Introduction to transverse beam dynamics II

Restricted to the LINEAR BEAM OPTICS ➔ THE IDEAL WORLD

Content of the course

TBD 1

- Charge particle motion in a magnetic field
- Equations of motion ➔ derivation and assumptions
- Type of magnets

TUTO 1

- Rigidity formula
- Relativistic equations
- Create a storage ring with the Earth Magnetic field

TBD 2

- Particle trajectory
- Transfer Matrices
- Thin lens approximation
- Betatron oscillations
- Betatron tune
- Dispersion

TUTO 2

- Application of transfer matrices
- Thin lens
- FoDo cell

TBD 3

- Phase space ellipse
- Emittance
- Beam size
- Aperture
- Beta function evolution
- Periodic lattices

TUTO 3

- Beam size and aperture calculations

TBD 4

- Effect field errors
- Resonances
- Coupling
- Chromaticity

TUTO 4

- How the tune changes from a quadrupole defect
- Optimize beta beating
- Orbit bumps

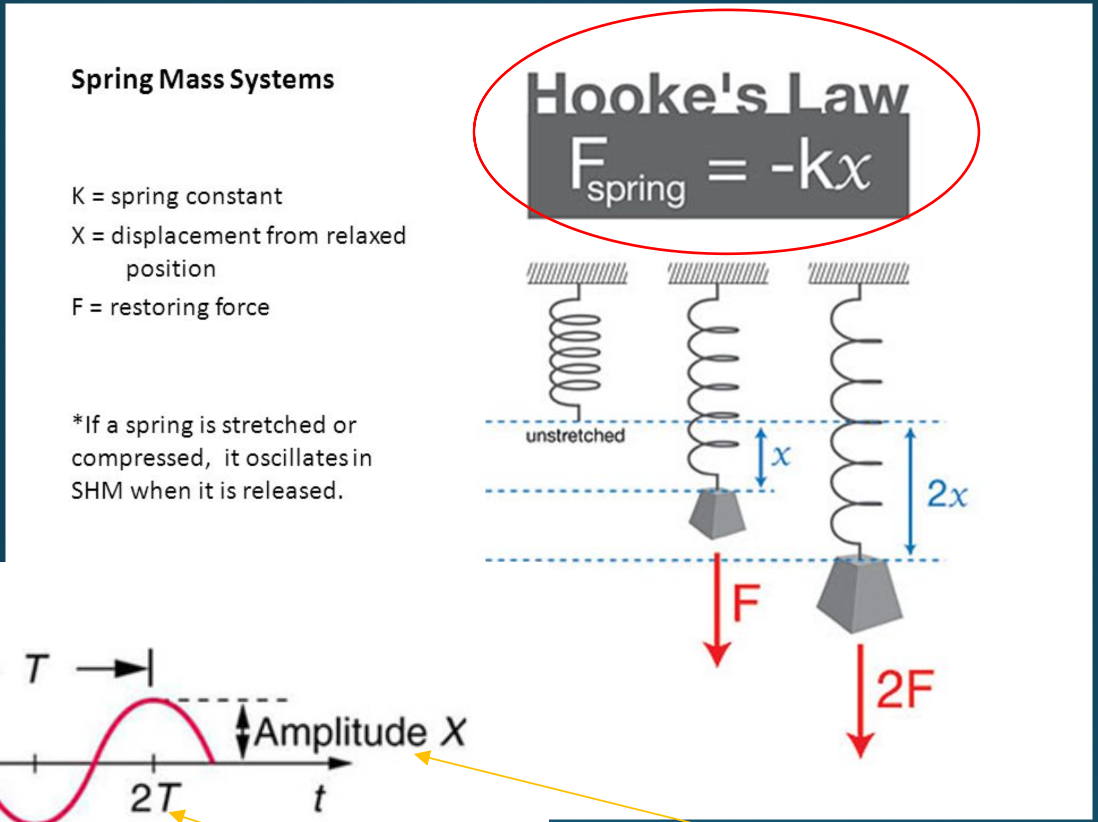
Equation of motion in the linear approximation

