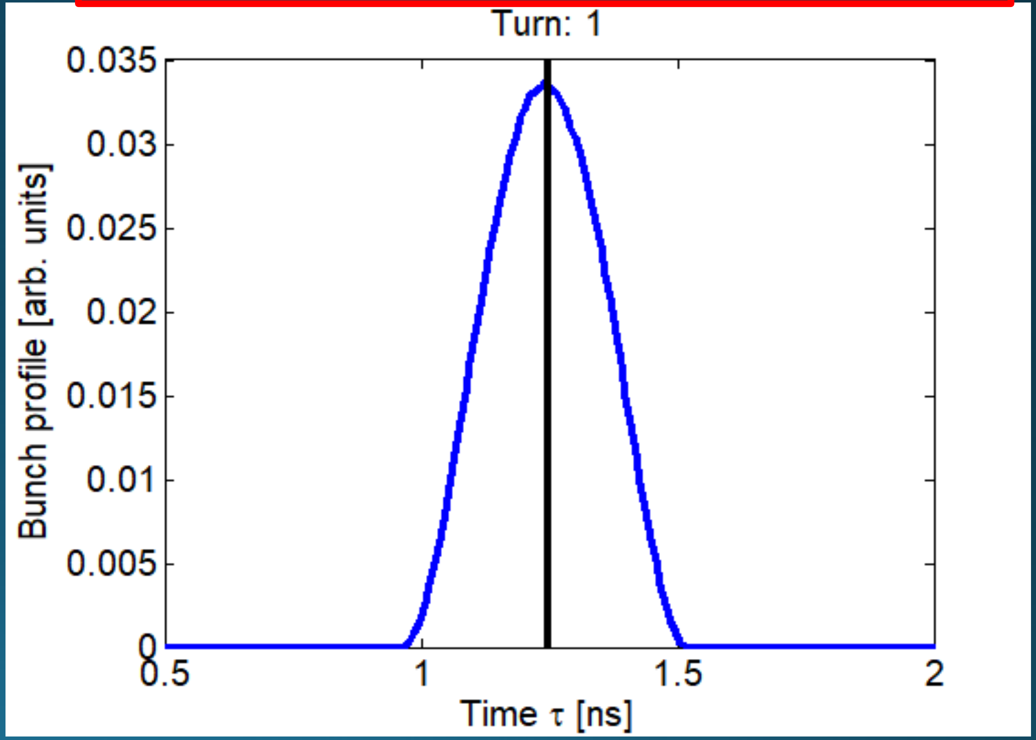
Note that the size of the bunch in this example is much smaller than the one that is normally injected from the SPS into the LHC ($\tau_{4\sigma} \sim 1.4 - 1.6$ ns)

Phase error at injection – Dipole oscillations

- The bunch will initially start to **oscillate as a whole with frequency close to the zero amplitude synchrotron frequency f_{s0}.**
- This bunch motion is called **dipole oscillation**



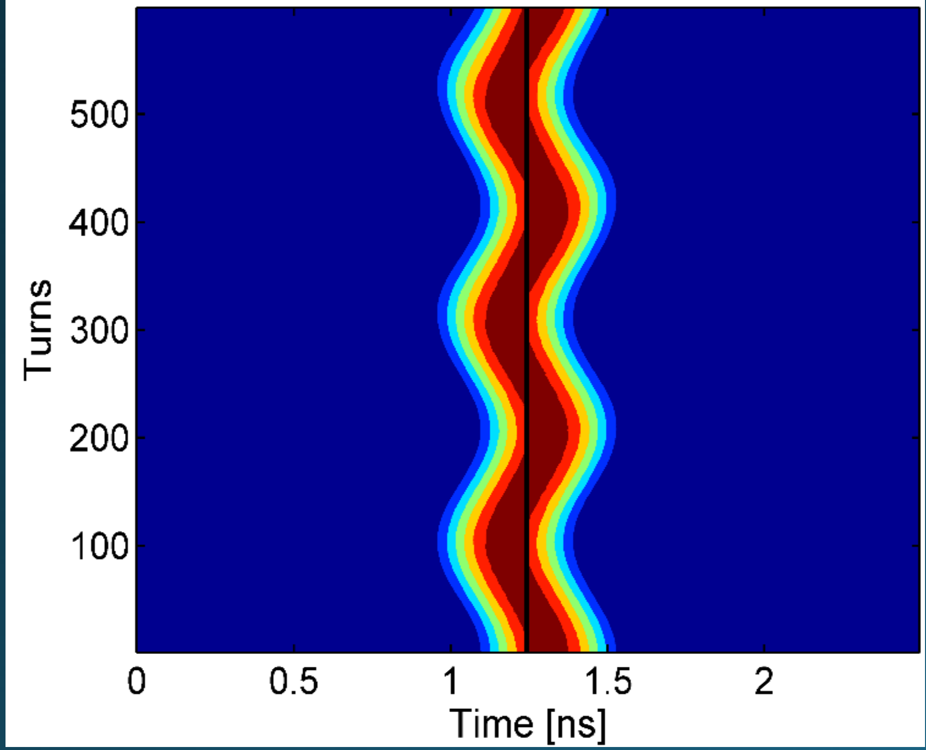
Projection into the Time axis. This is what **can be measured in the machine** using a wall-current monitor and a fast sampling oscilloscope.



