Introduction to transverse beam dynamics IV

Restricted to the LINEAR BEAM OPTICS → THE IDEAL WORLD

Content of the course

TBD 1

- Charge particle motion in a magnetic field
- Equations of motion ➔ derivation and assumptions
- Type of magnets

TUTO 1

- Rigidity formula
- Relativistic equations
- Create a storage ring with the Earth Magnetic field

TBD 2

- Particle trajectory
- Transfer Matrices
- Thin lens approximation
- Betatron oscillations
- Betatron tune
- Dispersion

TUTO 2

- Application of transfer matrices
- Thin lens
- FoDo cell

TBD 3

- Phase space ellipse
- Emittance
- Beam size
- Aperture
- Beta function evolution
- Periodic lattices

TUTO 3

- Beam size and aperture calculations

TBD 4

- Effect field errors
- Resonances
- Coupling
- Chromaticity

TUTO 4

- How the tune changes from a quadrupole defect
- Optimize beta beating
- Orbit bumps

Effect of magnetic field errors on beam optics

- Ideal magnets that agree with the hard-edge model cannot be built
- Manufacture tolerances, not perfect pole ends, etc, produce a non-perfect magnetic field and introduce field errors

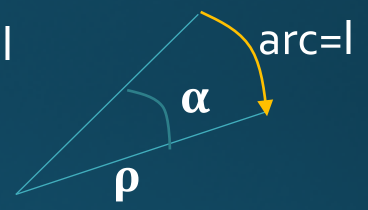
- We'll study two types of errors:
  - Dipole errors → Orbit distortion and emittance growth
  - Quadrupole errors → Orbit distortion and emittance growth
    → Tune shift
    → Beta beating
    → Chromaticity

Dipole field error ➔ extra dipole kick

- Assume a dipole field error of strength ΔB acting over a length l

$$\Delta\alpha = \Delta x' = \frac{\Delta B l}{p/q}$$

In general



Particle trajectory in a magnetic field, B

For small angles: arc=**α** x **ρ**
So the angle described by the particle is **α**=arc/**ρ**
If we multiply and divide by B:

$\alpha = \frac{Bl}{B\rho}$ ➔Rigidity formula ➔ $\alpha = \frac{Bl}{p/q}$

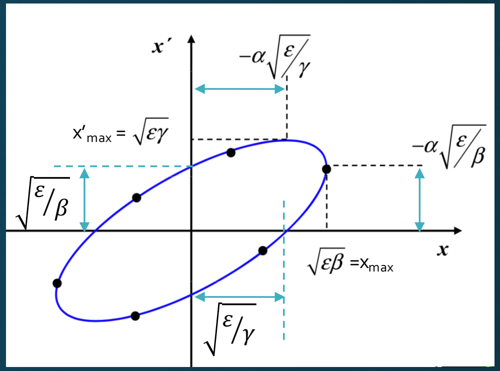
Dipole field error → extra dipole kick

$$\Delta\alpha = \Delta x' = \frac{\Delta B l}{p/q}$$

- If *l* is not too long, the disturbance can be described by a localized angular kick right in the middle of the disturbing field, i.e. *l*/2, and we can approximate it to a infinitesimally short field disturbance
- Consider a particle travelling exactly along the orbit in front of the disturbance with the trajectory vector (x, x') = (0, 0)

- (x, x′) = (o, o) ➔ this particle has, therefore, zero emittance
- But immediately after the field disturbance, it travels at an angle $\Delta x'$ w.r.t. orbit
- The trajectory vector is now (o, Δx′)
- This deflection will lead to betatron oscillations as a result of the focusing elements in the lattice
- And now the particle emittance is not zero but

- And now the particle emittance is not zero but



Remember $$\gamma(s)x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s) = \frac{Area}{\pi} = \varepsilon$$

(x=0, Δx′)

$$\beta(s)\Delta x'^2(s) = \varepsilon_{error}$$

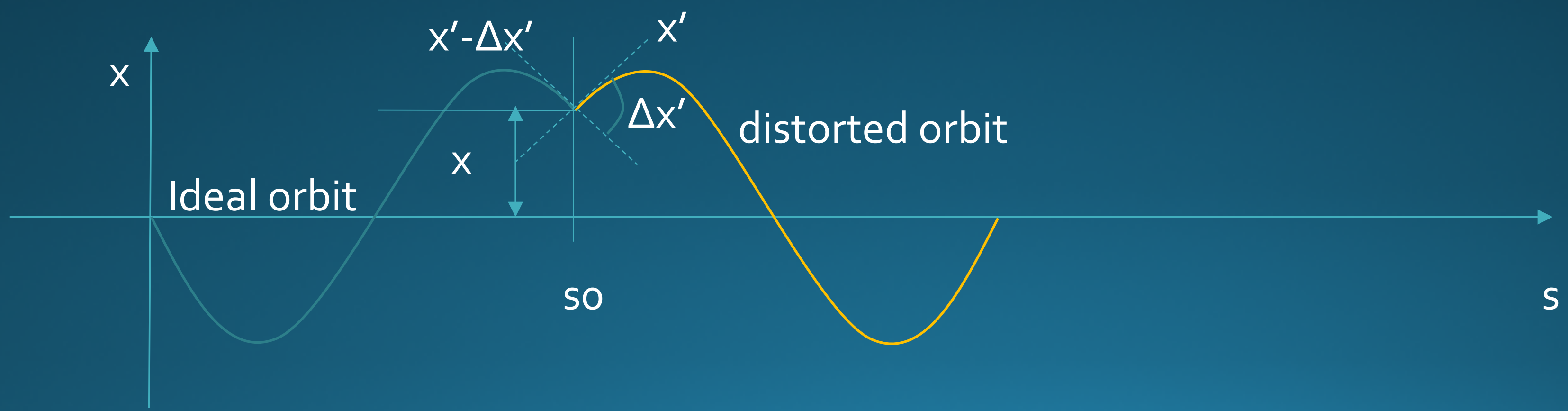
For a given field error, the increase in emittance is proportional to the beta function at the point of the disturbance!!!!

It is a fundamental property of beam optics that the effect of a field error increases with the beta function

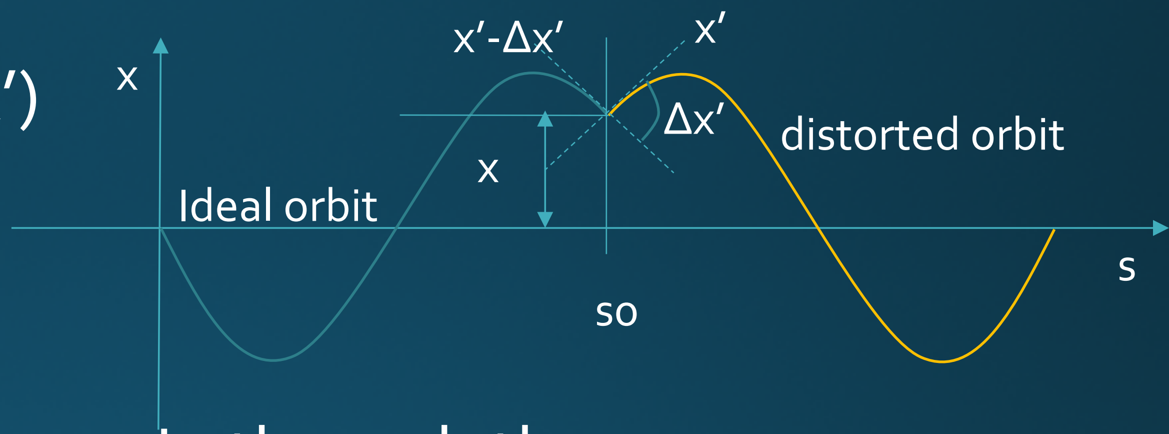
- Therefore, special care has to be taken when designing magnets that will be placed in regions of large beta ➔ e.g. the LHC inner triplets ➔ very very high quality magnets (with collision optics beta = 4.5 km) ➔ very very expensive

- In circular accelerators the beam passes over and over again through the same disturbance and it is deflected with the same angle each time

- After many revolutions a stable equilibrium is established resulting in a new distorted orbit ➔ a static betatron oscillation about the unperturbed ideal orbit, with a phase or angular shift as follows



- In equilibrium the perturbed orbit has a displacement x from the ideal orbit at s_o
- The angle right before de perturbation is x'-Δx', and x' after the perturbation
- Hence at s_o the vector trajectory is (x,x')



- After one revolution the particle passes again through the same disturbance and arrives with a trajectory vector (x, x'-Δx'), the distorted orbit can be evolved using

(TBD 3)

$$\begin{pmatrix} x \\ x'-\Delta x' \end{pmatrix} = M_{turn} \begin{pmatrix} x \\ x' \end{pmatrix}$$

$$M_{turn} = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ \frac{-(1+\alpha_s^2)\sin\psi_{turn}}{\beta_s} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

- $\psi_{turn} = 2\pi Q$
- Because of the periodicity condition the beta function and its derivative are the same at the beginning and end of a revolution

$$M_{turn} = \begin{pmatrix} cos2\pi Q + \alpha(s_0)sin2\pi Q & \beta(s_0)sin2\pi Q \\ -\gamma(s_0)sin2\pi Q & cos2\pi Q - \alpha(s_0)sin2\pi Q \end{pmatrix}$$

- Putting this matrix in the equation of the trajectory, we get for x and x'

$$x = \Delta x' \frac{\beta(s_0)}{2tan\pi Q}$$

$$x' = \frac{\Delta x'}{2}\left(1 - \frac{\alpha(s_0)}{tan\pi Q}\right)$$

Distorted orbit at any point, s, around the ring

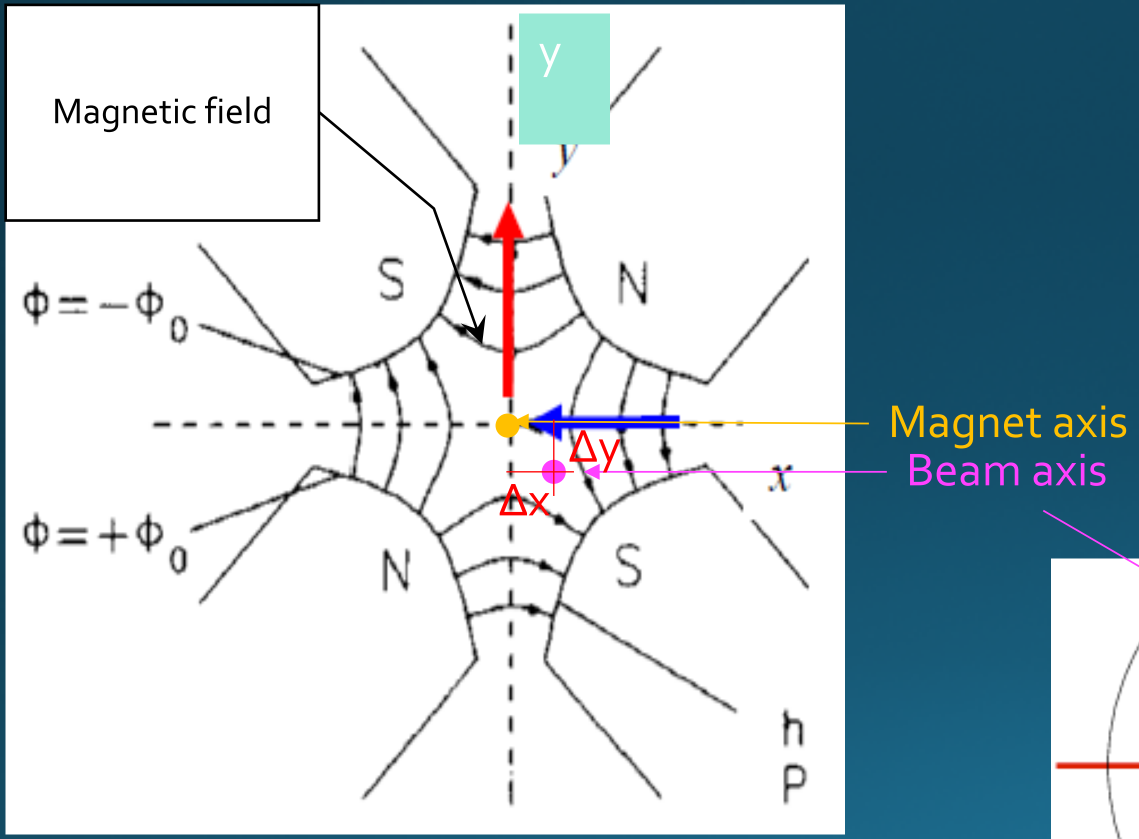
For integer tunes the orbit distortion grows without bound ➔ INTEGER RESONANCES