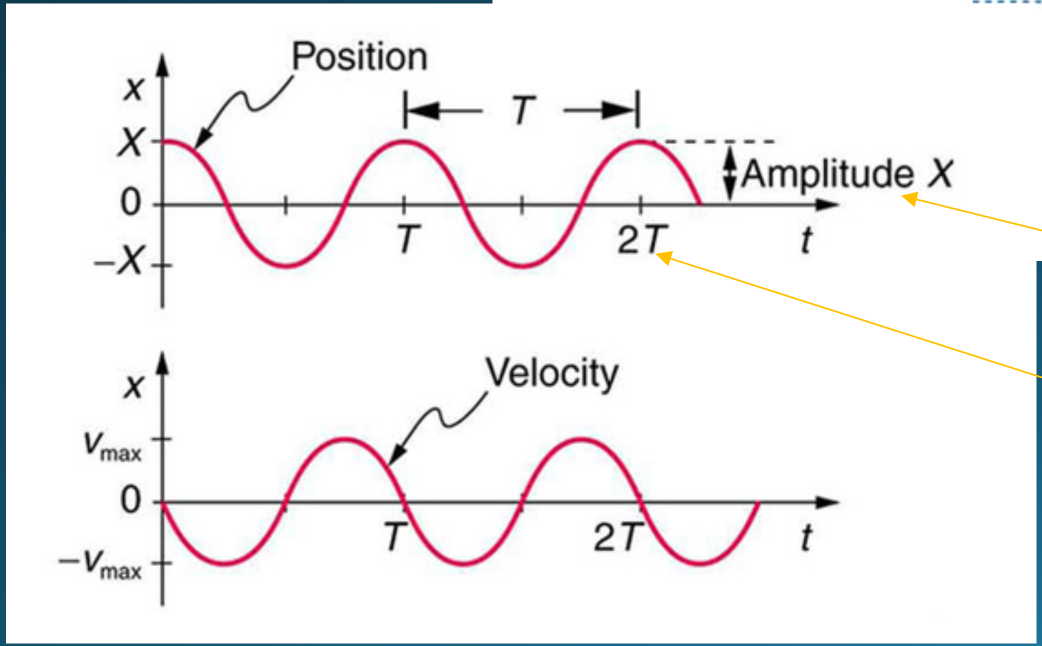
$$x'' + \left(\frac{1}{\rho^2} - k\right)x = 0$$

$$y'' + ky = 0$$

Simple Harmonic Motion

A simple harmonic oscillator is an oscillator that is neither driven nor damped. It consists of a mass *m*, which experiences a single force *F*, which pulls the mass in the direction of the point *x* = 0 and depends only on the position *x* of the mass and *k*.





$$x(t) = A\cos(\omega t + \varphi)$$

$$\omega = \sqrt{\frac{k}{m}} \qquad T = \frac{2\pi}{\omega} \qquad f = \frac{\omega}{2\pi}$$

Type of magnets

$$\frac{q}{p}B_y(s) = \frac{q}{p}B_{y0} + \frac{q}{p}\frac{dB_y}{dx}x + \frac{1}{2!}\frac{q}{p}\frac{d^2B_y}{dx^2}x^2 + \frac{1}{3!}\frac{q}{p}\frac{d^3B_y}{dx^3}x^3 + \cdots$$

LINEAR BEAM DYNAMICS!!

$$\frac{q}{p}B_y(s) = \frac{1}{\rho} + kx + \frac{1}{2!}mx^2 + \frac{1}{3!}ox^3 + \cdots$$

