Introduction to transverse beam dynamics III

Restricted to the LINEAR BEAM OPTICS ➔ THE IDEAL WORLD

Content of the course

TBD 1

- Charge particle motion in a magnetic field
- Equations of motion ➔ derivation and assumptions
- Type of magnets

TUTO 1

- Rigidity formula
- Relativistic equations
- Create a storage ring with the Earth Magnetic field

TBD 2

- Particle trajectory
- Transfer Matrices
- Thin lens approximation
- Betatron oscillations
- Betatron tune
- Dispersion

TUTO 2

- Application of transfer matrices
- Thin lens
- FoDo cell

TBD 3

- Phase space ellipse
- Emittance
- Beam size
- Aperture
- Beta function evolution
- Periodic lattices

TUTO 3

- Beam size and aperture calculations

TBD 4

- Effect field errors
- Resonances
- Coupling
- Chromaticity

TUTO 4

- How the tune changes from a quadrupole defect
- Optimize beta beating
- Orbit bumps

Phase space ellipse

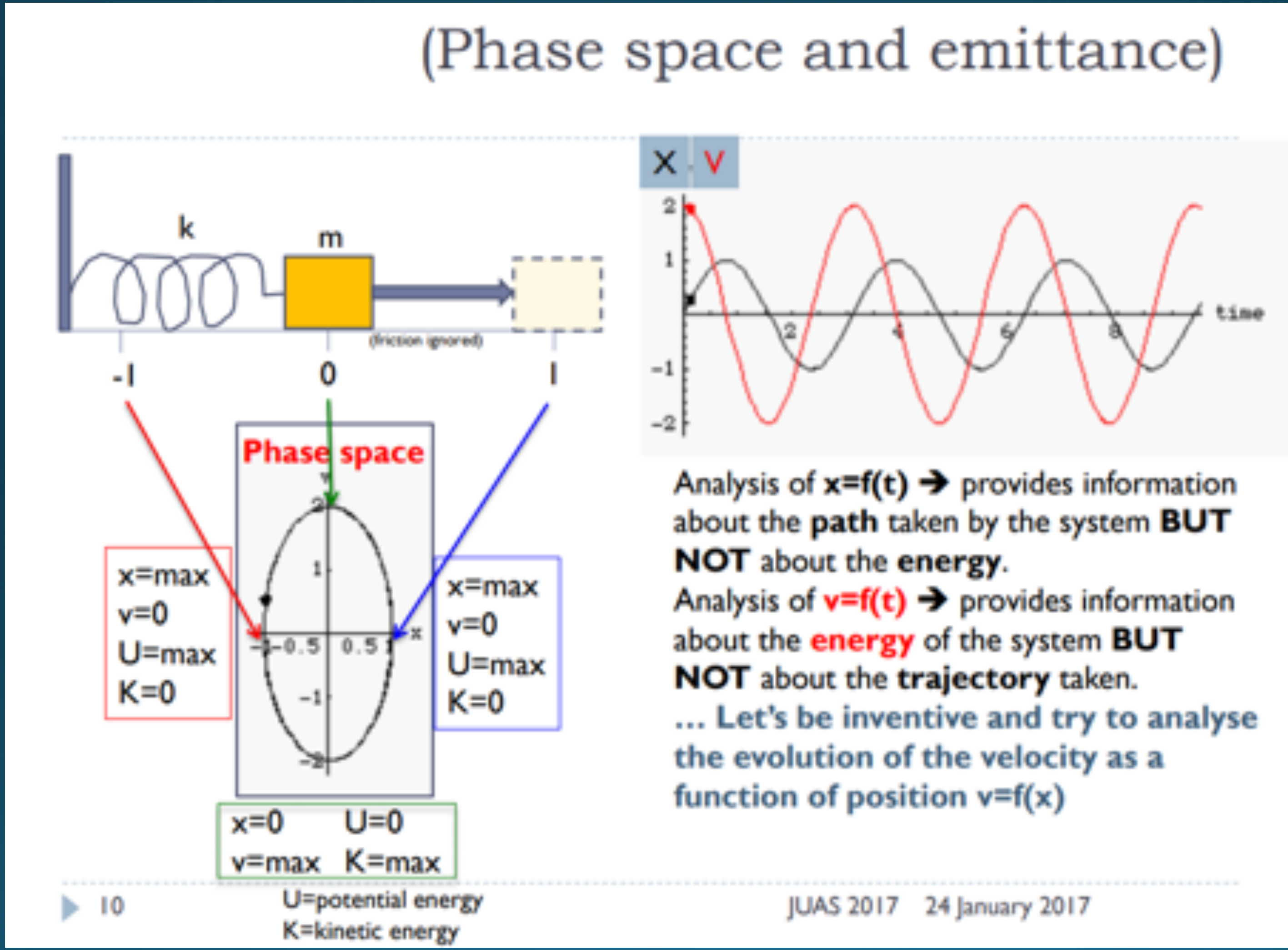
- In the previous lecture we introduced the following:

- *Second we replace the amplitude factor A by $\sqrt{\varepsilon}$ ➔ a constant term called* *emittance*

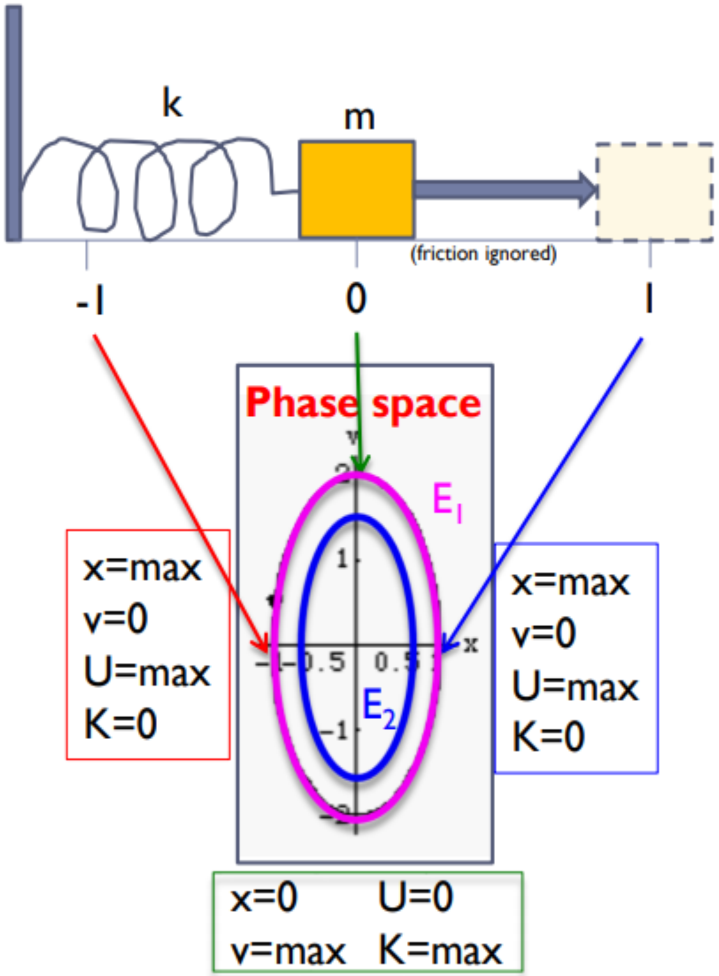
$$x(s) = \sqrt{\varepsilon\beta(s)}\cos(\psi(s) + \phi)$$ Hill's equation solution

- Let's try to understand what emittance is by going to phase space

- Why is convenient to work in phase space?



(Phase space and emittance)



U=potential energy
K=kinetic energy

- Each point (x,v) in the ellipse represents an STATE of the physical system with well define position and velocity.
- All the points (x,v) in the ellipse have the SAME ENERGY (E_1)
- If the initial elongation is smaller, then we get a smaller ellipse with energy E_2 (E_2<E_1).
- If we change K the ellipse shape will change.

:spring constant

A **beam of charged particles** in an accelerator subjected to **focusing and defocusing forces** have the same dynamics as the system above. The beam dynamics also **reproduces an ellipse in phase space …**

(In linear approximation)

- We take the position and its derivative (angle≡velocity)

$$x(s) = \sqrt{\varepsilon\beta(s)}\cos(\psi(s) + \phi)$$

$$x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\alpha(s)\left[\cos(\psi(s) + \phi) + sin(\psi(s) + \phi)\right]$$

- Where $\alpha(s) \equiv -\frac{\beta'(s)}{2}$
- To arrive to an expression describing the phase space motion (x,x') we have to eliminate the terms which depend on phase advance $\psi(s)$

$$\cos(\psi(s) + \phi) = \frac{x(s)}{\sqrt{\varepsilon\beta(s)}} \qquad \sin(\psi(s) + \phi) = \frac{\sqrt{\beta(s)}x'(s)}{\sqrt{\varepsilon}} + \frac{\alpha(s)x(s)}{\sqrt{\varepsilon\beta(s)}}$$

- If we now use the general relation $sin^2\theta + cos^2\theta = 1$

- If we now use the general relation $sin^2\theta + cos^2\theta = 1$

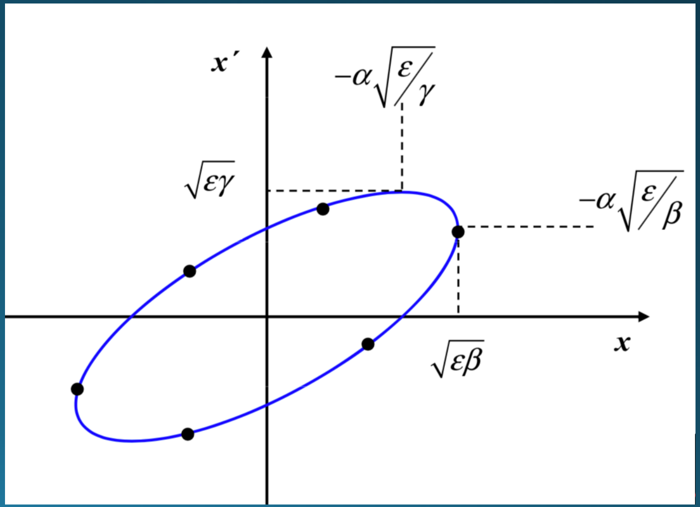
$$\frac{x^2(s)}{\beta(s)} + \left(\frac{\alpha(s)}{\sqrt{\beta(s)}}x(s) + \sqrt{\beta(s)}x'(s)\right)^2 = \varepsilon$$