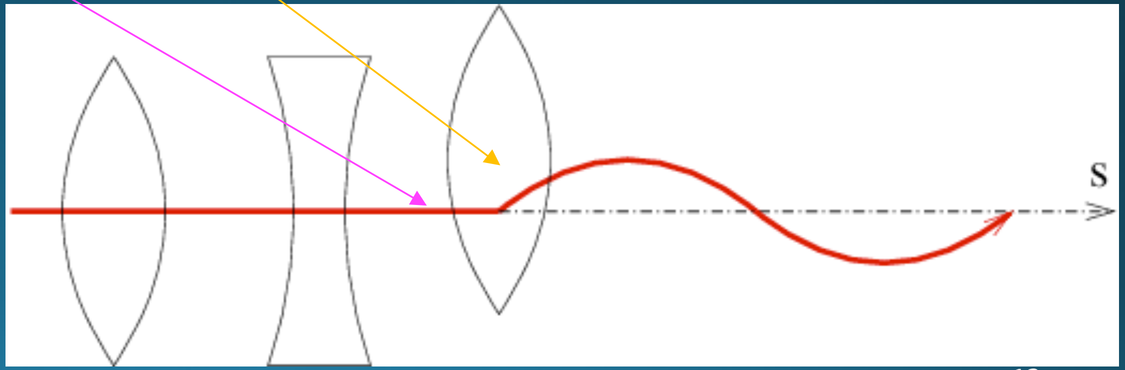
Don't design your accelerator with an integer tune

Effect of quadrupole field errors

- The simplest case of a quadrupole field error is a transverse misalignment

Horizontal focusing quadrupole





$$\boxed{\frac{q}{p}B_y(s)} = \frac{q}{p}B_{y0} + \boxed{\frac{q}{p}\frac{dB_y}{dx}x} + \frac{1}{2!}\frac{q}{p}\frac{d^2B_y}{dx^2}x^2 + \frac{1}{3!}\frac{q}{p}\frac{d^3B_y}{dx^3}x^3 + \cdots$$

- For a quadrupole with a gradient $g = \frac{\partial B_y}{\partial x}$

- At the point where the error is located, there is the following error field

$$\begin{pmatrix}\Delta B_x \\ \Delta B_y\end{pmatrix} = g\begin{pmatrix}\Delta y \\ \Delta x\end{pmatrix}$$

- This leads to a deflection in both planes of ➔ $\Delta\alpha = \Delta x' = \frac{\Delta B l}{p/q}$ (slide 5)

$$\begin{pmatrix}\Delta x' \\ \Delta y'\end{pmatrix} = \frac{q}{p}l\begin{pmatrix}\Delta B_y \\ \Delta B_x\end{pmatrix} = \frac{q}{p}gl\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix} = kl\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix}$$

- This leads to a deflection in both planes of

$$\begin{pmatrix}\Delta x' \\ \Delta y'\end{pmatrix} = \frac{q}{p} l \begin{pmatrix}\Delta B_y \\ \Delta B_x\end{pmatrix} = \frac{q}{p} g l \begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix} = kl \begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix}$$

- Like a dipole field error, this misalignment causes an angular deflection of the trajectory in both planes

- Therefore we have an orbit distortion that can be calculated

$$x = \Delta x' \frac{\beta(s_0)}{2 tan\pi Q}$$

$$x' = \frac{\Delta x'}{2}\left(1 - \frac{\alpha(s_0)}{tan\pi Q}\right)$$

(And the equivalent equation for the y plane)

The orbit distortion is proportional to the **size of the misalignment (Δx, Δy)** and the **beta function** at the quadrupole position

Quadrupole error: tune shift

- A quadrupole misalignment gives an orbit distortion

- What about an error in the quadrupole field?
  - It changes the focusing properties and therefore the beta function and therefore it changes the tune
  - It can be demonstrated (K. Wille page 116-118) that the tune shift due to a quadrupole of finite length *l* with a very small gradient error Δk is

$$\Delta Q = \frac{1}{4\pi}\int_{s_0}^{s_0+l} \Delta k \beta(s) ds$$

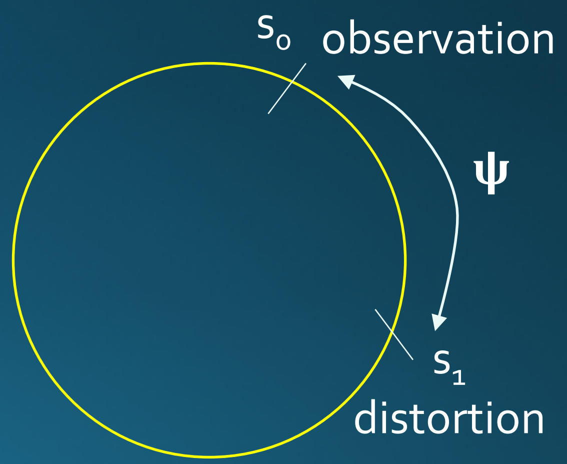
- ΔQ is proportional to the $\boldsymbol{\beta}$-function at the quadrupole
- Field quality, power supply tolerances, etc are much tighter at places where the $\boldsymbol{\beta}$-function is large: mini-beta quads $\boldsymbol{\beta}$≈km, arc quads $\boldsymbol{\beta}$≈ m
- $\boldsymbol{\beta}$ is a measure for the sensitivity of the beam

Quadrupole error: beta beat

- On top of generating a tune shift, a quadrupole error changes the beta function (demonstration K. Wille pages 118-120)

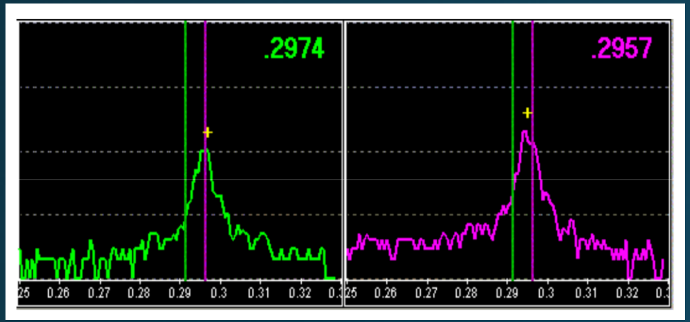
$$\Delta\beta(s_0) = -\frac{\beta_0}{2sin2\pi Q}\int_{s_1}^{s_1+l}\beta(s)\Delta k(s)cos[2(\psi(s)-\psi_0)-2\pi Q]ds$$

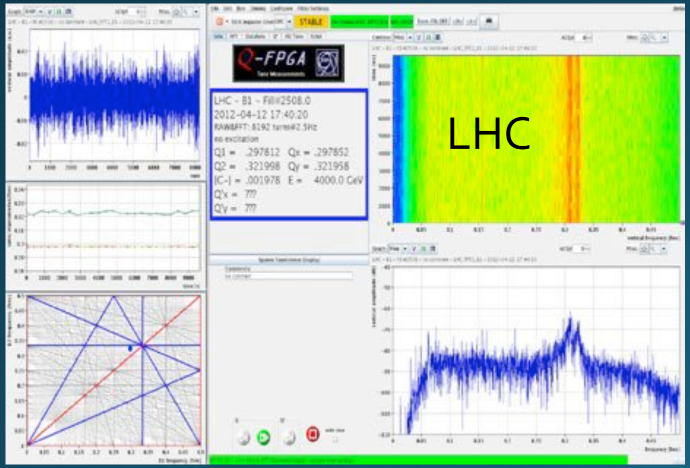
- We see that **Δβ** grows without bounds if sin2**π**Q ➔0
- The tune must NOT HAVE INTEGER OR HALF-INTEGER values
- Unlimited growth of **β** ➔ unlimited growth of beam size ➔ beam losses