DIPOLE

QUADRUPOLE

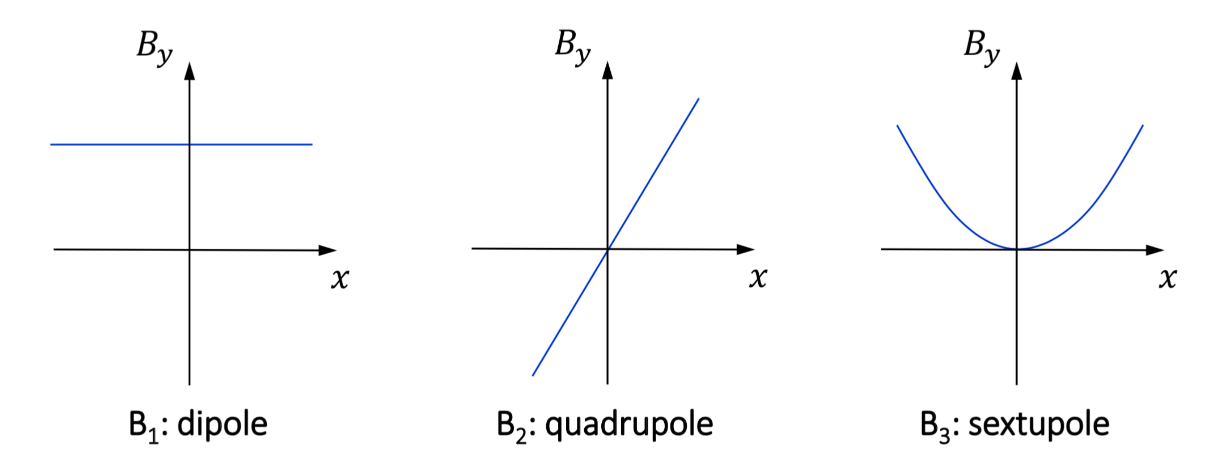
SEXTUPOLE

OCTUPOLE

$$k_0 = \frac{1}{\rho} = \frac{B}{B\rho}\left(\frac{1}{m}\right)$$

$$k_1 = \frac{q}{p}\frac{dB_y}{dx} = \frac{1}{B\rho}\frac{dB_y}{dx} = \frac{1}{B\rho}g\left(\frac{1}{m^2}\right)$$

$$k_2 = \frac{q}{p}\frac{d^2B_y}{dx^2} = \frac{1}{B\rho}\frac{d^2B_y}{dx^2}\left(\frac{1}{m^3}\right)$$



Particle trajectories

- We have seen in the previous course how to generate magnetic fields in the region of the beam
- Let's now find a solution for the equations of motion
- In dipoles and quadrupole magnets there is no coupling (in first order) between the horizontal and vertical plane (we have two different equations for x and y)
- Then, let's solve just one plane, e.g. the x-s
- To simplify even more, let's assume the B field ends abruptly at the beginning and end of the magnets "hard-edge model" (we ignore the edge field issue)
- Within the magnets we assume constant B field, i.e. $1/\rho$= cte and k=cte

- We can solve this equation, section by section, either within a magnet or within a field-free region = drift region

$$x'' + \left(\frac{1}{\rho^2} - k\right)x = 0$$

Solution within a focusing quadrupole

- A quadrupole is characterized by its strength, k, and its length, l
- There is no bending ➔ 1/ρ= 0

$$x'' + \left(\frac{1}{\rho^2} - k\right)x = 0$$

$$x'' - kx = 0$$ (k=cte)

Homogeneous and linear-second-order differential equation

- Convention is that defocusing magnets k > 0 and focusing magnets k<0
- For a focusing quadrupole the solution is

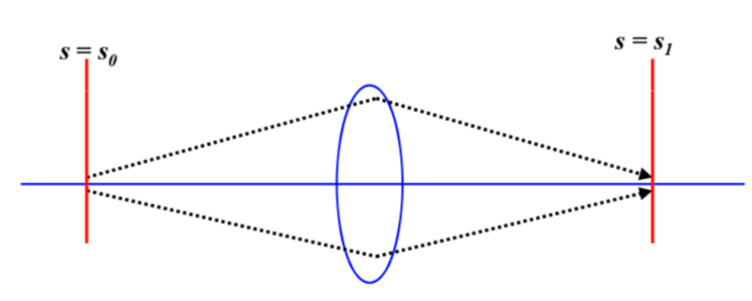
$$x(s) = A cos\sqrt{|k|}s + B sin\sqrt{|k|}s$$

$$x'(s) = -\sqrt{|k|}A sin\sqrt{|k|}s + \sqrt{|k|}B cos\sqrt{|k|}s$$

- The integration constants A and B are determined by the initial conditions
- Initial conditions → at the beginning of the magnet s=0, the particle trajectory has position x_0 and angle x'_0
- Inserting these initial conditions in the solutions we get: $A=x_0$, $B=x'_0/\sqrt{k}$
- We can also express the solution in a more elegant way using matrices:

FOCUSING

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cos\sqrt{|k|}s & {}^{1}\!/\!_{\sqrt{|k|}} \sin\sqrt{|k|}s \\ -\sqrt{|k|}\sin\sqrt{|k|}s & \cos\sqrt{|k|}s \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \quad k<0$$