- If we introduce the definition $\gamma(s) \equiv \frac{1+\alpha^2(s)}{\beta(s)}$ (Attention this is not relativistic gamma)

$$\gamma(s)x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s) = \frac{Area}{\pi} = \varepsilon$$

(Proof Appendix C of "The Physics of Particle Accelerators" K. Wille)

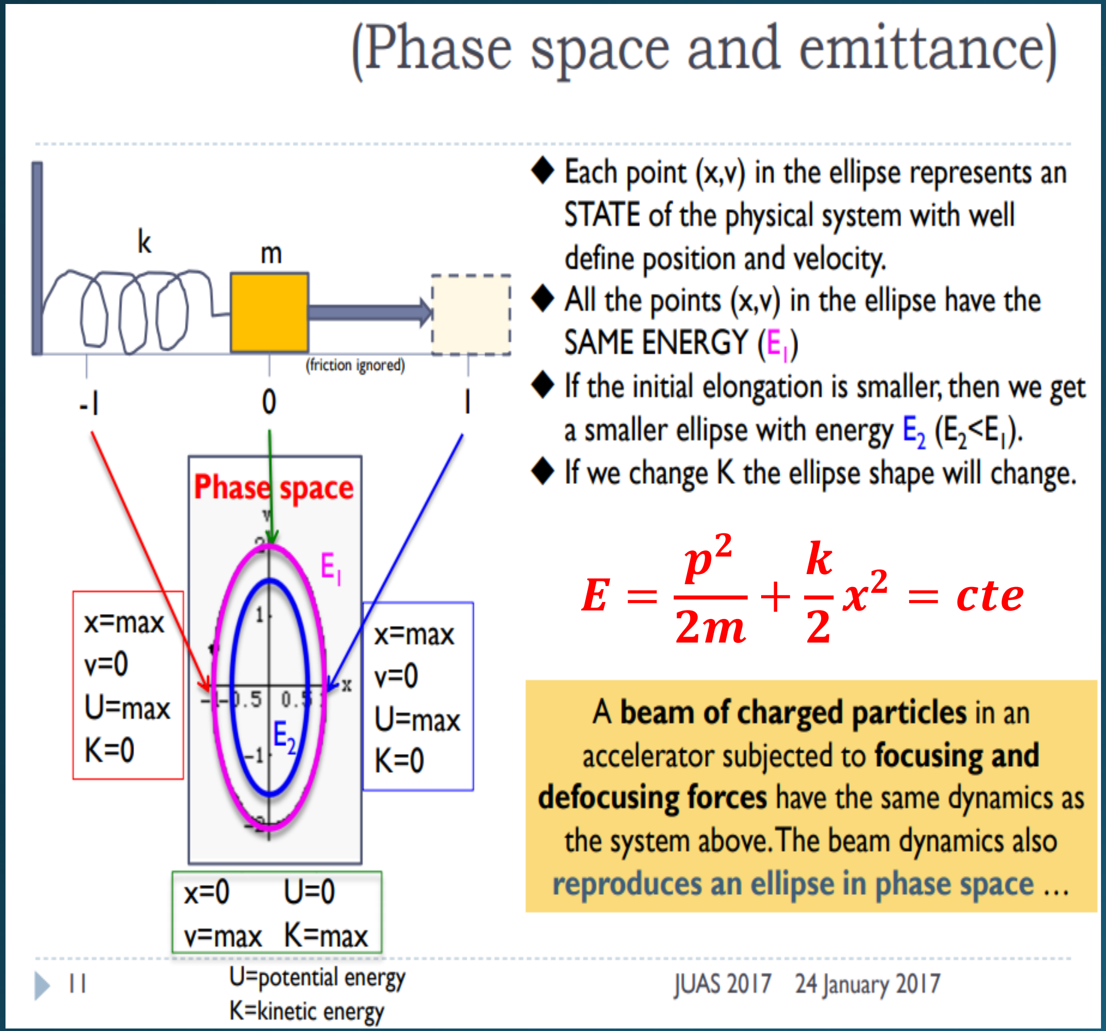
- The emittance, ε, introduced originally as a constant of integration, has now an obvious meaning ➔ it is related to the area of the ellipse, and it is constant ⟵➔ if conservative system, like the one presented two slides ago, and in linear beam dynamics



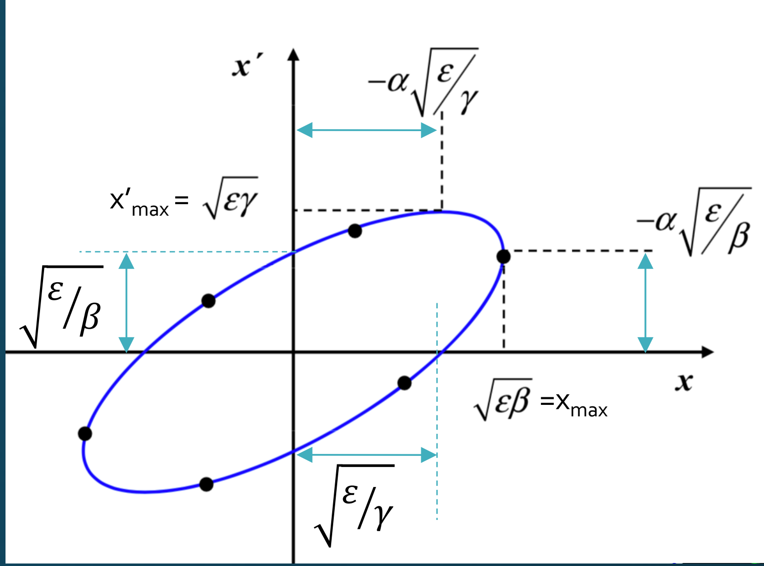
Twiss functions

$$\boldsymbol{\alpha(s)} \equiv -\frac{\beta'(s)}{2}, \boldsymbol{\beta(s)} \,\&\, \boldsymbol{\gamma(s)} \equiv \frac{1+\alpha^2(s)}{\beta(s)}$$