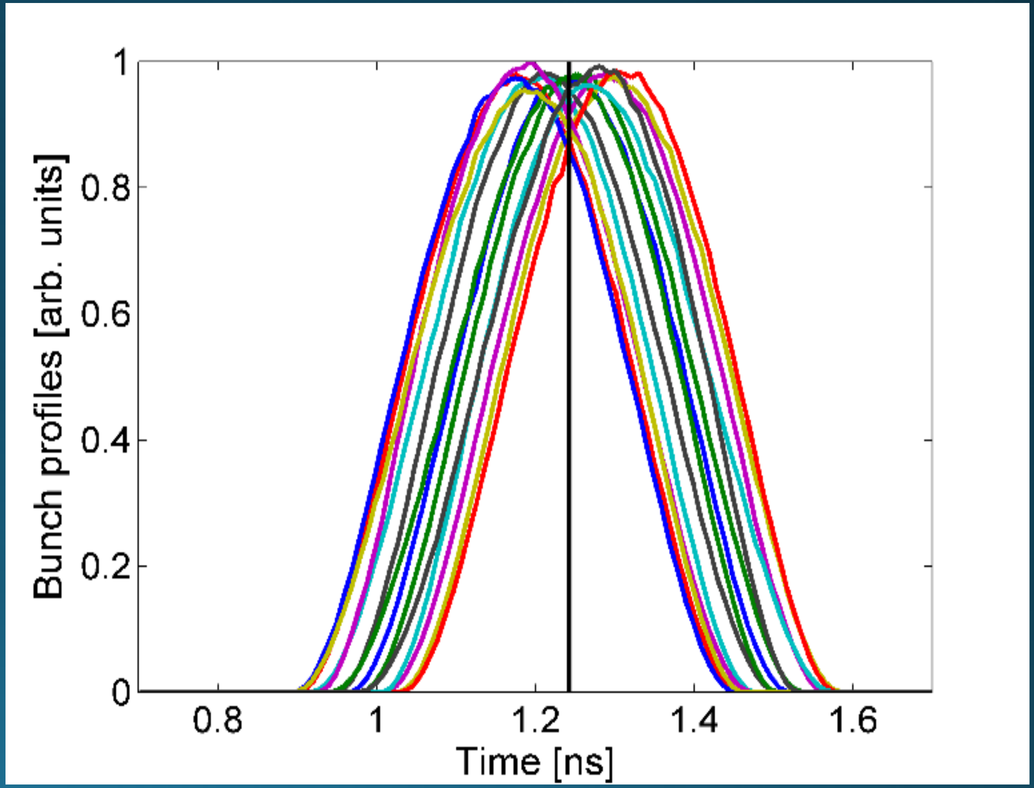
Phase error at injection – Dipole oscillations

- Summary plots of the bunch profiles are used to illustrate the bunch oscillations. The black lines on the plots show the centre of the RF bucket.

Waterfall plot: Each horizontal line corresponds to one acquisition of the same bunch. The colour code depicts the bunch profile density in arb. units.