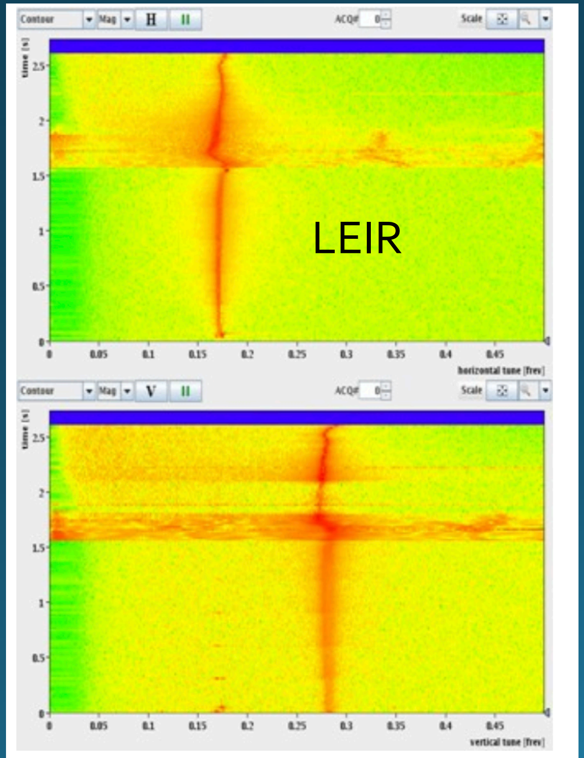


Tune measurement examples

HERA Q_H HERA Q_V

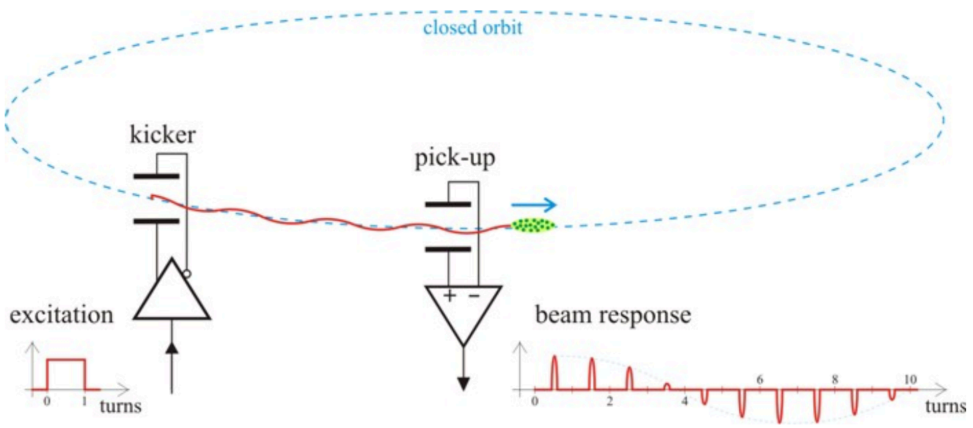




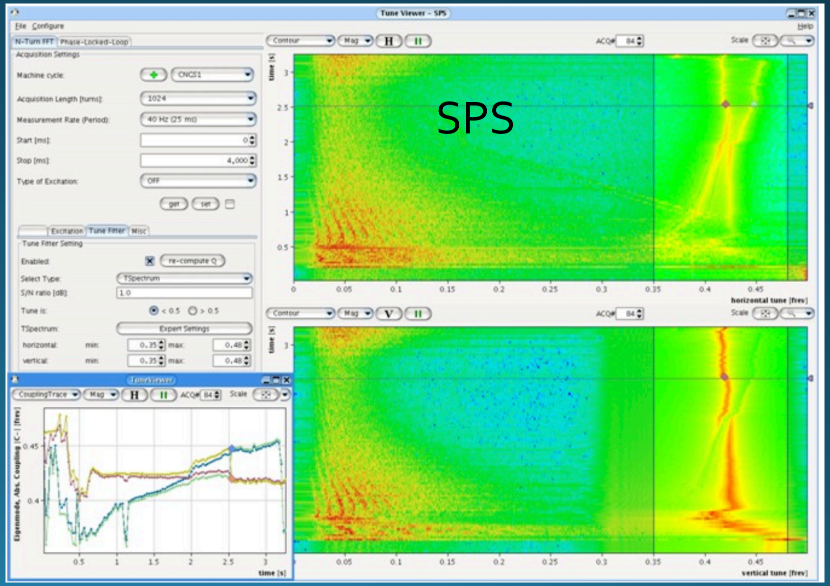


NOMINAL, regular kick 500 V, every 10 ms

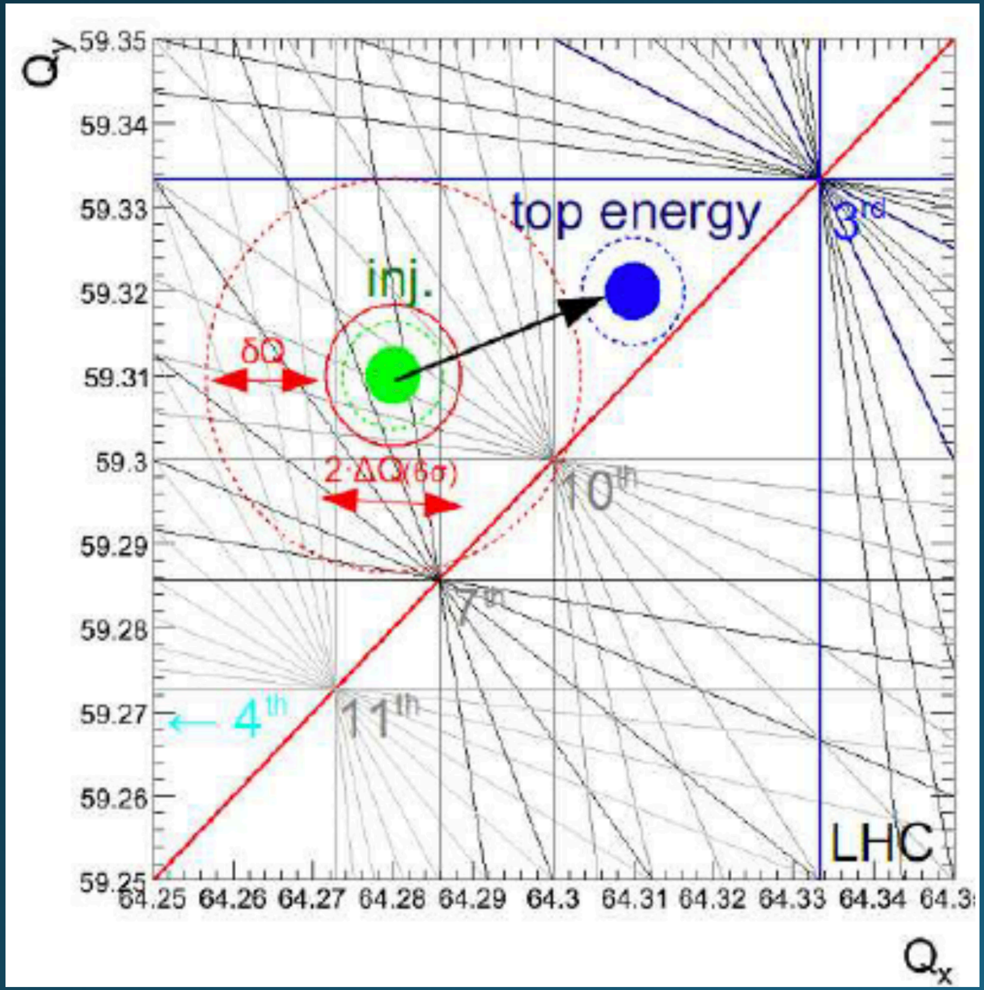
Tune measurement – The principle



- Beam oscillations are observed on a position pick-up
- Oscillations of individual particles are incoherent – an excitation needed for "synchronization"
- Small beam oscillation signals in the presence of large revolution frequency content due to the fact that each bunch appears in the pick-up only once per revolution
- Oscillations are usually observed in the frequency domain (separation from the strong background)



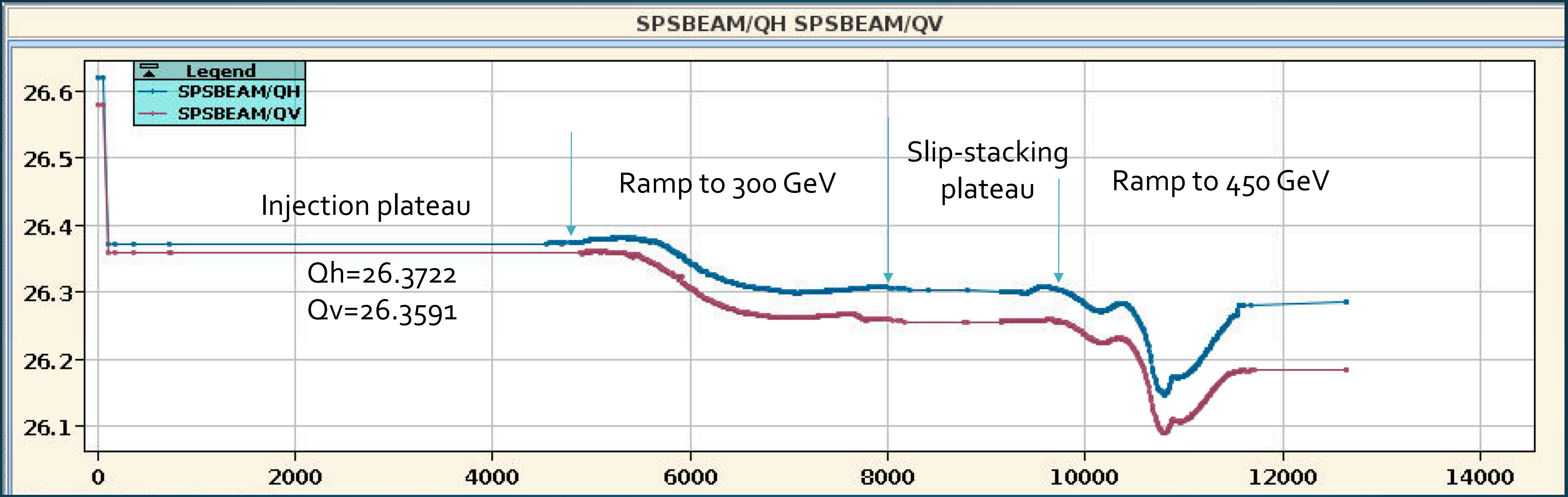
LHC tune diagram



SPS LHC protons parameters

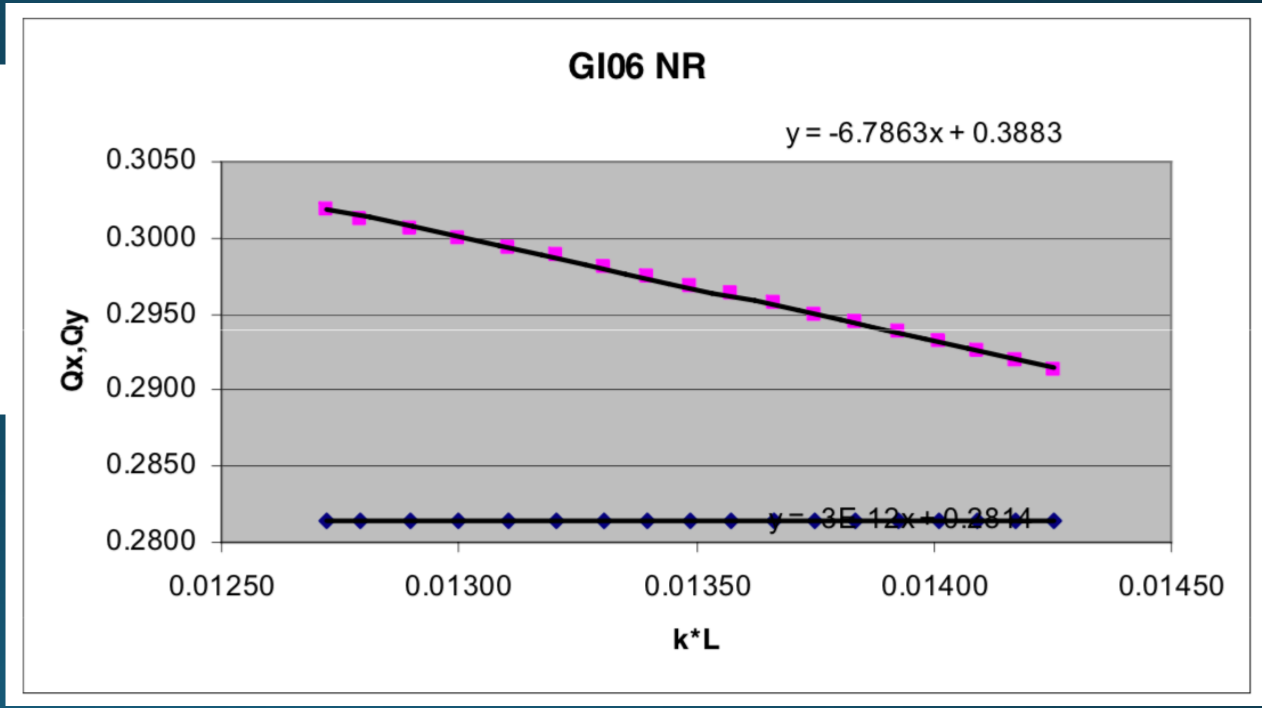
RMS Horizontal Spot Size (mm)	2
RMS Vertical Spot Size (mm)	2
RMS Bunch Length (cm)	30
Horizontal Box Size (mm)	80
Vertical Box Size (mm)	40
Bunch Population	10^{11}
Horizontal Emittance (μm)	0.1
Vertical Emittance (μm)	0.1
Momentum Spread	2.48E-3
Beam Momentum (GeV/c)	26
Circumference (km)	6.9
Horizontal Betatron Tune	26.22
Vertical Betatron Tune	26.18
Synchrotron Tune	0.005
Electron Cloud Density (cm^{-3})	10^6 - 10^7
Number of Grids	128×64×64
Number of Beam Particles	1048576
Number of Electron cloud Particles	16384