Determine the shape and orientation of the ellipse

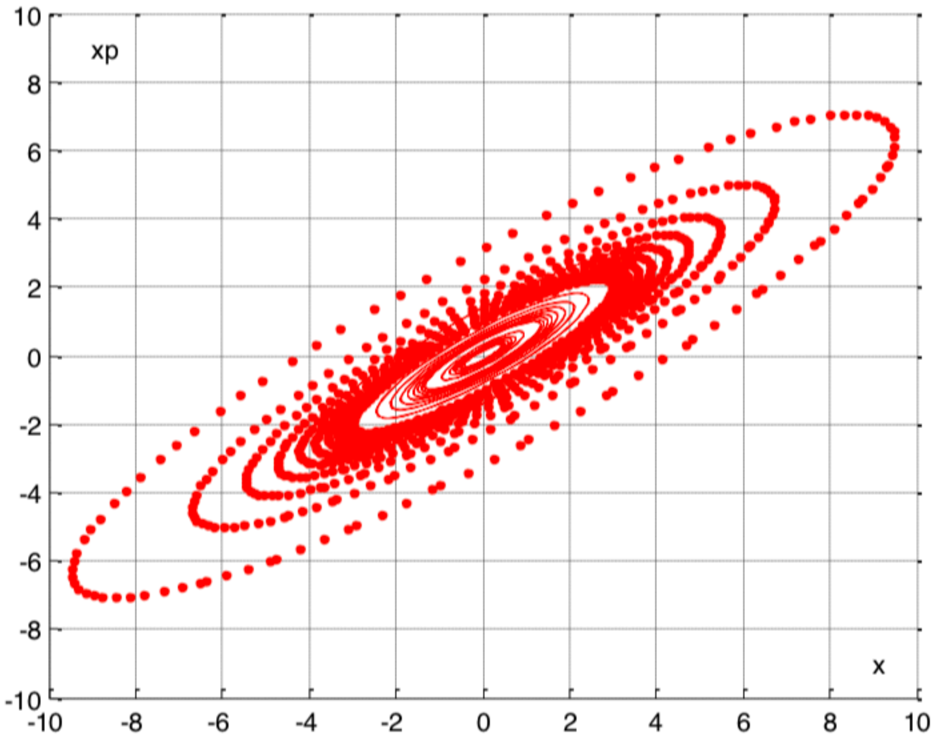


$$\frac{Area}{\pi} = \varepsilon$$



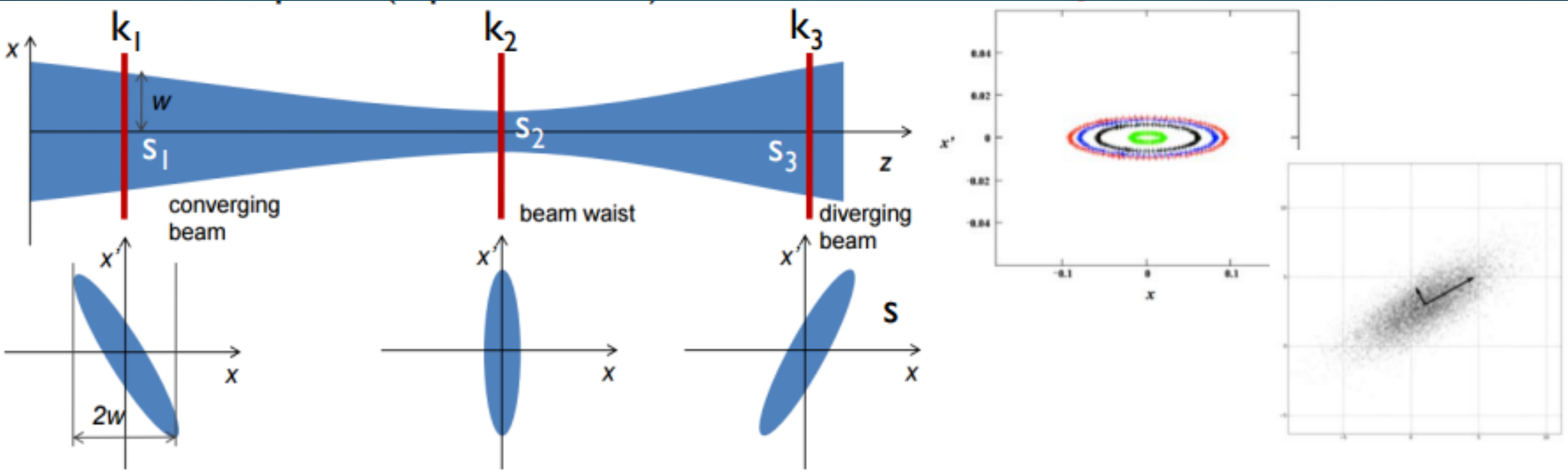
- In both cases, the energy is conserved, the **area of the ellipse is conserved, is an invariant of motion over time**.
- In the case of the mass attached to the spring, the ellipse has always the same shape for a given initial conditions.
- In the case of a particle in an accelerator, since k=k(s), as the particle moves along the closed orbit, the shape and position of the ellipse changes according to the amplitude function β(s). **BUT THE AREA REMAINS CONSTANT**.

Liouville's theorem



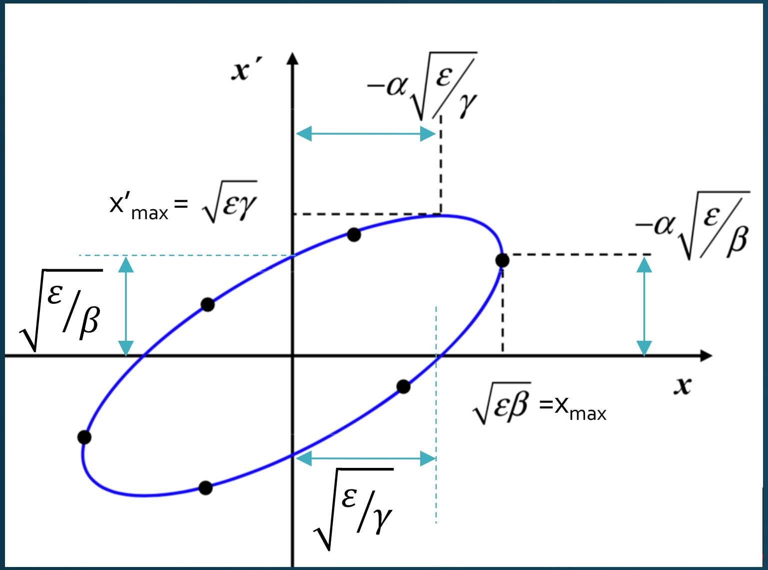
Each particle, in absence of non conservative forces, has a constant invariant.
Under the influence of conservative forces the particle density in phase space is constant.
Magnetic fields of dipoles and quadrupoles are conservative:
In a beam the phase space is maintained constant

Beam size $x_{max} = \sqrt{\beta(s)\varepsilon}$ $x'_{max} = \sqrt{\gamma(s)\varepsilon}$



Along a beam line, the orientation and aspect ratio of the ellipse varies, **BUT THE AREA** remains **CONSTANT** in the absence of non-linear forces or acceleration

AREA ≈ EMITTANCE (Ɛ)

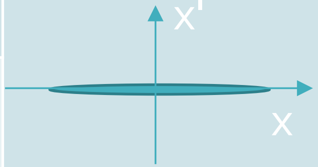


$$x(s) = \sqrt{\varepsilon\beta(s)}\cos(\psi(s)+\phi) \quad =1$$

$$x_{max}(s) = \sqrt{\varepsilon\beta(s)}$$

How much is x'(s) at this position? (put $x_{max}(s)$ in the ellipse equation and solve for x')

$$x_{max}(s) = \sqrt{\varepsilon\beta(s)} \qquad x'_{min}(s) = -\alpha\sqrt{\varepsilon/\beta(s)}$$

	Large β	In the middle of a foc quadrupole β=max & α=0	Ellipse
$x_{max} = \sqrt{\varepsilon\beta(s)}$	Large beam size	Max beam size	
$x'_{min} = -\alpha\sqrt{\varepsilon/\beta(s)}$	Small divergence	Zero divergence	

- So far we have considered the trajectory of a single particle and defined the phase space ellipse and emittance
- However, a beam is made of many particles (10^{11} protons for LHC) each injected with different position and angle, and therefore moving with different amplitudes and describing different ellipses

- This raises the question of what we mean by the AVERAGE EMITTANCE of a beam consisting of an assembly of many particles

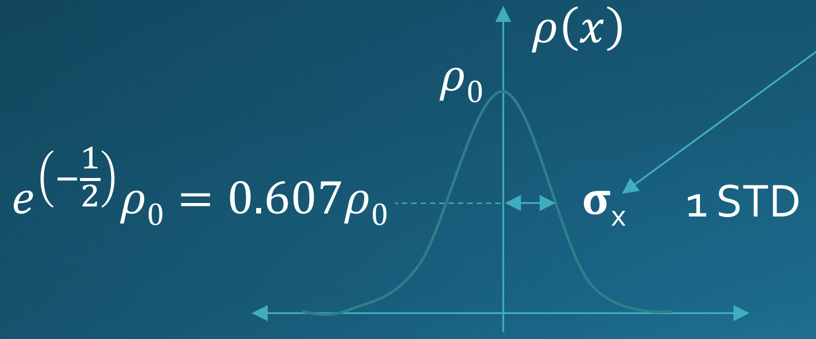
- To answer the question let's think about the equilibrium distribution of particles in a beam
- In most cases the Gaussian distribution is a good description of the transverse density function

$$\rho(x,y) = \frac{Ne}{2\pi\sigma_x\sigma_y} e^{\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right)}$$

N: number of particle of charge *e* in the beam

σx,y: horizontal, vertical beam sizes

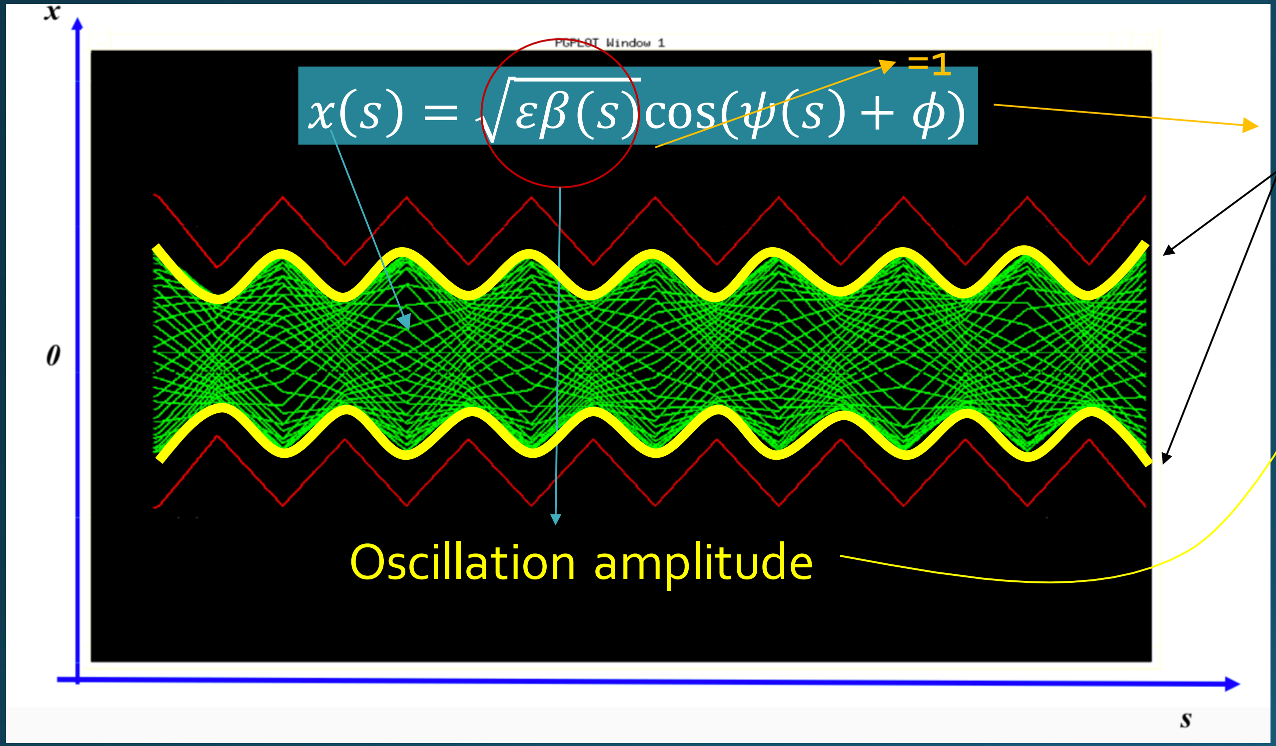
- The horizontal distribution can be obtained by setting y=0



All particles which lie exactly one standard deviation σ from the beam axis may be assigned a precise emittance $\varepsilon_{\mathrm{STD}}$ via the relation

$$\sigma(s) = \sqrt{\varepsilon_{STD}\beta(s)}$$

Emittance of the whole beam $\Rightarrow$ $\varepsilon_{STD} = \dfrac{\sigma^2(s)}{\beta(s)}$