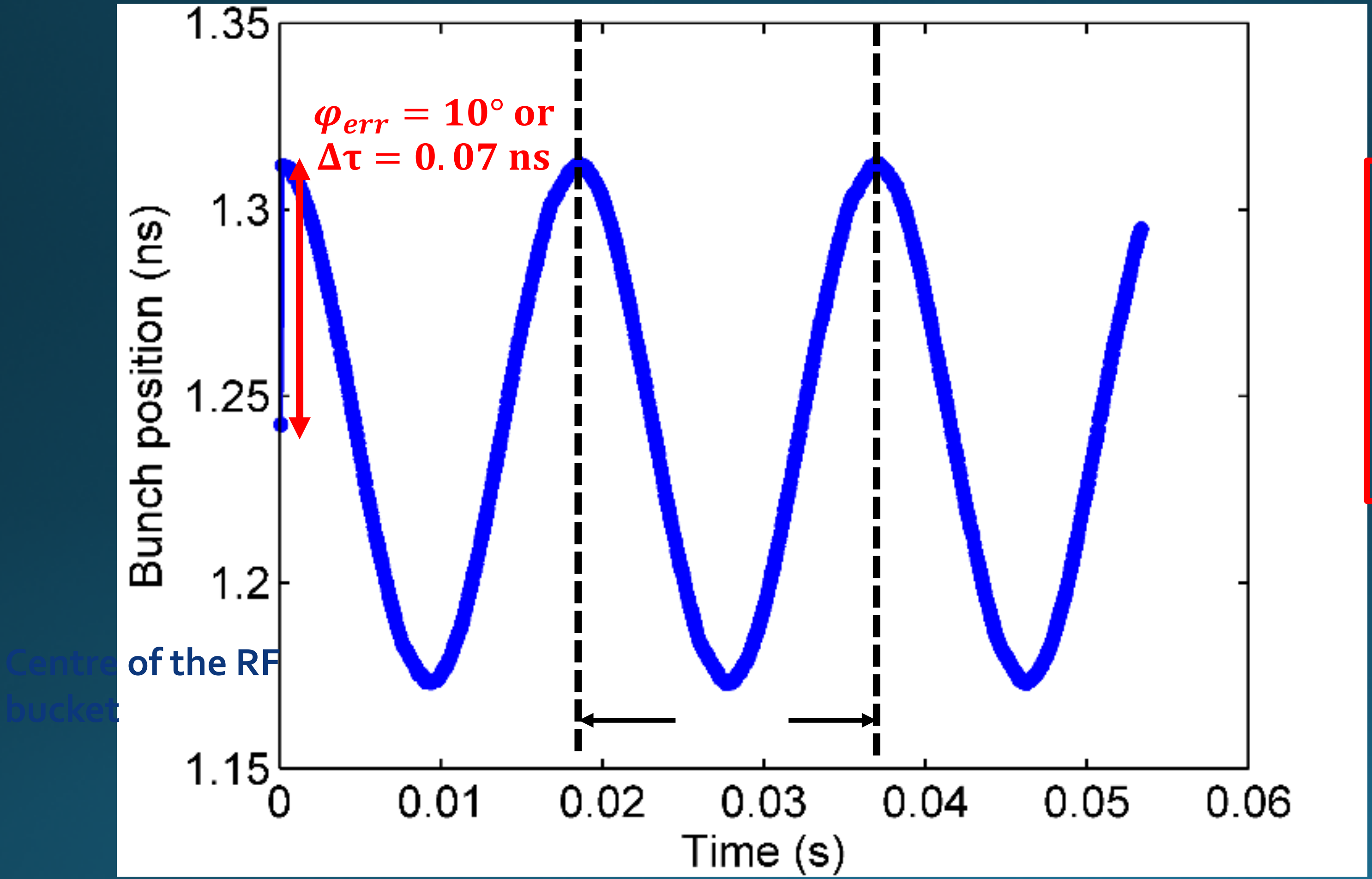


Overlaying of all bunch profiles. Different colours correspond to different turns.