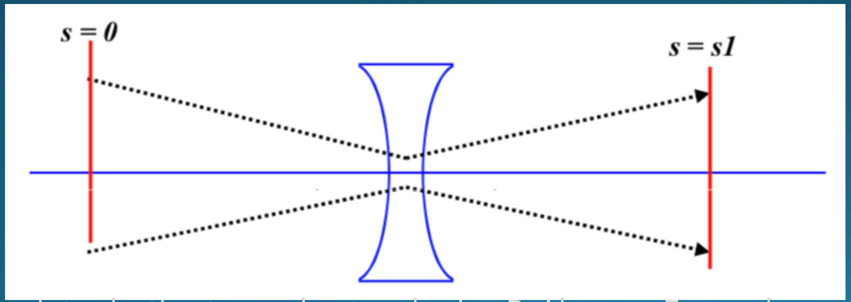


Solution within a defocusing quadrupole

$$x(s) = x_0 cosh\sqrt{k}s + {x'_0}/{\sqrt{k}} sinh\sqrt{k}s$$

$$x'(s) = x_0\sqrt{k} sinh\sqrt{k}s + x'_0 cosh\sqrt{k}s$$

DEFOCUSING $\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} cosh\sqrt{k}s & {1}/{\sqrt{k}} sinh\sqrt{k}s \\ \sqrt{k} sinh\sqrt{k}s & cosh\sqrt{k}s \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$ k>0



Solution for a zero-field drift region

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} cos\sqrt{|k|}s & {}^{1}\!/_{\sqrt{|k|}}\, sin\sqrt{|k|}s \\ -\sqrt{|k|}sin\sqrt{|k|}s & cos\sqrt{|k|}s \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

k=o, sin$\alpha \widehat{=} \alpha$

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

Solution for a dipole

- Dipole with constant bending radius 1/ρ=cte and no gradient, k=0

$$x'' + \left(\frac{1}{\rho^2} - \cancel{k}\right)x = 0$$

$$x'' + \frac{1}{\rho^2}x = 0$$

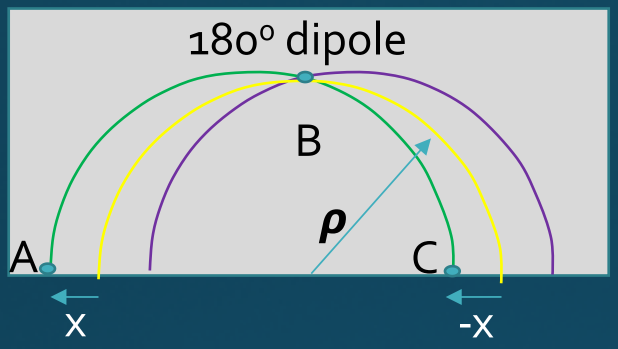
- The solution is like a focusing quadrupole

DIPOLE

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cos {}^{s}/_{\rho} & \rho \sin {}^{s}/_{\rho} \\ -{}^{s}/_{\rho} \sin {}^{s}/_{\rho} & \cos {}^{s}/_{\rho} \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

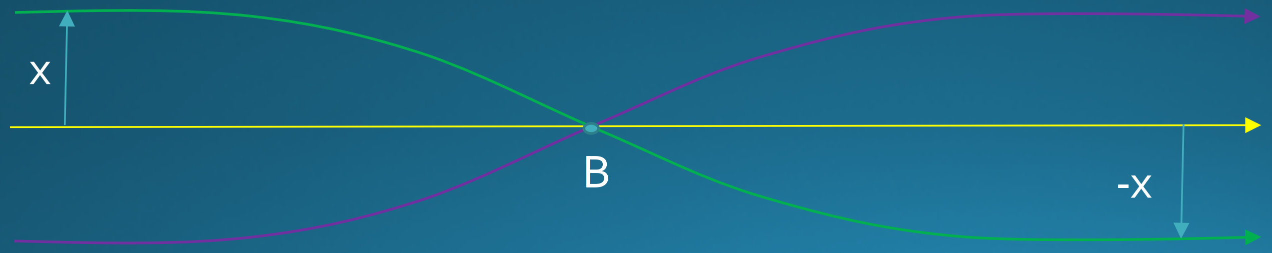
BEAM FOCUSING!!