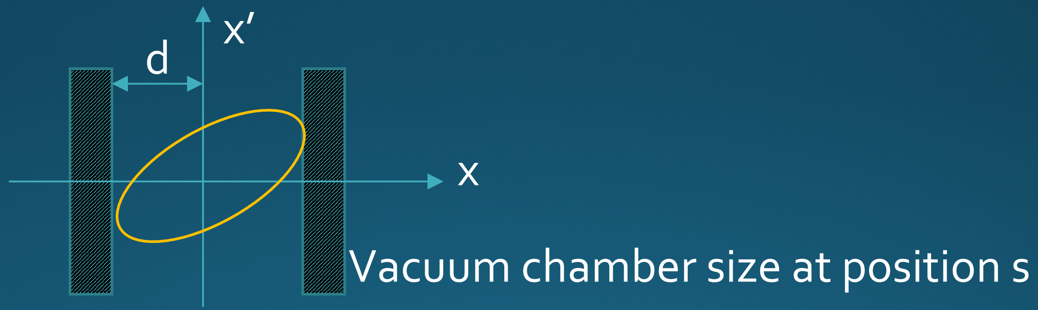


Beam envelope

Since all particle trajectories lie inside this envelope, it defines the <u>BEAM SIZE</u>

Aperture

- When designing an accelerator is important to ensure that the beam has sufficient room available in the transverse phase space
- Even particles undergoing extremely large betatron oscillations can then still circulate stably
- This raises the question how large the phase space ellipse of a particle is allowed to be before it collides with the wall of the vacuum chamber and it is lost



Is this an acceptable solution?

Evolution of the $\boldsymbol{\beta}$ function through the lattice

- The $\boldsymbol{\beta}$ function is an important quantity in linear beam dynamics
- It allows to calculate the beam size along the magnet structure
- It allows to calculate the phase advance of the betatron oscillations in between two points
- Now we will learn how to calculate the evolution of the $\boldsymbol{\beta}$ function itself through the storage ring
- As usual we need initial conditions, in this case the value of the $\boldsymbol{\beta}$ function at the starting point in the lattice $\boldsymbol{\beta}(s_o)$
- Starting from this initial value we can calculate step by step the evolution of $\boldsymbol{\beta}$ by the use of appropriate transformations

- There are two methods, we will briefly try to understand them

METHOD 1

- We describe the trajectory of the particle by the vector ***X***, which travels around the phase space ellipse during the motion of the particle around the orbit
- At the beginning of the magnet structure, s=s$_0$=0, we have **X**=**X**$_0$

$$X_0 = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \qquad X_0^T = \begin{pmatrix} x_0 & x_0' \end{pmatrix}$$

(Transpose matrix)

- We define the beta matrix

$$B_0 \equiv \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix}$$

$\boldsymbol{\alpha(s)} \equiv -\frac{\beta'(s)}{2}, \boldsymbol{\beta(s)}$ & $\boldsymbol{\gamma(s)} \equiv \frac{1+\alpha^2(s)}{\beta(s)}$ ⇨ Twiss functions

- If we calculate the product $X_0^T B_0^{-1} X_0 = \gamma_0 x_0^2 + 2\alpha_0 x_0 x_0' + \beta_0 x_0'^2 = \varepsilon$
- We also know that the trajectory vector at any position s can be calculated as

$$X_1 = M X_0$$

Where $M^{-1}M = M^T(M^T)^{-1} = 1$

- Let's now play with the matrices

$$\varepsilon = X_0^T B_0^{-1} X_0 = X_0^T M^T \underbrace{(M^T)^{-1} B_0^{-1} M^{-1}}_{} M X_0$$

$$A^T B^T = (BA)^T \qquad \& \quad A^{-1} B^{-1} = (BA)^{-1}$$

$$\varepsilon = X_0^T B_0^{-1} X_0 = X_0^T M^T ((M^T)^{-1} (M B_0\,)^{-1}) M X_0$$

$$= X_0^T M^T (M B_0 M^T)^{-1} M X_0$$

$$= \underbrace{(\mathrm{M}X_0)^T}_{X_1^T} (M B_0 M^T)^{-1} \underbrace{M X_0}_{X_1}$$

$$\varepsilon = X_0^T B_0^{-1} X_0 = X_1^T \boxed{(M B_0 M^T)^{-1}} X_1 = X_1^T \boxed{B_1^{-1}} X_1$$

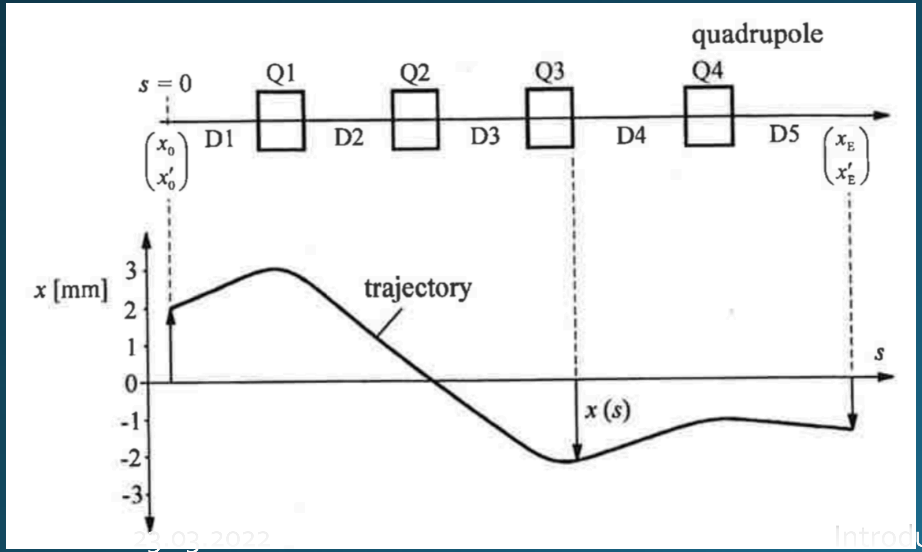
Since at point s=s1 the particle trajectory is given by X1 and the B1 matrix, then it follows

$$B_1 = M B_0 M^T$$

$$X_1 = M X_0$$

$$B_1 = M B_0 M^T$$

Evolution of the beta function along the lattice uses the same cos(h), sin(h) matrices as the particle trajectory



M=

$$\mathbf{X}_E = \boxed{\mathbf{M}_{D5} \cdot \mathbf{M}_{Q4} \cdot \mathbf{M}_{D4} \cdot \mathbf{M}_{Q3} \cdot \mathbf{M}_{D3} \cdot \mathbf{M}_{Q2} \cdot \mathbf{M}_{D2} \cdot \mathbf{M}_{Q1} \cdot \mathbf{M}_{D1}} \cdot \mathbf{X}_0$$

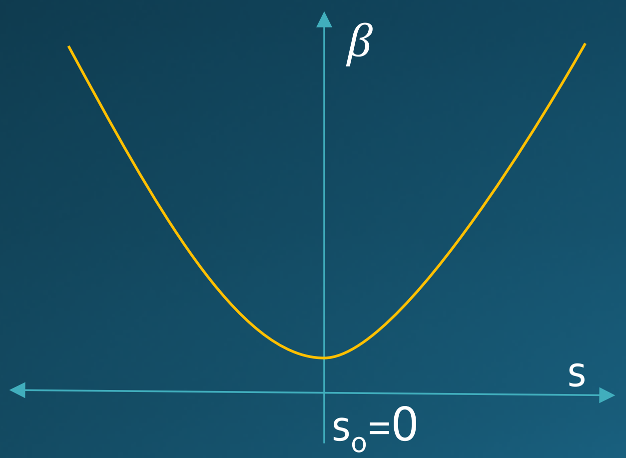
Example: calculate the beta function around a symmetry point in a field-free drift section

- The symmetry point is $s=s_o=o$
- All the coordinates at this point will have the label *
- Here the gradient of the beta function is zero, i.e. $\alpha^* = 0$
- The beta function at this position is β^*

$$X_1 = M X_0$$

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \underbrace{\boxed{\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}}}_{=M} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

DRIFT
k=o



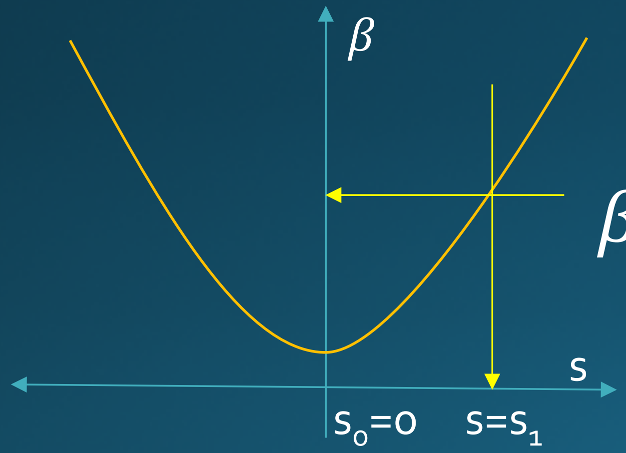
$$B_1 = M B_0 M^T$$

$\beta^*, \alpha^* = 0$

$$B_0 \equiv \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix}$$

Beta function around a symmetry point

$$B_1(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta^* & 0 \\ 0 & \frac{1}{\beta^*} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} \boxed{\beta^* + \frac{s^2}{\beta^*}} & \frac{s}{\beta^*} \\ \boxed{\frac{s}{\beta^*}} & \frac{1}{\beta^*} \end{pmatrix}$$