Phase error at injection – Dipole oscillations

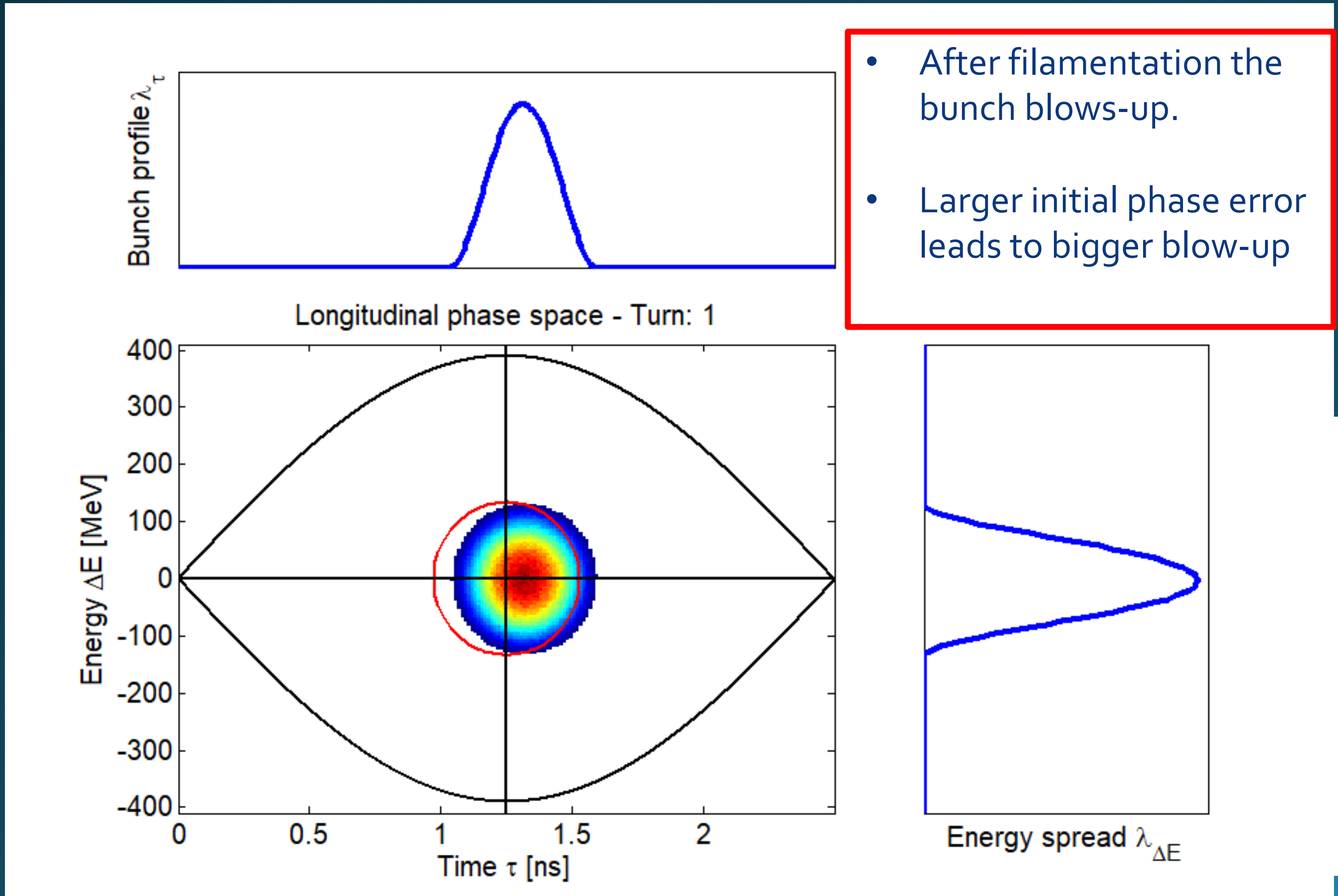
- After analysing the profiles i.e. fitting the bunches and getting the bunch parameters for each acquisition, we can plot the motion of the centroid of the bunch.