SPS LHC ions parameters with slip-stacking

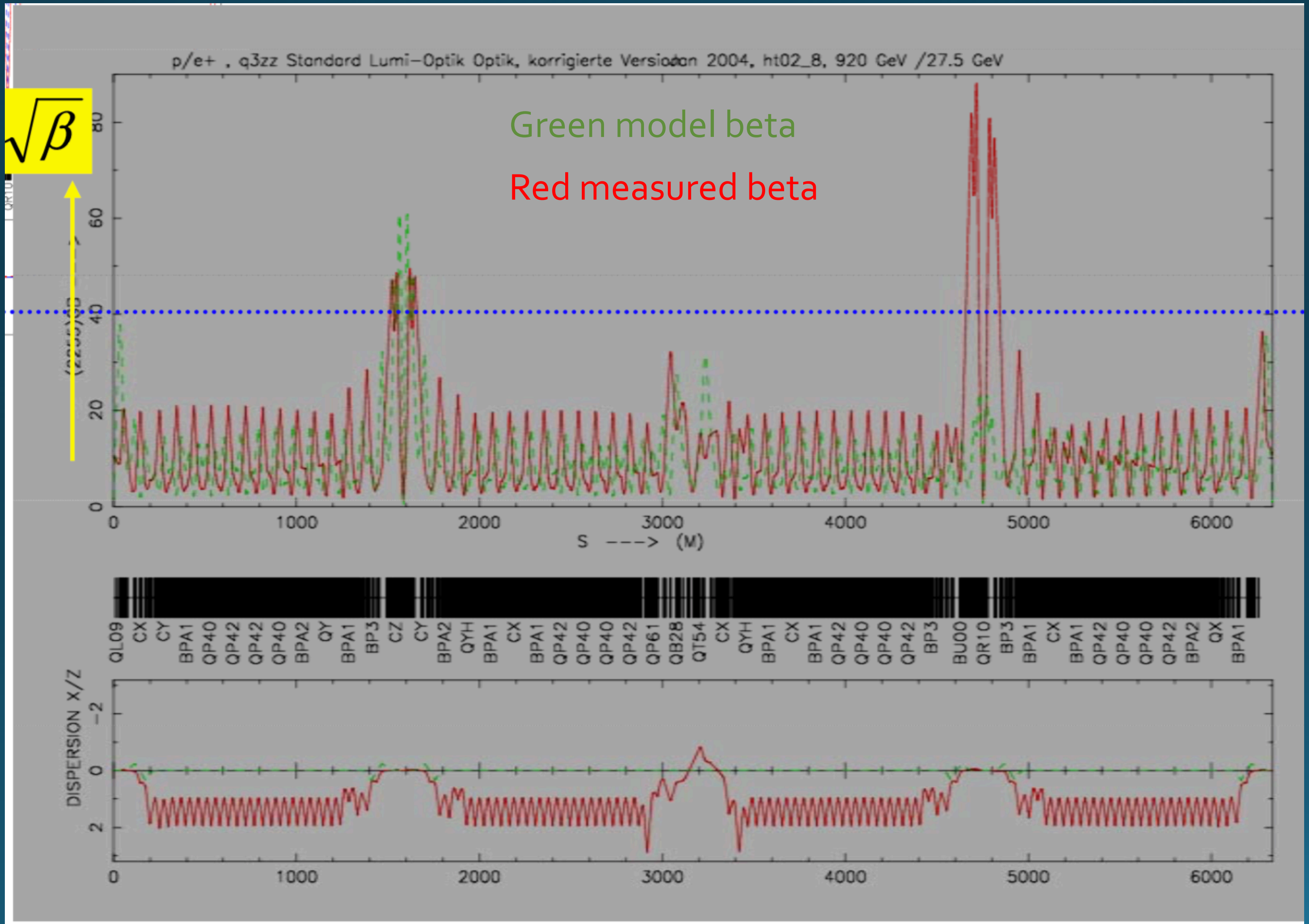


How could we measure the beta function?

$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta k\,\beta(s)}{4\pi}\,ds \approx \frac{\Delta k l_{quad}\,\overline{\beta}}{4\pi}$$



Examples of beta beat measurements



Quadrupole error: chromaticity

- A quadrupole error when $\frac{\Delta p}{p} \neq 0$ is called CHROMATICITY

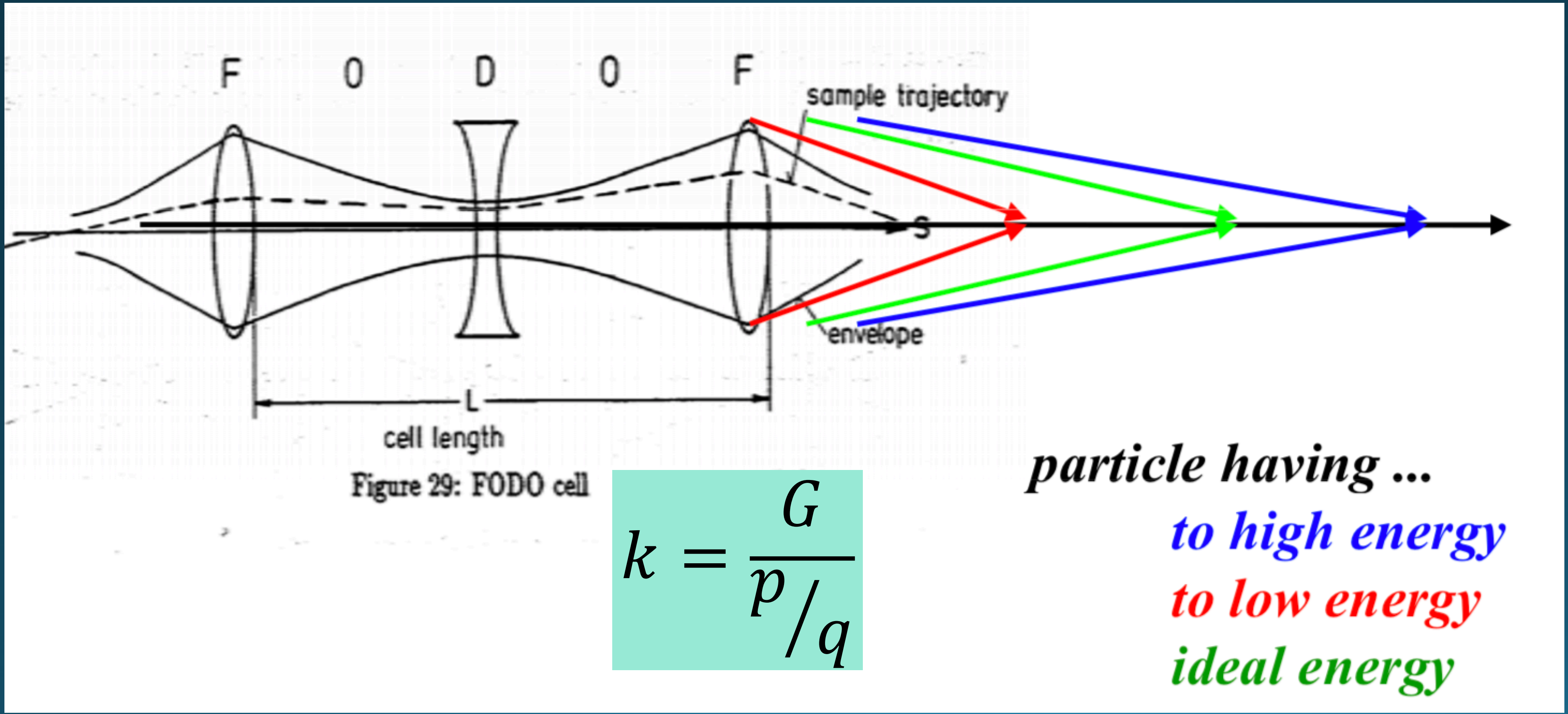


Figure 29: FODO cell

$$k = \frac{G}{p/q}$$

In case of momentum spread: $p = p_o + \Delta p$

$$k = \frac{qG}{p_o + \Delta p} \approx \frac{qG}{p_o}\left(1 - \frac{\Delta p}{p}\right) = k_o + \Delta k$$

$$\frac{qG}{p_o} = k_o \qquad \Delta k = -k_o \frac{\Delta p}{p_o}$$

Δk acts as a quadrupole error in the machine and leads to a tune spread:

$$\Delta Q = \frac{1}{4\pi}\int_{s_0}^{s_0+l} \Delta k \beta(s) ds \quad \Rightarrow \quad \Delta Q = -\frac{1}{4\pi}\int_{s_0}^{s_0+l} k_o \frac{\Delta p}{p_o} \beta(s) ds$$

Definition of chromaticity:

$$\Delta Q = Q' \frac{\Delta p}{p_o}$$

$$Q' = -\frac{1}{4\pi}\oint k(s)\beta(s) ds$$

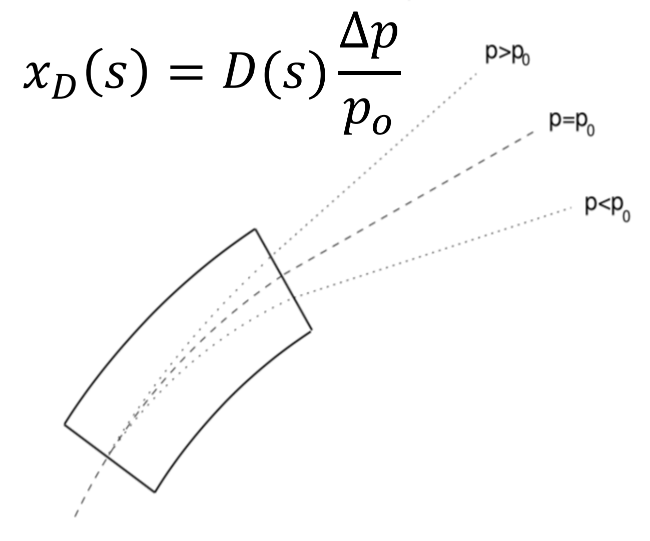
$$Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds$$

Natural chromaticity

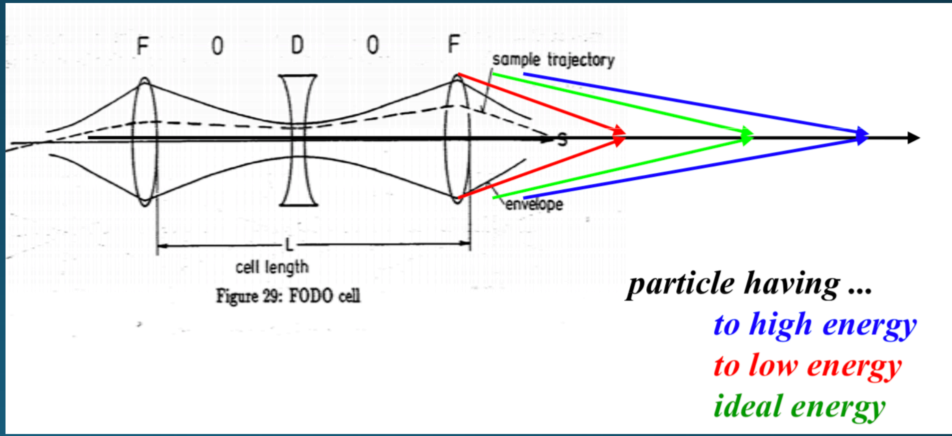
- Chromaticity is generated by the lattice itself!!!
- Q' is a number indicating the size of the tune spot in the tune diagram
- Q' is always created if the beam is focused ➔ is determined by the focusing strength *k* of all quadrupoles
- Because due to chromaticity the tune spot is a *pancake*, some particles get close to resonances and are lost

Chromaticity correction

1. We need to sort the particles as a function of the momentum:



2. We need additional quadrupole strength for each momentum deviation: $\frac{\Delta p}{p_o}$



We have to apply a magnetic field that raises quadratically with increasing x → sextupole

Type of magnets

$$\frac{q}{p}B_y(s) = \frac{q}{p}B_{y0} + \frac{q}{p}\frac{dB_y}{dx}x + \frac{1}{2!}\frac{q}{p}\frac{d^2B_y}{dx^2}x^2 + \frac{1}{3!}\frac{q}{p}\frac{d^3B_y}{dx^3}x^3 + \cdots$$

$$\frac{q}{p}B_y(s) = \frac{1}{\rho} + kx + \frac{1}{2!}mx^2 + \frac{1}{3!}ox^3 + \cdots$$