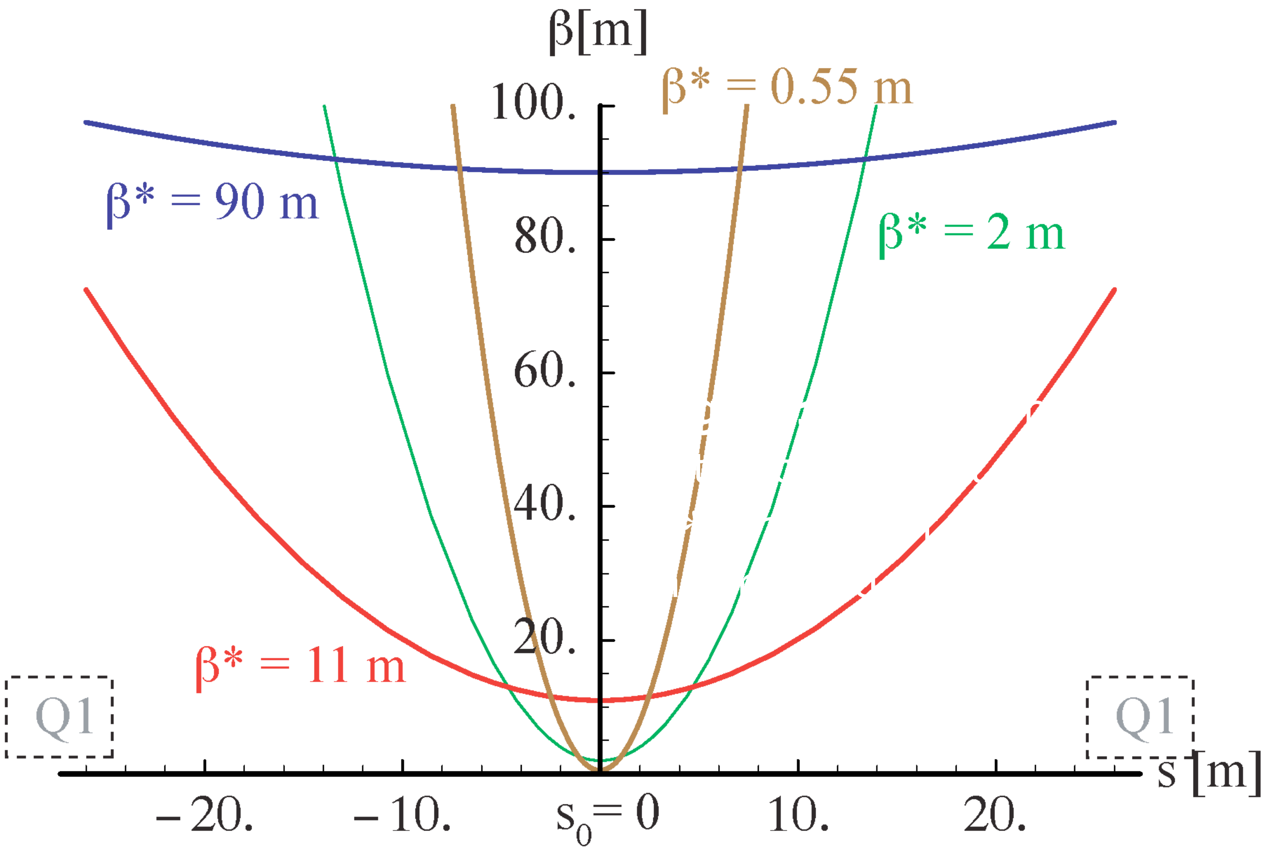
$\beta^*, \alpha^* = 0$

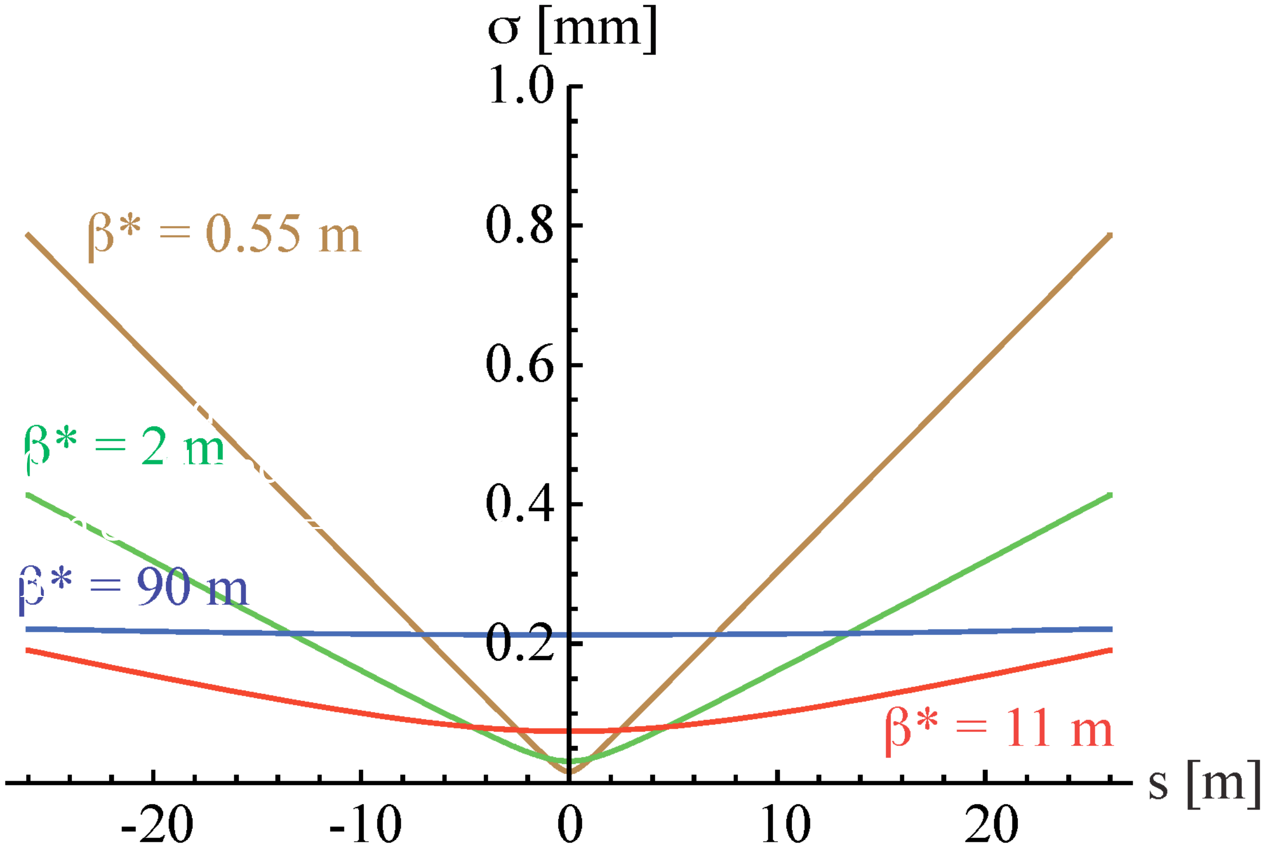
$$\beta_1 = \beta^* + \frac{s_1^2}{\beta^*}$$

-α around a symmetry point

The beta function grows quadratically with the distance from the symmetry point and grows faster the smaller the β

LHC





β-functions and beam sizes σ at distance s from the interaction point at $s_0=0$, for $\beta^*=0.55,2,11$ and 90 m up to $L^*=\pm 26$ m, for the LHC design beam energy $Eb=7$TeV and normalized emittance $\epsilon_N=3.75\mu$m.

METHOD 2

- We start with two trajectory vectors at different positions in the lattice, s_o and s, and its optical functions at s_o and s

$$X_0 = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \qquad X = \begin{pmatrix} x \\ x' \end{pmatrix} \qquad \beta_0, \alpha_0, \gamma_0 \qquad \beta, \alpha, \gamma$$

- At s_o and s we have the same invariant of motion

$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2 = \beta_0 {x_0}'^2 + 2\alpha_0 x_0 {x_0}' + \gamma_0 {x_0}^2$$

- We also know $X = MX_0$ with $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$

$X_0 = M^{-1}X$ with $M^{-1} = \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$

- Solving for $\boldsymbol{X_o}$ and bringing x_o and x'_o as a function of $\boldsymbol{X}$ and the matrix elements to the invariant equation we obtain

$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2 = \beta_0(-m_{21}x + m_{11}x')^2$$
$$+2\alpha_0(m_{22}x - m_{12}x')(-m_{21}x + m_{11}x')$$
$$+\gamma_0(m_{22}x - m_{12}x')^2$$

- Rearranging terms we end up

$$\beta = m_{11}^2\beta_0 - 2m_{12}m_{11}\alpha_0 + m_{12}^2\gamma_0$$
$$\alpha = -m_{21}m_{11}\beta_0 + (m_{22}m_{11} + m_{12}m_{21})\alpha_0 - m_{22}m_{12}\gamma_0$$
$$\gamma = m_{21}^2\beta_0 - 2m_{22}m_{21}\alpha_0 + m_{22}^2\gamma_0$$

$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix} = \begin{pmatrix} m_{11}^2 & -2m_{12}m_{11} & m_{12}^2 \\ -m_{21}m_{11} & m_{22}m_{11} + m_{12}m_{21} & -m_{22}m_{12} \\ m_{21}^2 & -2m_{22}m_{21} & m_{22}^2 \end{pmatrix}\begin{pmatrix}\beta_0\\ \alpha_0\\ \gamma_0\end{pmatrix}$$