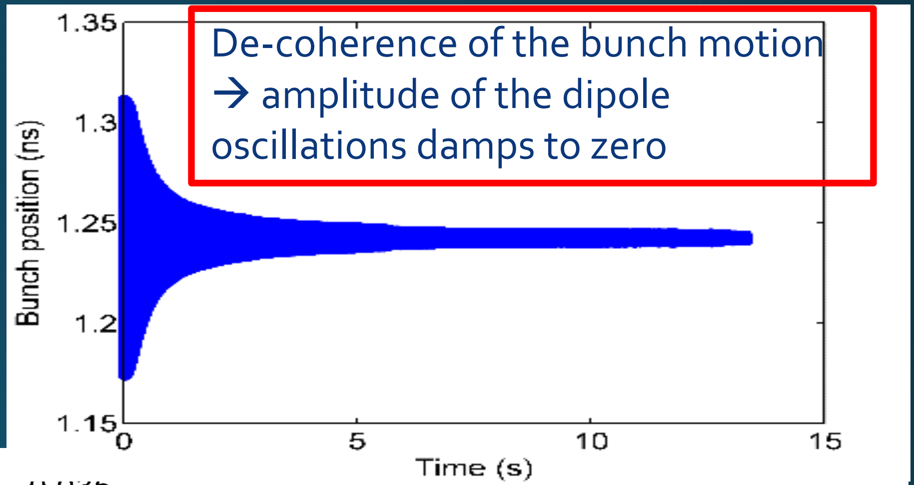


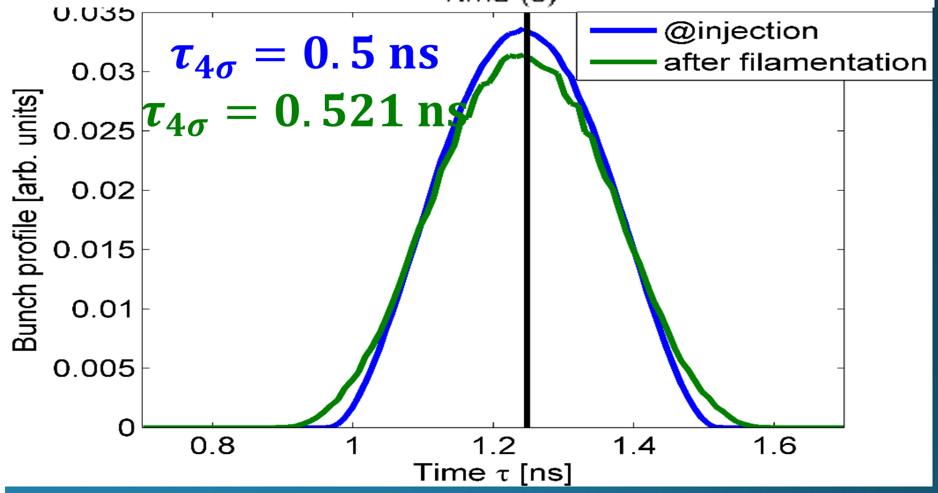
The motion of the bunch centroid after injection into the RF bucket with a phase error shows a clear harmonic oscillation with frequency close to the zero amplitude synchrotron frequency f_{s0}.

Phase error at injection – bunch filamentation

- Particles having different synchrotron frequencies (fast for the particles around the centre of the bucket and slow for the ones injected close to the separatrix) will lead to the **filamentation** of the bunch and finally to the **de-coherence of the bunch motion**. Particles will adapt to the corresponding trajectories in the phase-space at the moment of injection resulting to **observation of an emittance bunch blow-up**.