QUADRUPOLE

OCTUPOLE

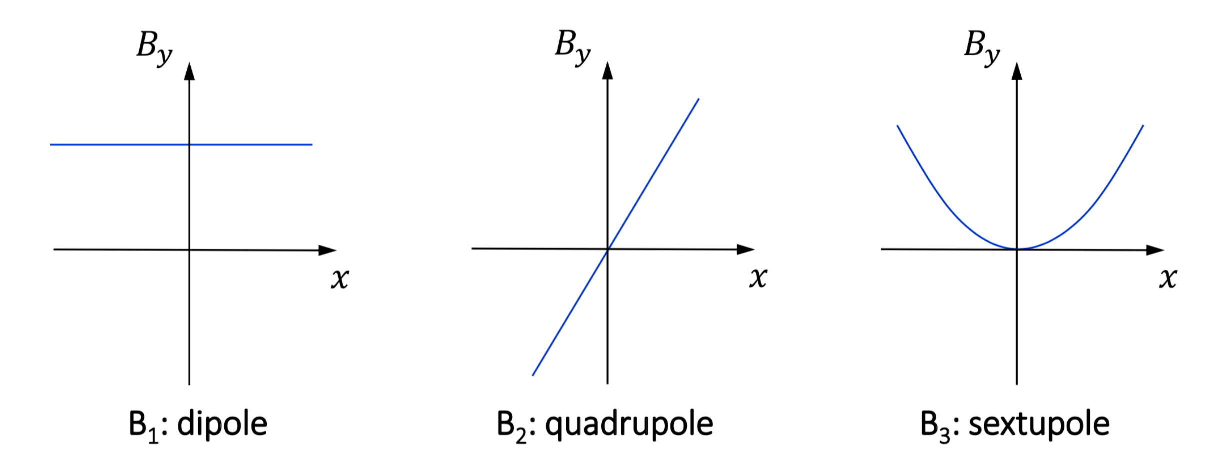
DIPOLE

SEXTUPOLE

$$k_0 = \frac{1}{\rho} = \frac{B}{B\rho}\left(\frac{1}{m}\right)$$

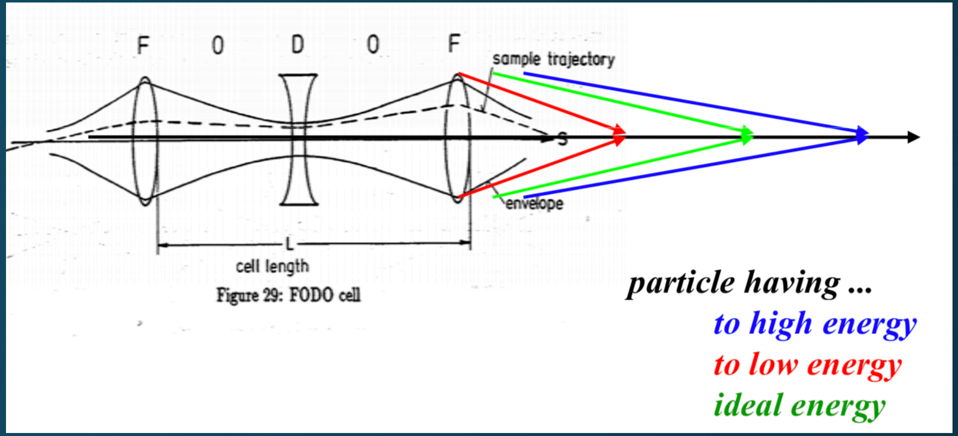
$$k_1 = \frac{q}{p}\frac{dB_y}{dx} = \frac{1}{B\rho}\frac{dB_y}{dx} = \frac{1}{B\rho}g\left(\frac{1}{m^2}\right)$$

$$k_2 = \frac{q}{p}\frac{d^2B_y}{dx^2} = \frac{1}{B\rho}\frac{d^2B_y}{dx^2}\left(\frac{1}{m^3}\right)$$



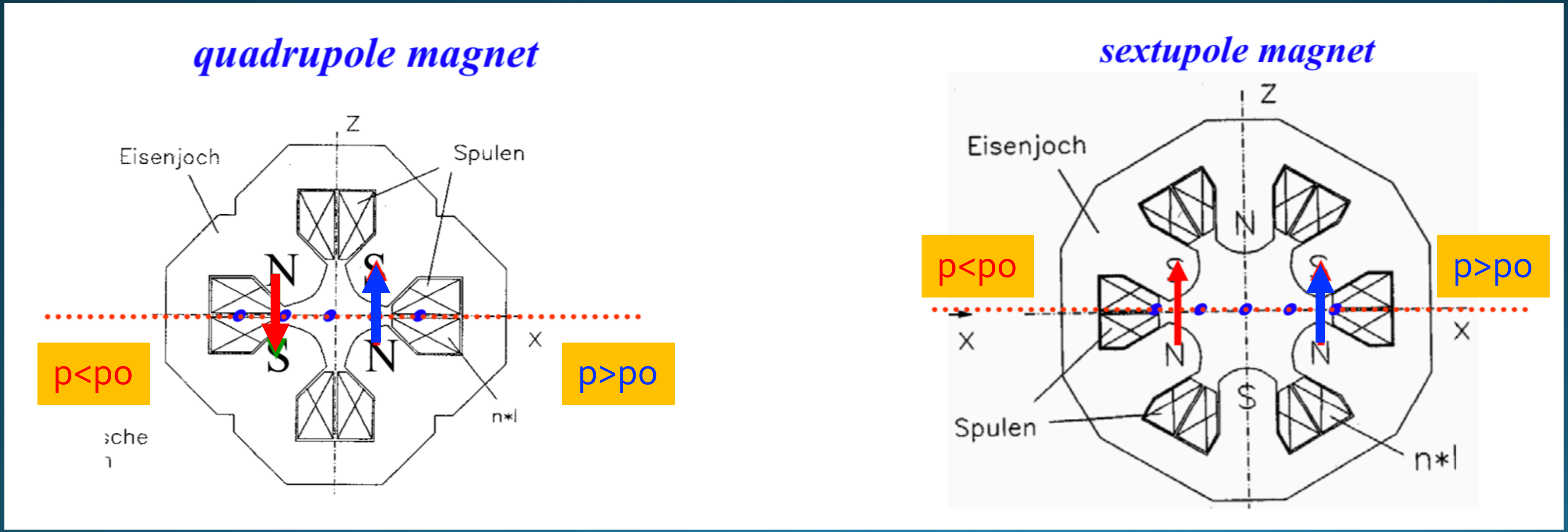
Normalized quadrupole strength for a sextupole:

$$k_{sext} = \frac{q}{p}\tilde{g}x = m_{sext}x = m_{sext}D\frac{\Delta p}{p_o}$$

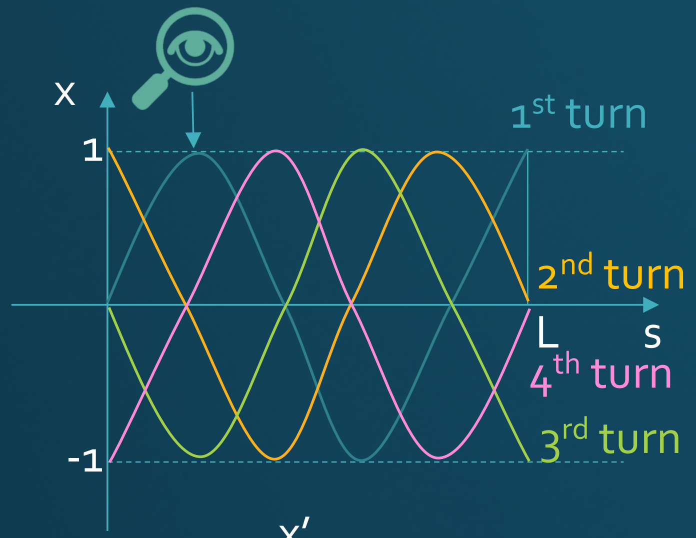


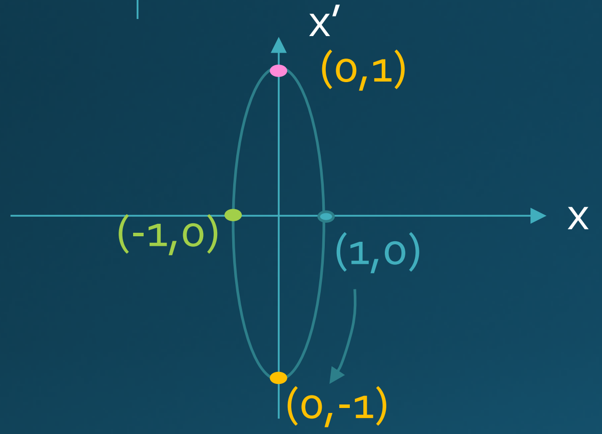
$$Q' = -\frac{1}{4\pi}\oint (k(s) - m_{sext}D)\beta(s)ds$$

Corrected chromaticity



Resonances





$$\psi_{turn} = 2\pi Q$$

Q=1.25

After a certain number of turns around the machine the phase advance of the betatron oscillation is such that the oscillation repeats itself.

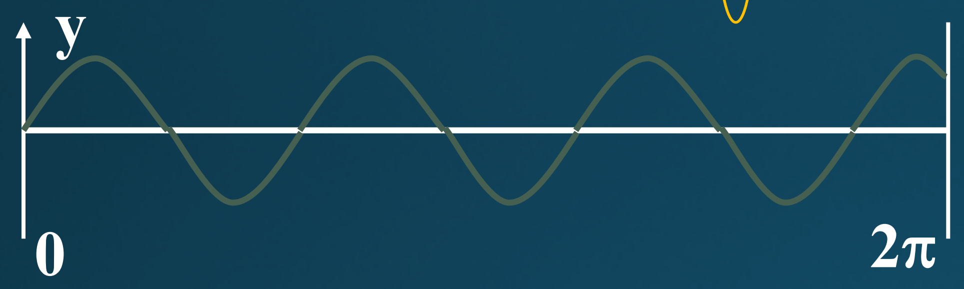
Q=1.25 ➔ q (fractional tune) = 0.25

4x0.25=1

Resonances

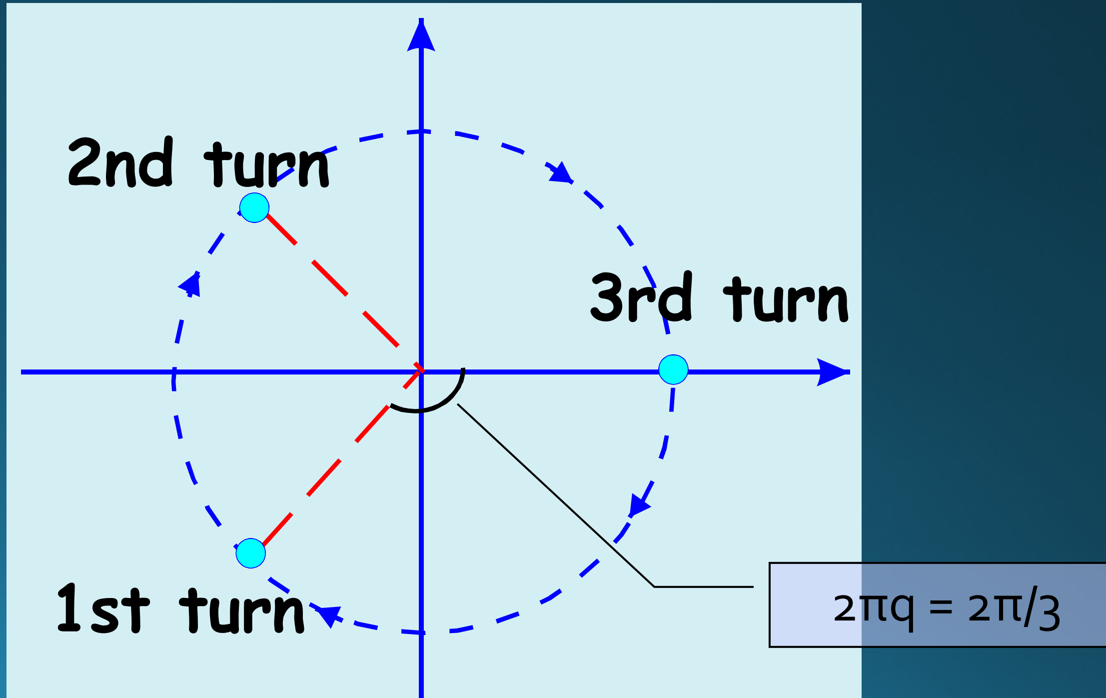
Third order resonance betatron oscillation