Dispersion calculation

- We said "dispersion is the orbit of a particle with $\Delta p/p_o$=1"

$$x(s) = D(s)\underbrace{\cancel{\frac{\Delta p}{p_o}}}_{=1} = \mathrm{D(s)}$$

$$x'' + \left(\frac{1}{\rho^2} - \cancel{k}\right)x = \frac{1}{\rho}\ \overbrace{\cancel{\frac{\Delta p}{p_o}}}^{=1}$$

$$D''(s) + \frac{1}{\rho^2}D(s) = \frac{1}{\rho}$$

Dispersion calculation

$$D''(s) + \frac{1}{\rho^2} D(s) = \frac{1}{\rho}$$

- This is an inhomogeneous differential equation, which we already solved in its homogeneous form (slide 12)
- Now we just need to obtain the inhomogeneous solution
- Since the right hand side of the equation is a constant, we can propose as solution: $D_{inh}=C$

$$\underbrace{D''(s)}_{0} + \frac{1}{\rho^2} \underbrace{D(s)}_{C} = \frac{1}{\rho} \Rightarrow C = \rho$$

$$x'' + \frac{1}{\rho^2} x = 0$$

Slide 12:
Dipole matrix

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cos {}^{s}/_{\rho} & \rho \sin {}^{s}/_{\rho} \\ -{}^{s}/_{\rho} \sin {}^{s}/_{\rho} & \cos {}^{s}/_{\rho} \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\mathrm{D}(s) = \underbrace{A\cos {}^{s}/_{\rho} + B\sin {}^{s}/_{\rho}}_{\text{homogeneous}} + \underbrace{\rho}_{\text{inhomogeneous}}$$

$$\mathrm{D}'(s) = -\frac{A}{\rho} \sin {}^{s}/_{\rho} + \frac{B}{\rho} \cos {}^{s}/_{\rho}$$

- The constants of integration A and B are determined by the initial conditions at s=0

$$D(0) = D_0 \qquad D'(0) = D_0'$$

- Inserting those in the equations for D and D' we get:

$$A = D_0 - \rho \qquad B = \rho D_0'$$

$$D(s) = D_0 \cos {}^{s}/_{\rho} + D_0' \rho \sin {}^{s}/_{\rho} + \rho\left(1 - \cos\frac{s}{\rho}\right)$$

$$D'(s) = -\frac{D_0}{\rho} \sin {}^{s}/_{\rho} + D_0' \cos {}^{s}/_{\rho} + \sin\frac{s}{\rho}$$

$$\begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \begin{bmatrix} \cos {}^{s}/_{\rho} & \rho \sin {}^{s}/_{\rho} & \rho(1 - \cos {}^{s}/_{\rho}) \\ -\frac{1}{\rho} \sin {}^{s}/_{\rho} & \cos {}^{s}/_{\rho} & \sin {}^{s}/_{\rho} \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} D_0(s) \\ D_0'(s) \\ 1 \end{pmatrix}$$

- What would be the trajectory in both planes through a dipole magnet?

$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \\ \Delta p/_{p_0} \end{pmatrix} = \begin{bmatrix} \cos {}^{s}/_{\rho} & \rho \sin {}^{s}/_{\rho} & 0 & 0 & \rho(1 - \cos {}^{s}/_{\rho}) \\ -\frac{1}{\rho} \sin {}^{s}/_{\rho} & \cos {}^{s}/_{\rho} & 0 & 0 & \sin {}^{s}/_{\rho} \\ 0 & 0 & 1 & s & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_0(s) \\ x_0'(s) \\ y_0(s) \\ y_0'(s) \\ \Delta p/_{p_0} \end{pmatrix}$$

Calculate transfer matrices from optical functions

- Up to now we have used the transfer matrix ***M*** to calculate uniquely optical functions at the end of a magnet structure from known initial conditions
- Now we want to use the known values of the optical functions at the beginning and end of the magnet structure to calculate how the particle trajectory evolves
- This has the advantage that many of the characteristics of the beam transport system can be discussed without knowing in detail the magnet structure
- I will save you the mathematics (see page 88 in K. Wille)
- I just bring you directly to the results, which is the useful thing

$$X_1 = MX_0$$

$$M = \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}(cos\psi + \alpha_0 sin\psi) & \sqrt{\beta\beta_0}sin\psi \\ \frac{(\alpha_0 - \alpha)cos\psi - (1 + \alpha_0\alpha)sin\psi}{\sqrt{\beta\beta_0}} & \sqrt{\frac{\beta_0}{\beta}}(cos\psi - \alpha sin\psi) \end{pmatrix}$$

ψ is the phase advance of the betatron oscillation between s_o and s

Periodic lattices

- Very simple, solve the matrix below for one turn, i.e. s_o=s

$$M = \begin{pmatrix} \cancelto{1}{\sqrt{\frac{\beta}{\beta_0}}}(\cos\psi + \alpha_0 \sin\psi) & \sqrt{\beta\beta_0}\sin\psi \\ \frac{\cancelto{0}{(\alpha_0 - \alpha)}\cos\psi - (1+\alpha_0\alpha)\sin\psi}{\sqrt{\beta\beta_0}} & \sqrt{\frac{\beta_0}{\beta}}(\cos\psi - \alpha\sin\psi) \end{pmatrix}$$

$$M_{turn} = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ \frac{-(1+\alpha_s^2)\sin\psi_{turn}}{\beta_s} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

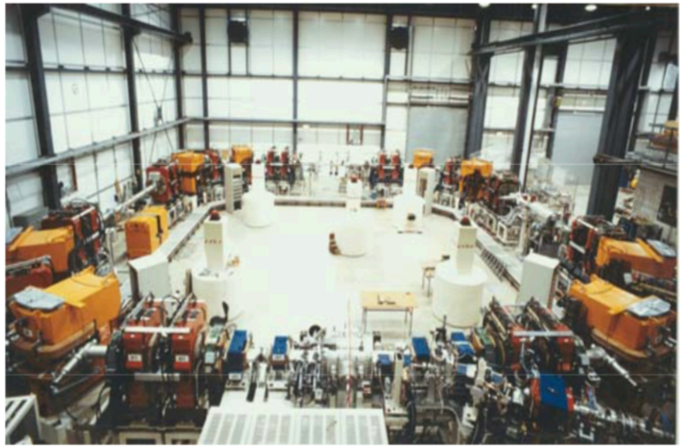
$$\psi_{turn} = \int_s^{s+L} \frac{ds}{\beta(s)} \Rightarrow Q_{x,y} = \frac{1}{2\pi}\int_s^{s+L} \frac{ds}{\beta(s)}$$

Betatron tune
Number of oscillations per turn

$$\psi_{turn} = 2\pi Q$$

Stability Criterion:

Question: ***what will happen, if we do not make too many mistakes and your particle performs one complete turn ?***



Matrix for 1 turn:

$$M=\begin{pmatrix} \cos\psi_{turn}+\alpha_s\sin\psi_{turn} & \beta_s\sin\psi_{turn} \\ -\gamma_s\sin\psi_{turn} & \cos\psi_{turn}-\alpha_s\sin\psi_{turn} \end{pmatrix} = \cos\psi\cdot\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\boldsymbol{1}} + \sin\psi\underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\boldsymbol{J}}$$

Matrix for N turns:

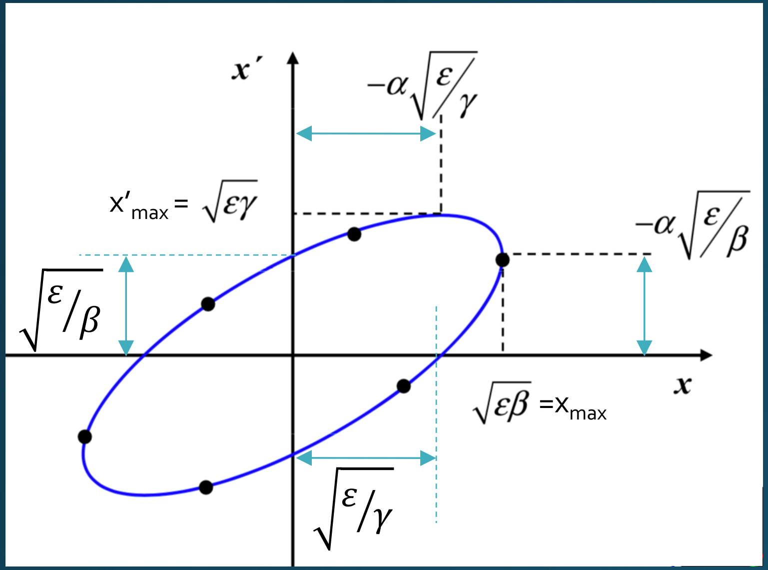
$$M^N = (1\cdot\cos\psi + J\cdot\sin\psi)^N = 1\cdot\cos N\psi + J\cdot\sin N\psi$$

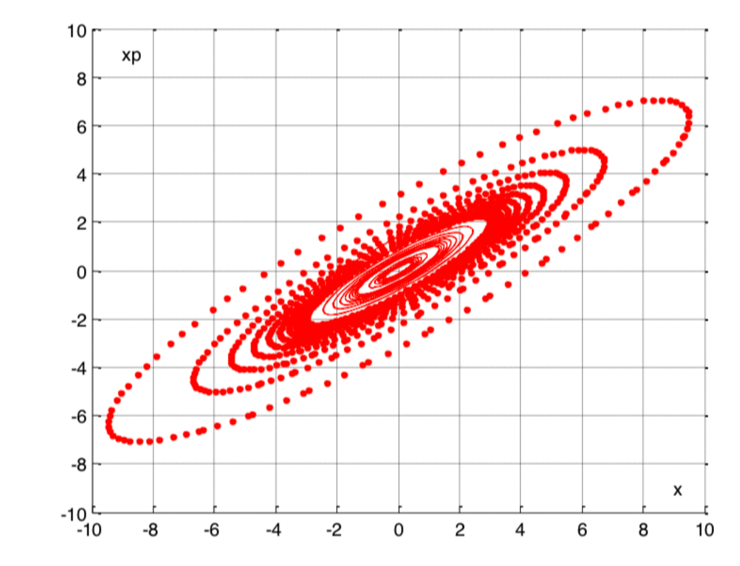
The motion for N turns remains bounded, if the elements of M^N remain bounded

$$\psi = real \quad \leftrightarrow \quad |\cos\psi| \leq 1 \quad \leftrightarrow \quad Tr(M) \leq 2$$

Measurement of the beam emittance

$$\frac{Area}{\pi} = \varepsilon$$





We see that all particles travel along their individual ellipses in phase space. If we now choose one with the largest phase ellipse within a particular beam, we know all particles within that ellipse will stay within that ellipse. Therefore we are able to describe the collective behavior of a beam formed by many particles by the dynamics of a single particle.