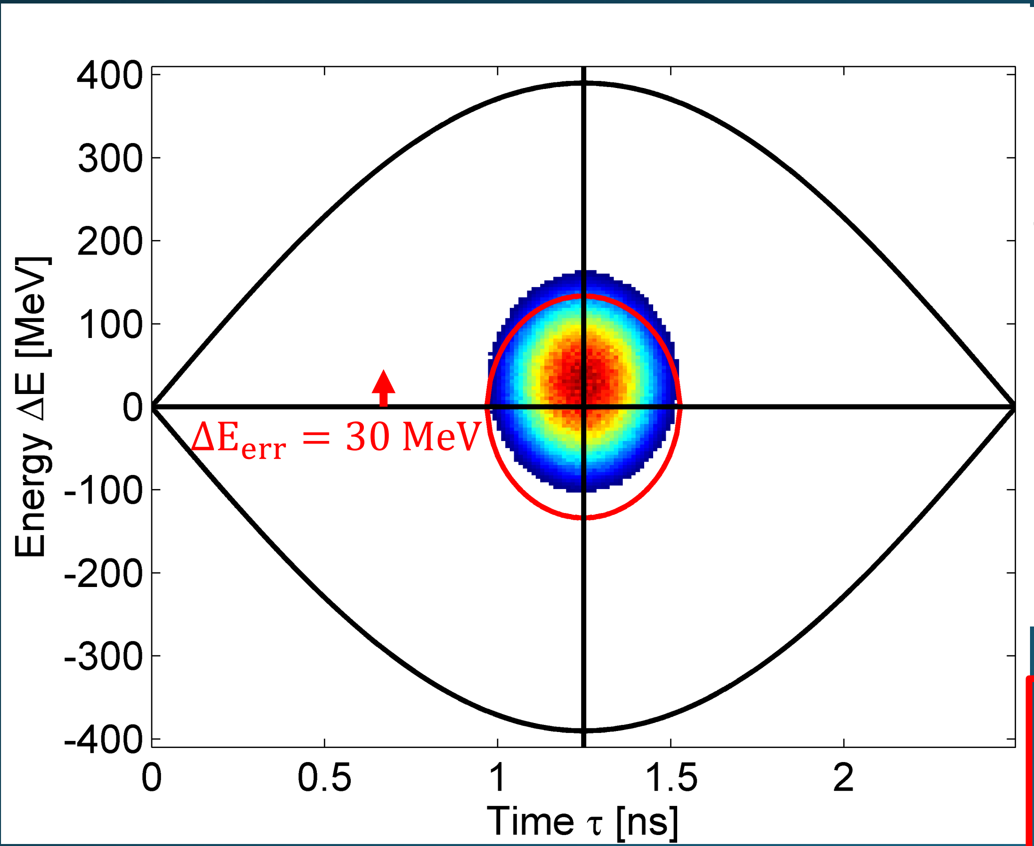


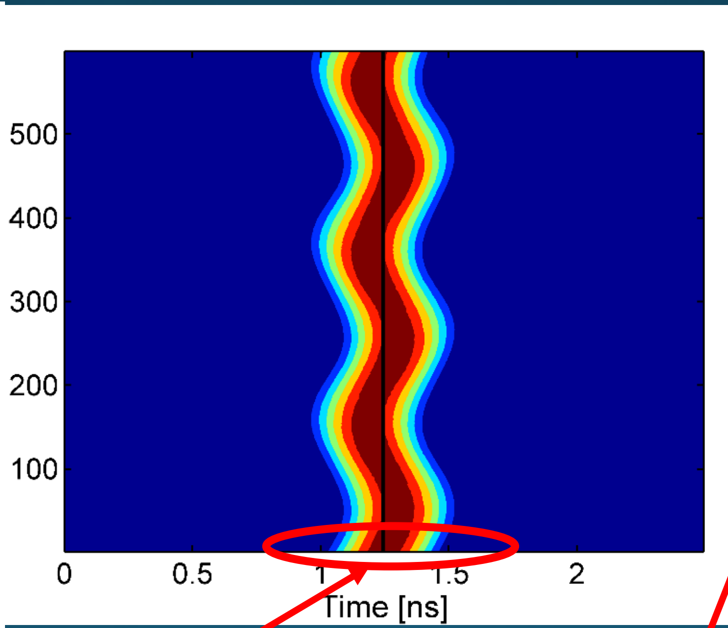


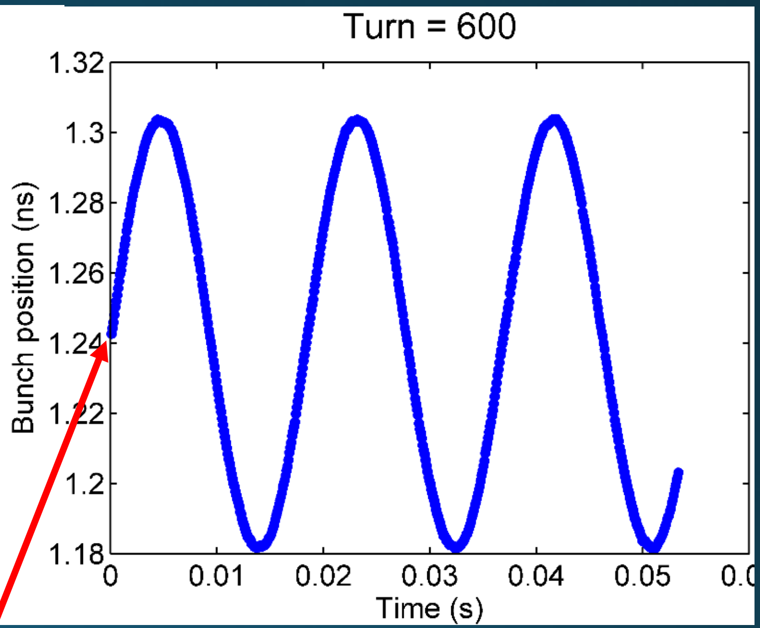


Energy error at injection

- Injection into the RF bucket with 30 MeV energy error → effect on the bunch is similar to the phase error case.







Note that the bunch is initial injected into the centre of the bucket (no phase error). These kind of plots can be used to distinguish between energy and phase errors.

Frequency Change during acceleration

- During the energy ramping, the particle velocity increases
  (… which is why we call it an accelerator)
- Which means the revolution frequency of the particles increases (except for e- which are fully relativistic)
- And we have to adjust the external RF frequency in order to stay synchronised.

- Revolution frequency and RF frequency are related by the "harmonic number".
- The harominic number is the number of RF wavelengths that fit around the ring.

$$f_r = \frac{f_{RF}}{h} = Function(B, R_s)$$

hence :

$$\frac{f_{RF}(t)}{h} = \frac{v(t)}{2\pi R_s} = \frac{1}{2\pi}\frac{e}{m} < B(t) > \quad \Rightarrow \quad \frac{f_{RF}(t)}{h} = \frac{1}{2\pi}\frac{ec^2}{E_s(t)}\frac{r}{R_s}B(t)$$

$$= \frac{1}{2\pi}\frac{e}{m}\frac{r}{R_s} * B(t) \qquad m = \frac{E_s}{c^2}$$

$$= \frac{1}{2\pi}\frac{ec^2}{E_s}\frac{r}{R_s} * B(t)$$