- Let's have now a tune of Q=3.333 (3Q=10) ➔ q=0.333 ➔ Q=$\frac{\psi turn}{2\pi}$



$$\psi_{turn} = 2\pi \cdot 0.33 = 2.0923 = 120^0$$

The betatron oscillation will repeat itself after 3 turns = 3x120°=360°

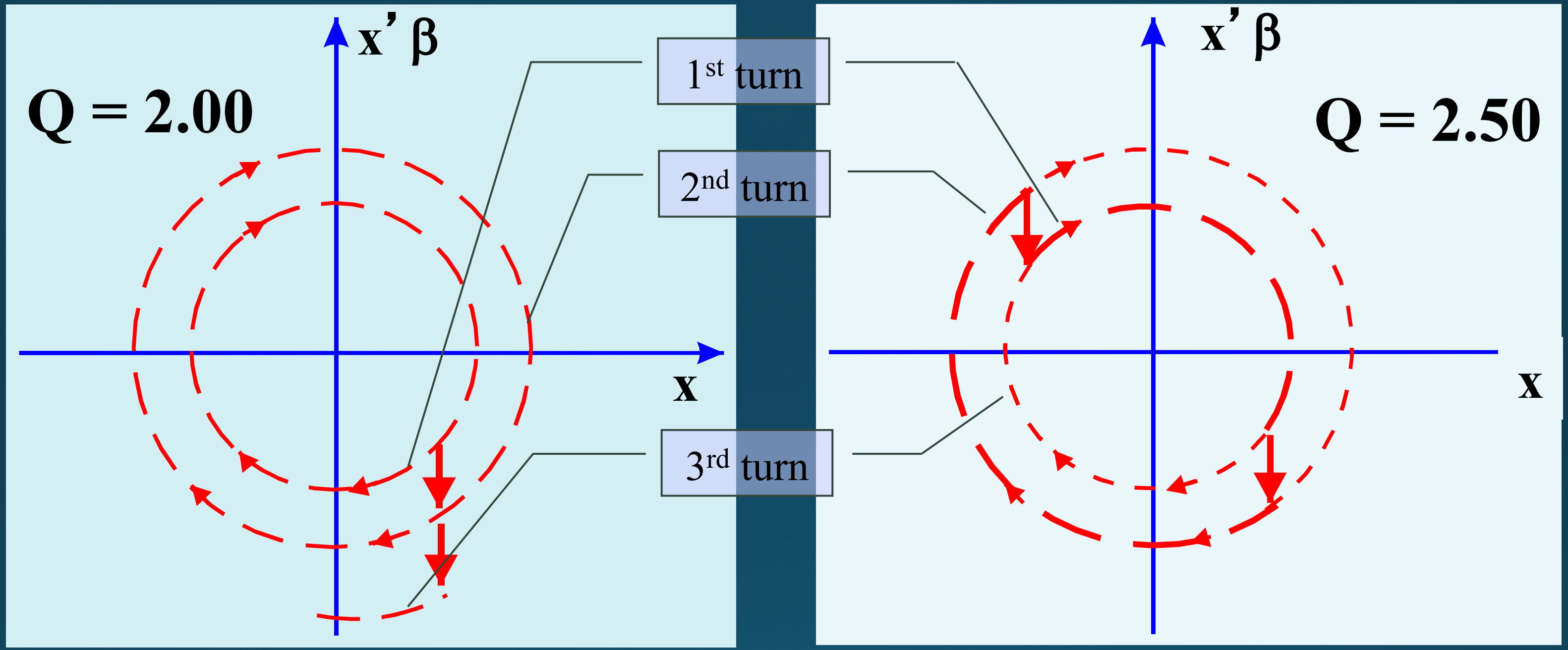


This could also be achieve by Q=2.333 ➔ 3Q=7

The order of the resonance is defined as:

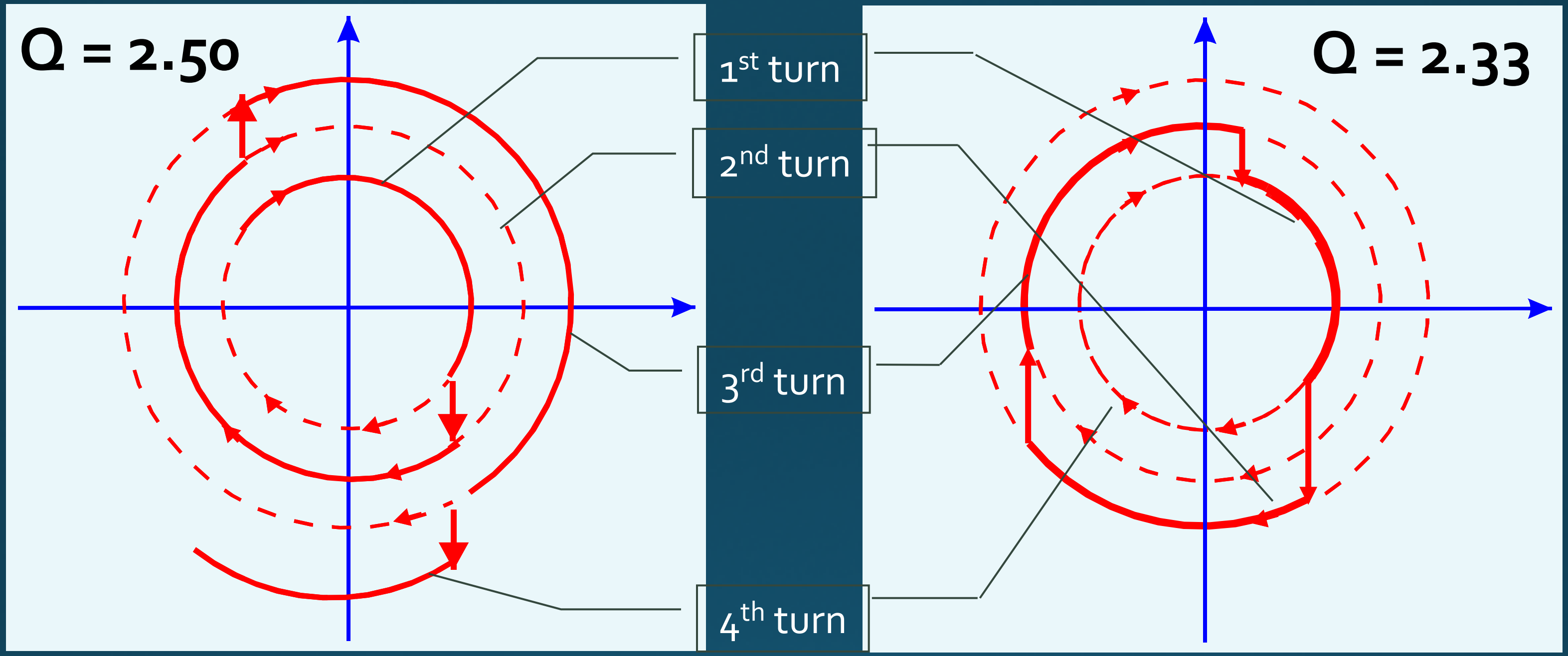
n x Q = integer

Dipole (deflection independent of position)



- ✓ For **Q = 2.00**: Oscillation induced by the **dipole kick** grows on each turn and the particle is lost (**1st order resonance Q = 2**).
- ✓ For **Q = 2.50**: Oscillation is cancelled out **every second turn**, and therefore the particle **motion is stable**.

Quadrupole (deflection ∝ position)

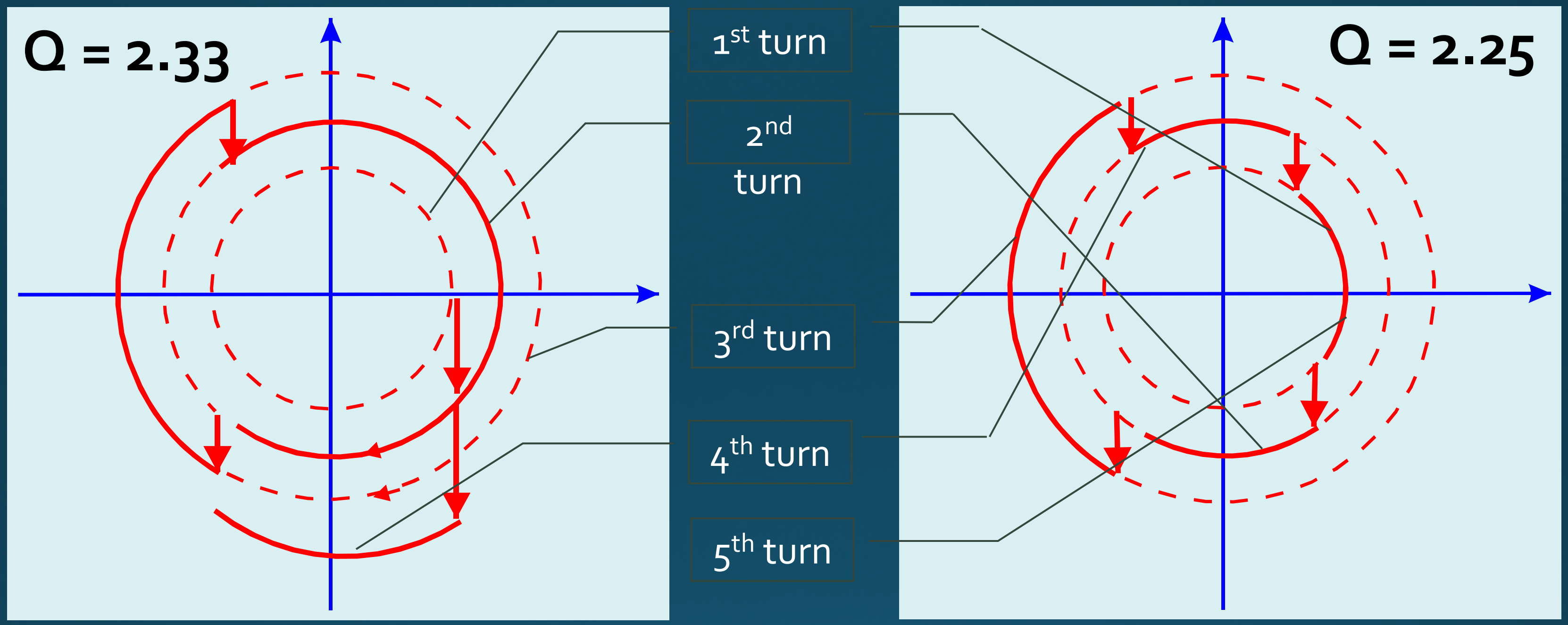


- ✓ For **Q = 2.50**: Oscillation induced by the **quadrupole kick** grows on each turn and the particle is lost

  (2nd order resonance 2Q = 5)

- ✓ For **Q = 2.33**: Oscillation is cancelled out **every third turn**, and therefore the particle **motion is stable**.

Sextupole (deflection ∝ position²)



- ✓ For **Q = 2.33**: Oscillation induced by the **sextupole kick** grows on each turn and the particle is lost

(3rd order resonance 3Q = 7)

- ✓ For **Q = 2.25**: Oscillation is cancelled out **every fourth turn**, and therefore the particle **motion is stable**.