Since all particles enclosed by a phase space ellipse stay within that ellipse, we only need to know how the ellipse parameters transform along the beam line to be able to describe the beam.

Let's define the beam matrix with the well known Twiss parameters:

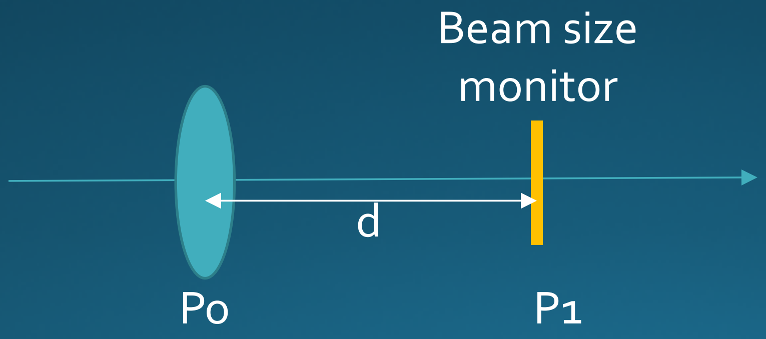
$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \varepsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

$$\left.\begin{aligned} \sigma_{11} &= \langle x_i^2 \rangle = \varepsilon\beta \\ \sigma_{22} &= \langle x_i'^2 \rangle = \varepsilon\gamma \\ \sigma_{12} &= \langle x_i x'_i \rangle = -\varepsilon\alpha \end{aligned}\right\} \quad \varepsilon^2 = \sigma_{11}\,\sigma_{22} - \sigma_{12}^2$$

If we find a way to determine the beam matrix, then we can measure the emittance

QUADRUPOLE SCAN METHOD TO MEASURE BEAM EMITTANCE

To determine the beam matrix at a place Po, we consider a beam transport line with one quadrupole at Po and a beam size monitor at P1. We vary the strength of the quadrupole and measure the beam size at P1 as a function of the quadrupole strength. This is equivalent to measure the beam size at a different locations in the line.



$$\begin{pmatrix} 1-d/f & d \\ -1/f & 1 \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

Matrix in thin lens

It can be demonstrated (H. Wiedemann, Particle Accelerator Physics, Chapter 5.1 Measurement of beam emittance) that form the beam matrix at Po, one can get the beam matrix element 11 at P1, i.e. the beam size at P1:

This is what we measure with the beam diagnostic device

$$\sigma_{1,11}(k) = \underbrace{\left(d^2 \ell_q^2 \sigma_{0,11}\right)}_{a} k^2 + \underbrace{\left(-2d\,\ell_q \sigma_{0,11} - 2d^2 \ell_q \sigma_{0,12}\right)}_{b} k + \underbrace{\left(\sigma_{0,11} + 2d\,\sigma_{0,12} + d^2 \sigma_{0,22}\right)}_{c} .$$

This we vary in steps

lq and d are known

Fitting $\sigma_{1,11}(k)$ to a parabola $\boxed{y = ak^2 + bk + c}$ will determine the whole beam matrix at Po

$$\sigma_{0,11} = \frac{a}{d^2 \ell_q^2},$$

$$\sigma_{0,12} = \frac{-b - 2d\ell_q \sigma_{0,11}}{2d^2 \ell_q},$$

$$\sigma_{0,22} = \frac{c - \sigma_{0,11} - 2d\sigma_{0,12}}{d^2}.$$

Geometrical emittance

$$\varepsilon^2 = \sigma_{11}\,\sigma_{22} - \sigma_{12}^2$$

Normalized emittance

$$\varepsilon_n = \gamma_{rel} \beta_{rel} \varepsilon$$

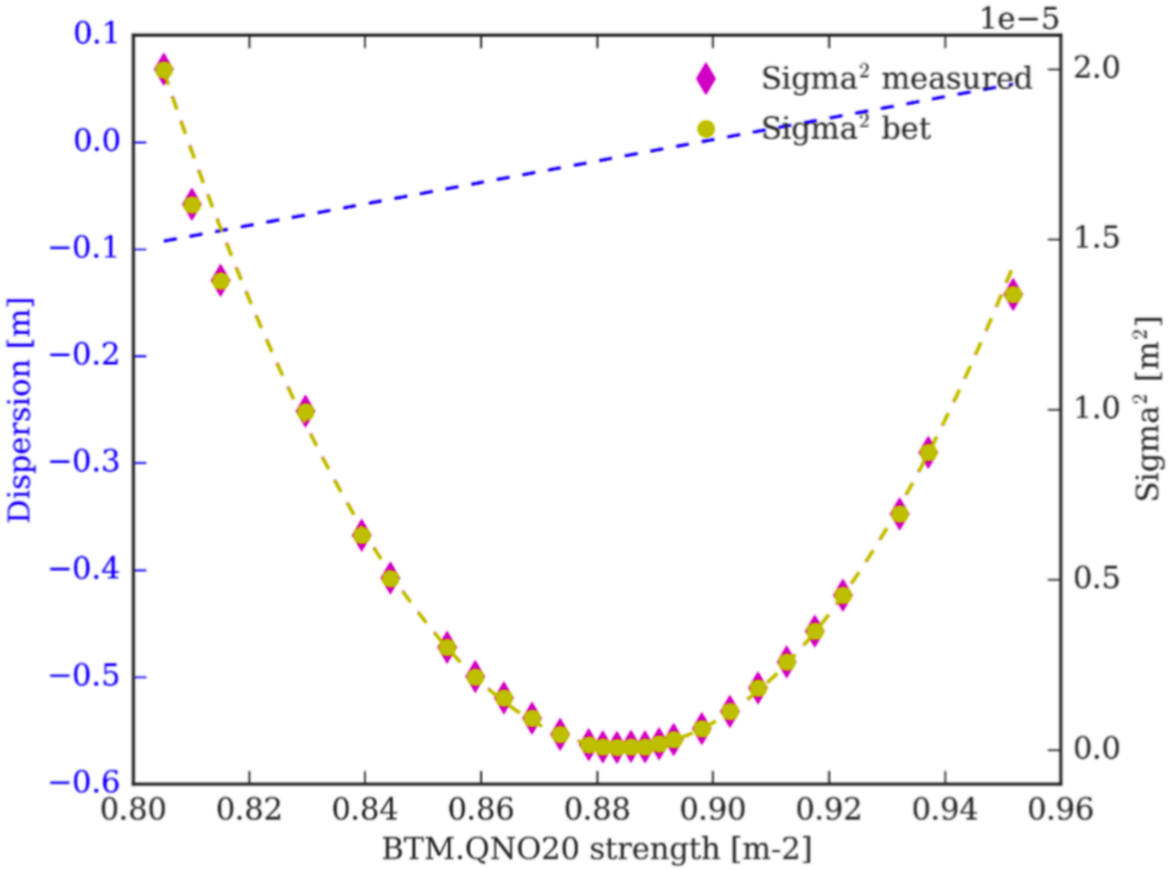
The beam matrix not only defines the beam emittance but also the betatron functions at the beginning of the quadrupole in this measurement. We gain with this measurement a full set of initial beam parameters $(\alpha_0, \beta_0, \gamma_0, \varepsilon)$ and may now calculate beam parameters at any point along the transport line.

$$\sigma_0 = \begin{pmatrix} \sigma_{0,11} & \sigma_{0,12} \\ \sigma_{0,21} & \sigma_{0,22} \end{pmatrix} = \varepsilon \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix}$$

COMMENTS

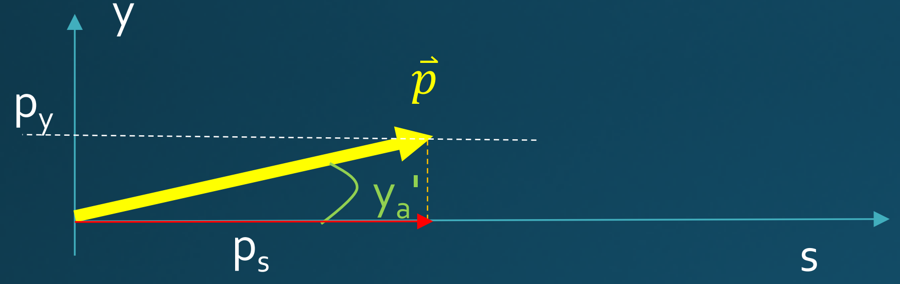
- Chose setting with **focus closed to** the **SEM grid**
- **Careful at the focus** – beam very small and possible space charge effects
- Guarantee **large beam size variation with quadrupole strength**, to be able to accurately fit the 3 parameters.

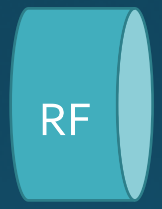
Disp-free optics

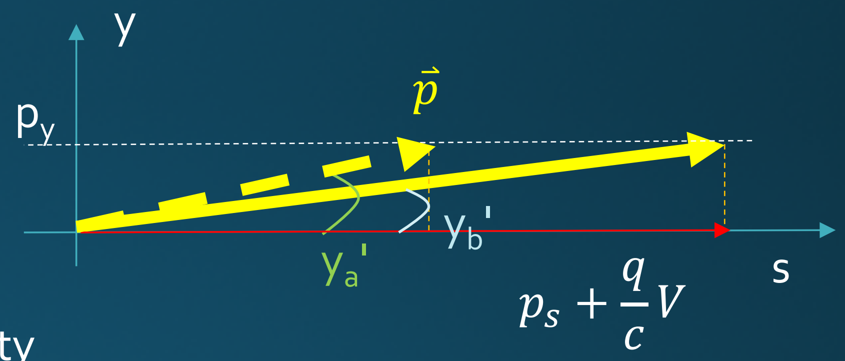


Emittance: geometrical and normalized emittance

- In these lectures I have used (x,x') and (y,y') as the phase space coordinates.