$$= \frac{ec^2}{2\pi}\frac{r}{R_s} * \sqrt{\frac{B(t)^{\,2}}{(mc^2)^2 + p^2c^2}}$$

Frequency Change during acceleration

$$f_r = \frac{f_{RF}}{h} = \frac{c}{2\pi R_s} * \sqrt{\frac{B(t)^2}{\frac{m^2 c^4}{(rec)^2} + \frac{p^2}{e^2 r^2}}}$$

$$B = \frac{p}{er}$$

$$= \frac{c}{2\pi R_s} * \sqrt{\frac{B(t)^2}{\frac{m^2 c^4}{(rec)^2} + B(t)^2}}$$

The RF frequency must follow the variation of the B field with the law
:

$$\frac{f_{RF}(t)}{h} = \frac{c}{2\pi R_s} \left\{ \frac{B(t)^2}{(m_0 c^2 / ecr)^2 + B(t)^2} \right\}^{1/2}$$

At relativistic energies, (as soon as ...) $B > \frac{m_0 c^2}{ecr}$

We can treat f_r as constant. $f_r = \frac{c}{2\pi R_s} = const$

which is true for LHC at high energy and for electrons from the start

Spares

It can be shown that:

$$\frac{\partial Hamiltonian}{\partial \Delta\varphi} = -\frac{\delta}{\delta t}\left(\frac{\Delta W}{\Omega_s}\right)$$ Eq. 49

$$\frac{\partial Hamiltonian}{\partial \left(\frac{\Delta W}{\Omega_s}\right)} = \frac{\delta}{\delta t}\Delta\varphi$$ Eq. 50

Canonical Hamiltonian equations

FIRST EQUATION OF MOTION $\dot{w} = \frac{qV_{max}\Omega_s}{2\pi W_s}\left(\sin(\Delta\varphi + \varphi_s) - sin\varphi_s\right)$ Eq. 25

SECOND EQUATION OF MOTION $\Delta\dot{\varphi} = \frac{\delta(\Delta\varphi)}{\delta t} = h\Gamma_s\Omega_s w$ Eq. 34