Resonance summary

- ✓ **Quadrupoles** excite **2nd** order resonances
- ✓ **Sextupoles** excite **1st** and **3rd** order resonances
- ✓ **Octupoles** excite **2nd** and **4th** order resonances

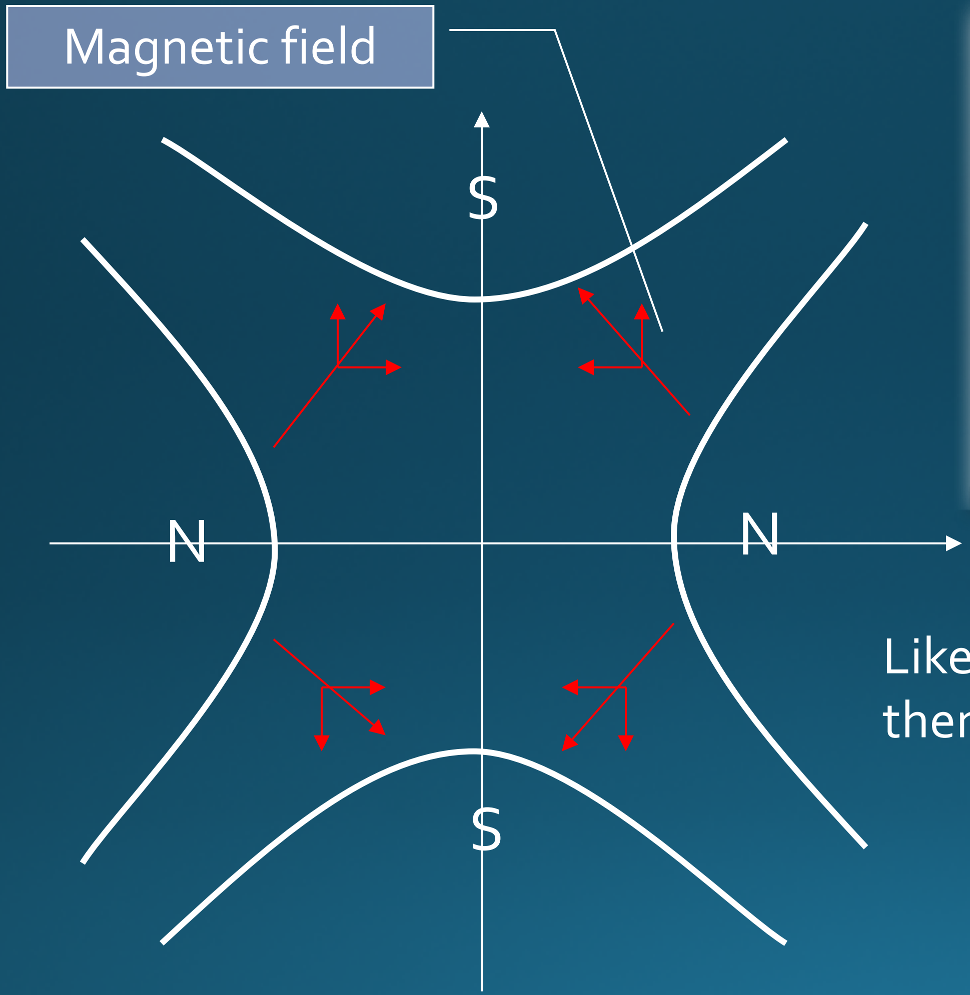
- ✓ This is true for small amplitude particles and low strength excitations

- ✓ However, for stronger excitations sextupoles will excite higher order resonance's (non-linear)

Coupling

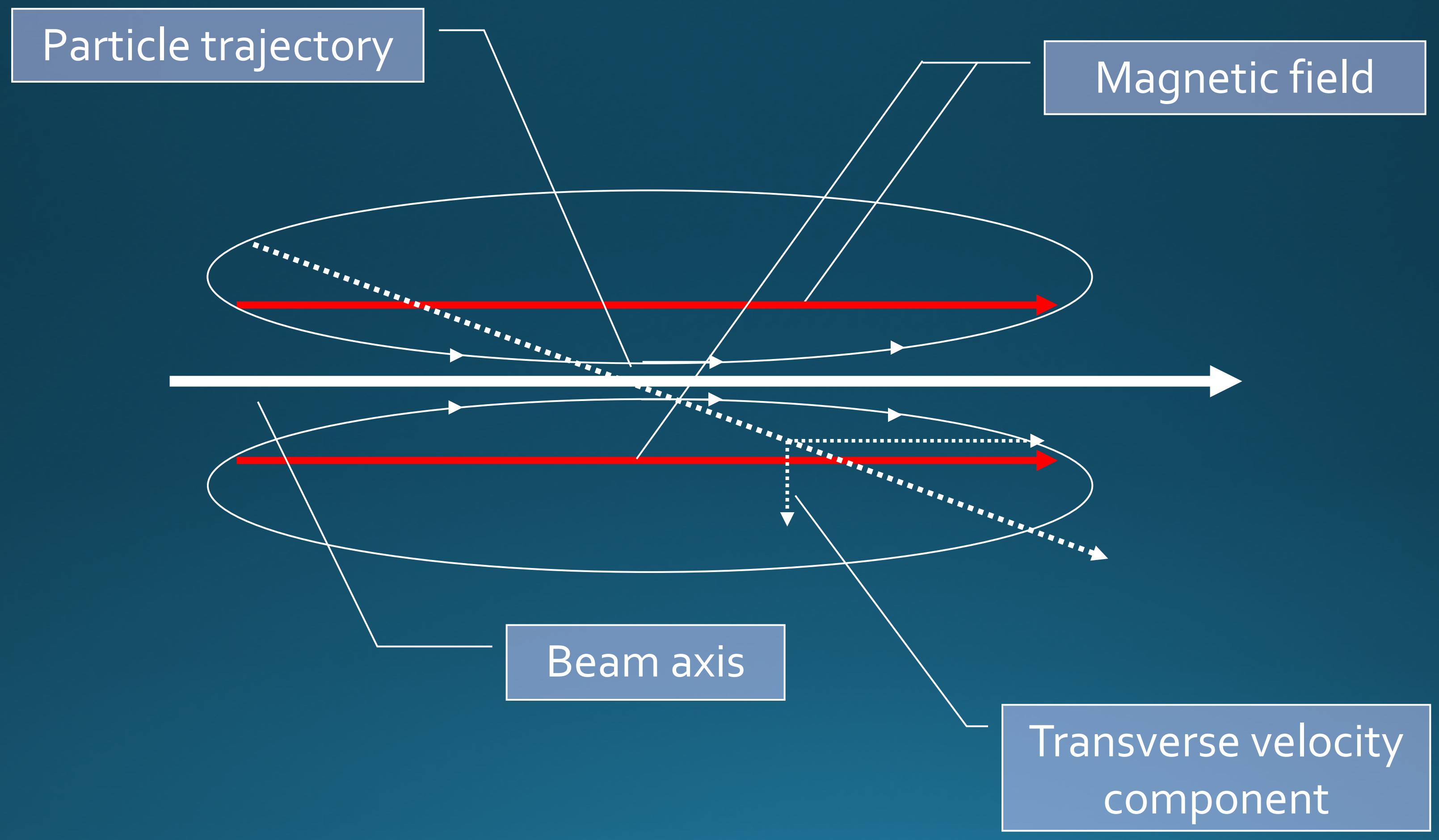
- ✓ Coupling is a phenomena, which converts betatron motion from one plane (horizontal or vertical) into motion in the other plane.
- ✓ Fields that will excite coupling are:
  - ✓ Skew quadrupoles, which are normal quadrupoles, but tilted by 45° about it's longitudinal axis.
  - ✓ Solenoidal (longitudinal magnetic field)

Skew Quadrupole



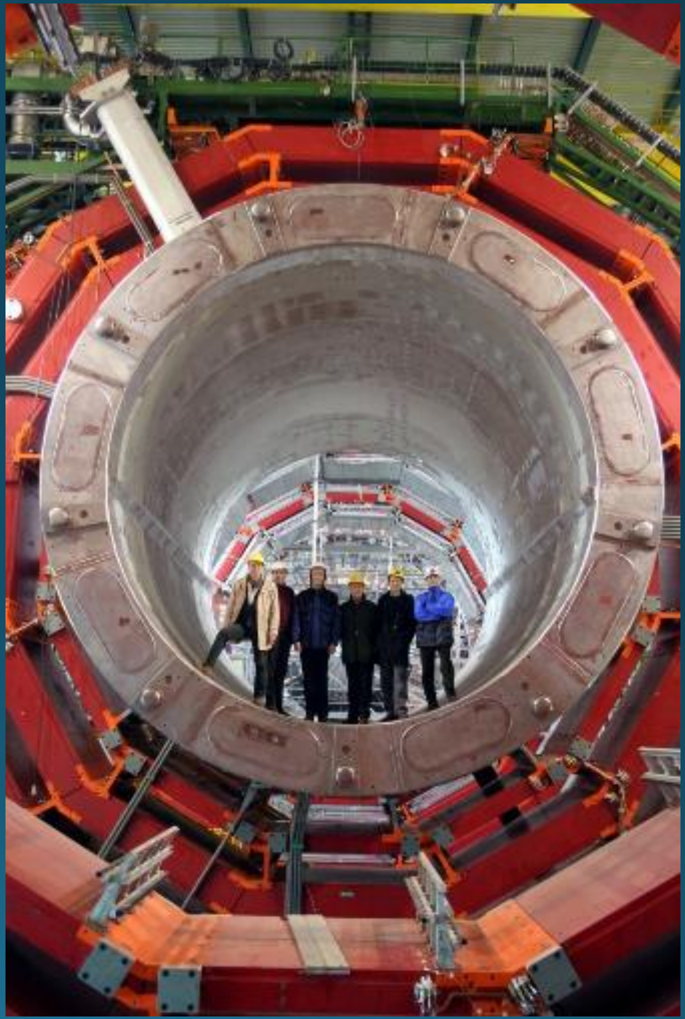
Like a normal quadrupole, but then tilted by 45°

Solenoid; longitudinal field (2)



Solenoid; longitudinal field (2)

Above:
The LPI solenoid that was used for the initial focusing of the positrons.
It was pulsed with a current of 6 kA for some 7 us, it produced a longitudinal magnetic field of 1.5 T.



At the right:
The somewhat bigger CMS solenoid

Coupling and Resonance

- ✓ This coupling means that one can transfer oscillation energy from one transverse plane to the other.

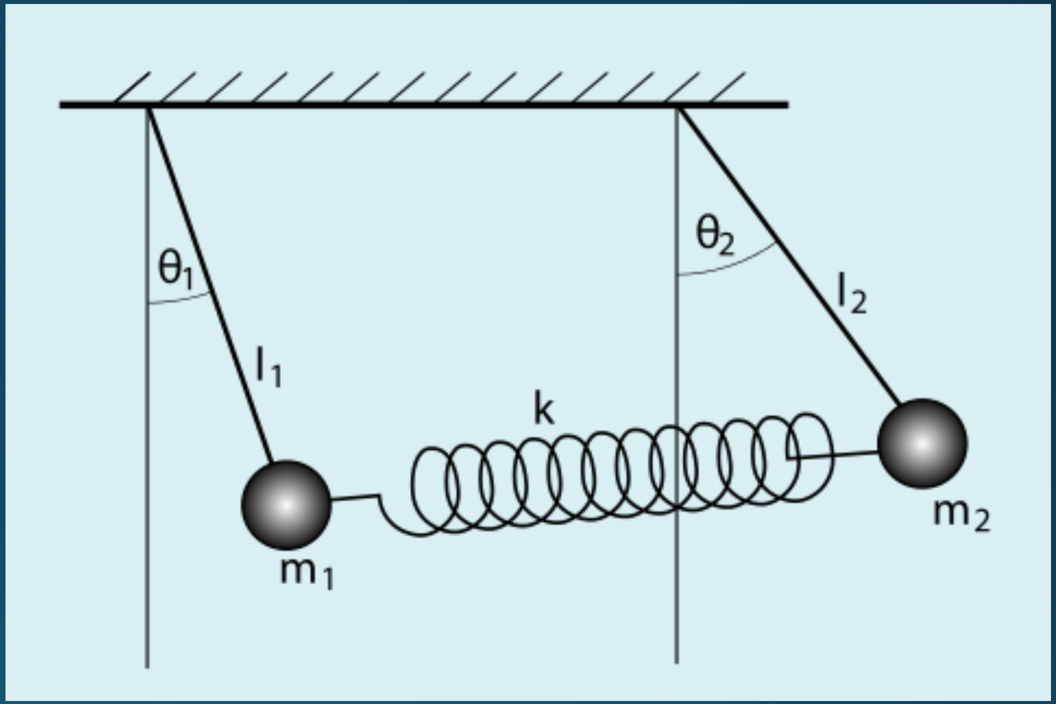
- ✓ Exactly as for linear resonances there are resonant conditions.

$$nQ_h \pm mQ_v = \text{integer}$$

- ✓ If we meet one of these conditions the transverse oscillation amplitude will again grow in an uncontrolled way.