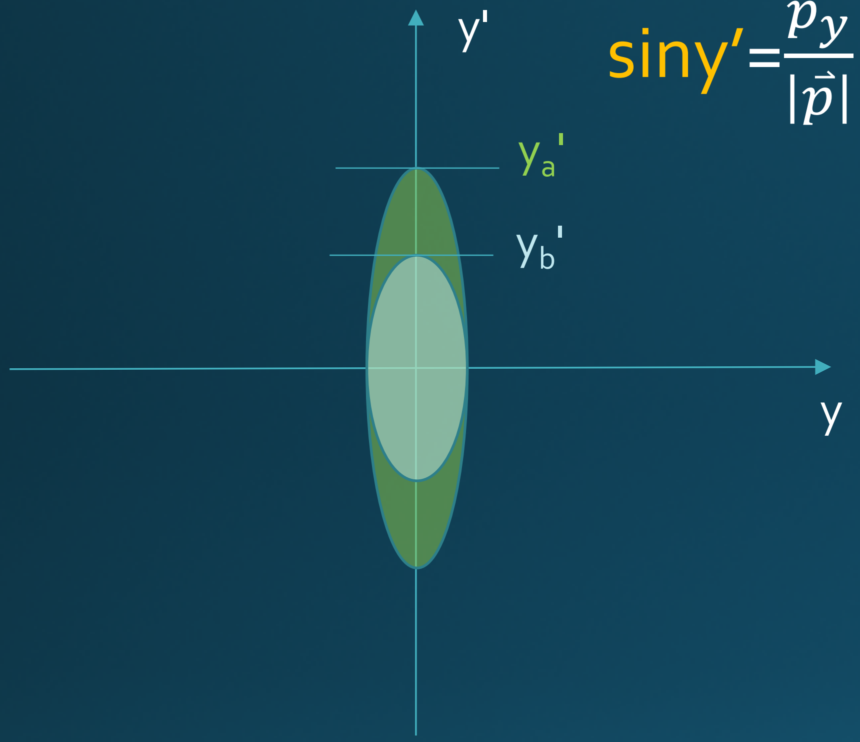


$\sin y' = \frac{p_y}{|\vec{p}|} \Rightarrow y' = \frac{p_y}{|\vec{p}|}$

After crossing the RF cavity the particle gains energy in the longitudinal direction, but **not in the transverse**

$y_b' < y_a'$

Phase space



$\sin y' = \frac{p_y}{|\vec{p}|}$ ➔ $y' = \frac{p_y}{|\vec{p}|} = \frac{p_y}{\gamma m_0 v} = \frac{p_y}{\gamma m_0 \beta c}$

If we choose as phase space coordinates (y, p_y), the phase space does not shrink because as $|\vec{p}|$ increases y' decreases and the product remains constant, i.e. p_y remains constant.

When we accelerate, as $|\vec{p}|$ increases y' decreases

As y' goes as $\frac{1}{\gamma\beta}$ to get an invariant emittance we have to multiply by $\gamma\beta$:

$$\varepsilon = \frac{A}{\pi} = a \cdot b = y \cdot y' = y \frac{p_y}{\gamma m_0 \beta c}$$

$$\varepsilon\gamma\beta = y \frac{p_y}{m_0 c} = \varepsilon_n$$

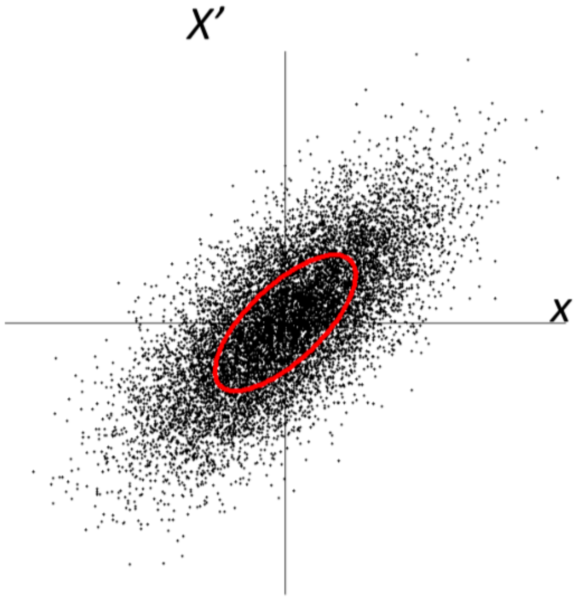
Spares

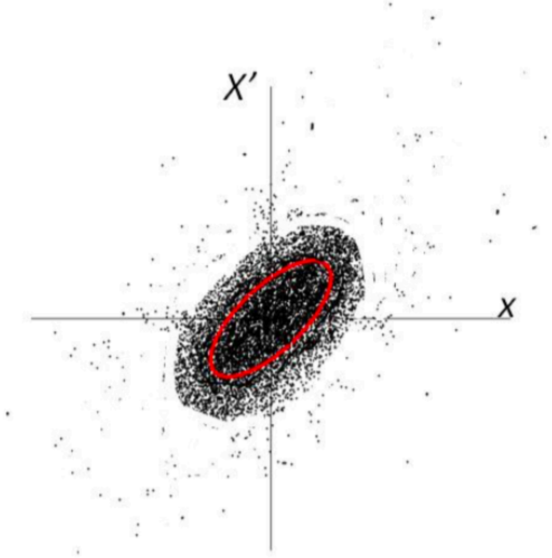
rms Emittance

The emittance is the area of the phase space occupied by all particles in a beam
Each particle has its own ‘invariant emittance’
rms emittance represents the beam characteristics, and is defined as:

$$\boxed{\varepsilon_x{}^{rms} = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle x\,x'\rangle^2}}$$

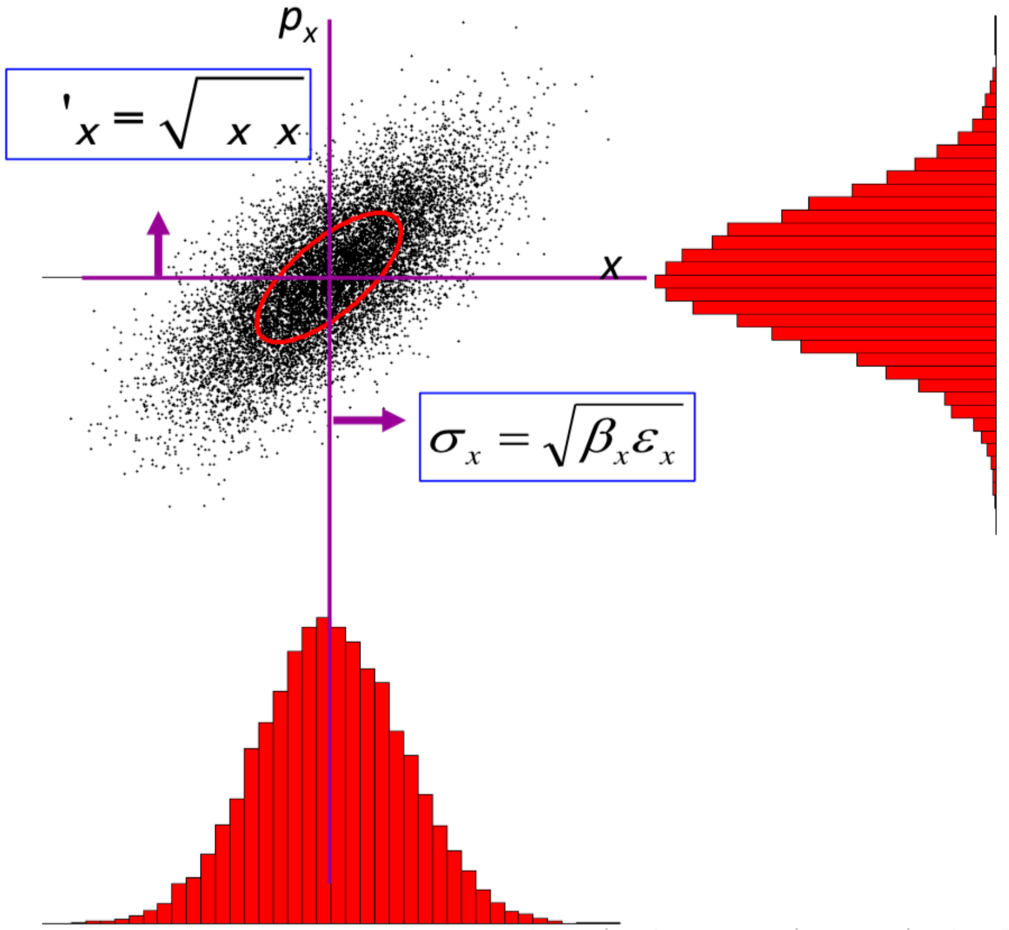
Rms values behave the same for all distributions in linear systems
Most usual beam distributions are gaussian





Emittance and beam dimensions

- The emittance is the area of the phase space occupied by the particles
  With the emittance and the Twiss parameters in a point of the accelerator, the beam dimensions are obtained : $\sigma_{x,y}$ e $\sigma'_{x,y}$



$$\varepsilon_x = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

Ellipse area= $\pi\varepsilon_x$

$$\langle x^2\rangle = \beta_x\varepsilon_x$$
$$\langle x'^2\rangle = \gamma_x\varepsilon_x$$
$$\langle xx'\rangle = -\alpha_x\varepsilon_x$$

$$\beta_x\gamma_x - \alpha_x^2 = 1$$