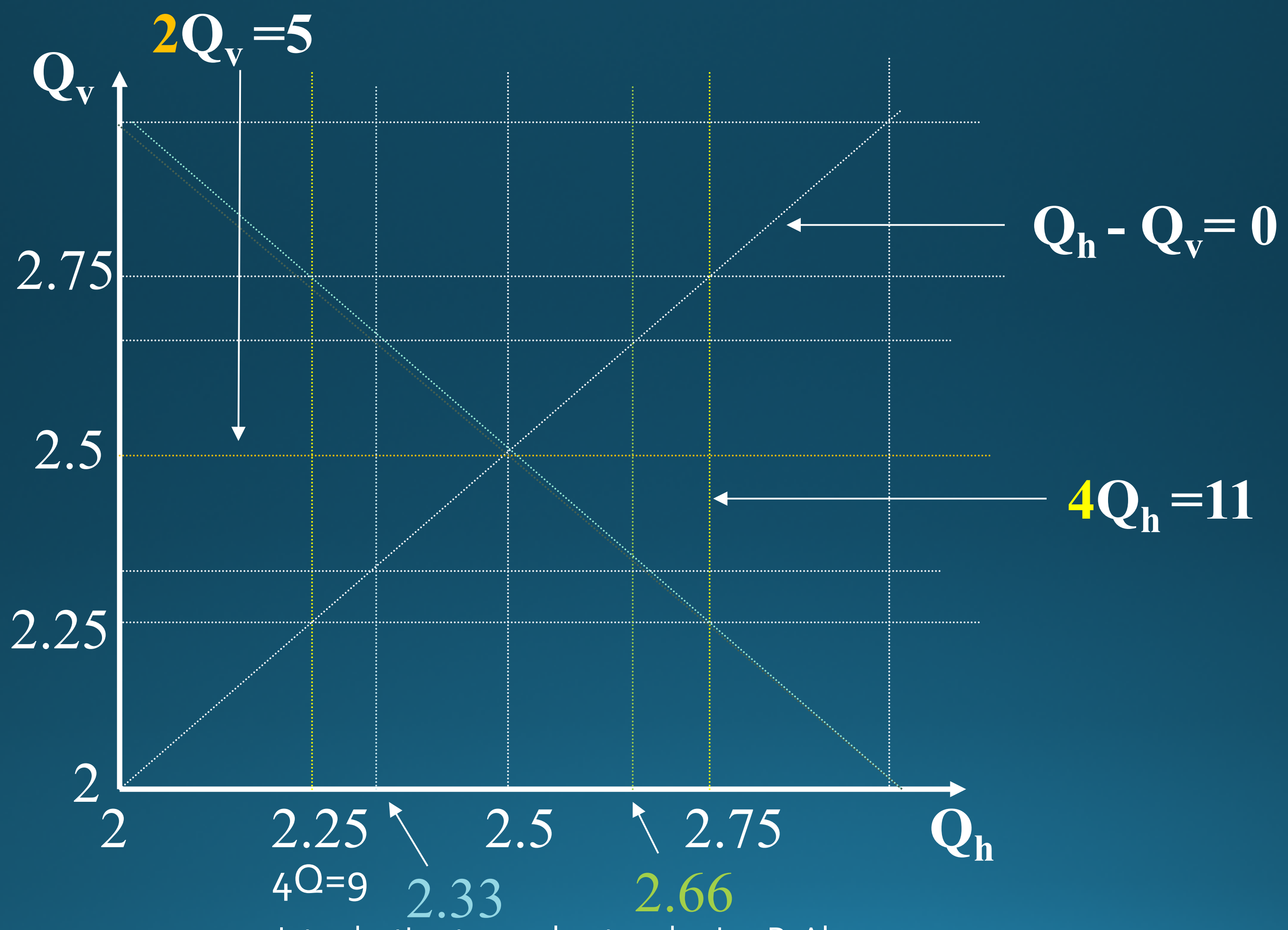
A mechanical equivalent

A beautiful demonstration of how energy can be transferred from one oscillator to another is provided by two weakly coupled pendulums

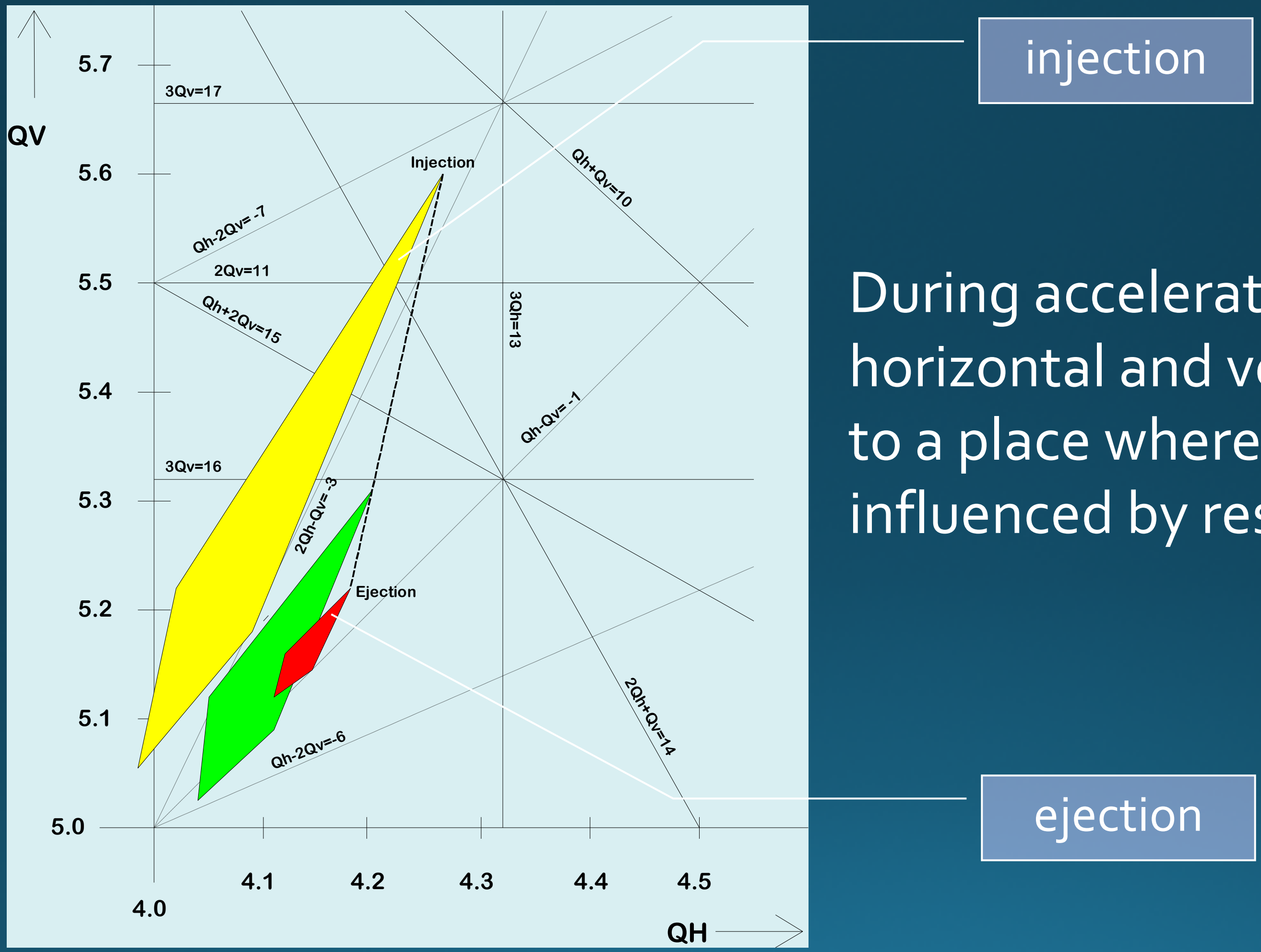


- We can transfer oscillation energy from one pendulum to the other depending on the strength 'k' of the spring

General tune diagram

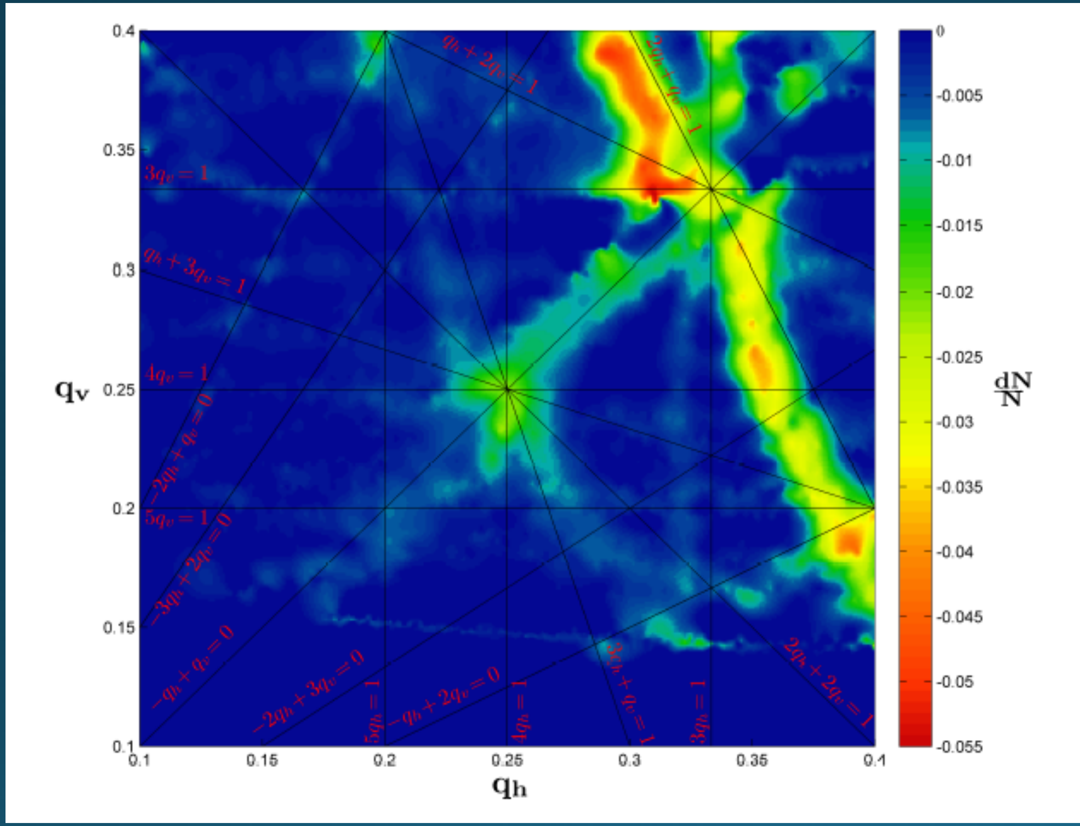


Realistic tune diagram



During acceleration we change the horizontal and vertical tune
to a place where the beam is the least influenced by resonances.

Measured tune diagram



Move a large emittance beam around in this tune diagram and measure the beam losses.

Not all resonance lines are harmful.

Conclusion

- ✓ There are many things in our machine, which will excite resonances:
  - ✓ The magnets themselves
  - ✓ Unwanted higher order field components in our magnets
  - ✓ Tilted magnets
  - ✓ Experimental solenoids (LHC experiments)

- ✓ The trick is to reduce and compensate these effects as much as possible and then find some point in the tune diagram where the beam is stable.