- How can a dipole with k=0 have a focusing effect?
- Let's consider again the trajectories of two particles within a 180° dipole



Weak focusing!!

- In this magnet all trajectories are semicircles with same radius ρ
- If we consider as reference trajectory the one described by YELLOW particle
- And analyze the trajectory of the second GREEN particle displaced by +x:
  - GREEN particle approaches the ideal trajectory and crosses it at point B
  - Then runs inside the orbit and exits with a displacement of -x



Weak focusing in x
Drift in y
(horizontal dipole)

FOCUSING

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cos\sqrt{|k|}s & {}^{1}/_{\sqrt{|k|}} \sin\sqrt{|k|}s \\ -\sqrt{|k|}\sin\sqrt{|k|}s & \cos\sqrt{|k|}s \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

k<o

DEFOCUSING

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cosh\sqrt{k}s & {}^{1}/_{\sqrt{k}} \sinh\sqrt{k}s \\ \sqrt{k}\sinh\sqrt{k}s & \cosh\sqrt{k}s \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

k>o

DRIFT

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

k=o

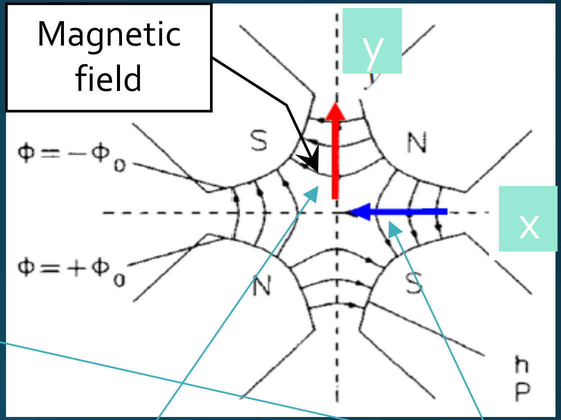
DIPOLE

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cos {}^{s}/_{\rho} & \rho\sin {}^{s}/_{\rho} \\ -{}^{s}/_{\rho}\sin {}^{s}/_{\rho} & \cos {}^{s}/_{\rho} \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

Four-dimensional trajectories

- In general a particle travelling along the accelerator moves in x and y, therefore we need to deal with four-dimensional vectors and matrices

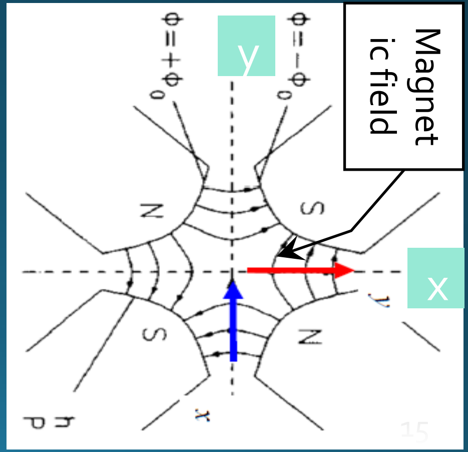
$$M_{QF} = \begin{pmatrix} \cos\sqrt{|k|}s & \frac{1}{\sqrt{|k|}}\sin\sqrt{|k|}s & 0 & 0 \\ -\sqrt{|k|}\sin\sqrt{|k|}s & \cos\sqrt{|k|}s & 0 & 0 \\ 0 & 0 & \cosh\sqrt{k}s & \frac{1}{\sqrt{k}}\sinh\sqrt{k}s \\ 0 & 0 & \sqrt{k}\sinh\sqrt{k}s & \cosh\sqrt{k}s \end{pmatrix}$$



Defocus in y

Focus in x

$$M_{QD} = \begin{pmatrix} \cosh\sqrt{k}s & \frac{1}{\sqrt{k}}\sinh\sqrt{k}s & 0 & 0 \\ \sqrt{k}\sinh\sqrt{k}s & \cosh\sqrt{k}s & 0 & 0 \\ 0 & 0 & \cos\sqrt{|k|}s & \frac{1}{\sqrt{|k|}}\sin\sqrt{|k|}s \\ 0 & 0 & -\sqrt{|k|}\sin\sqrt{|k|}s & \cos\sqrt{|k|}s \end{pmatrix}$$



$$M_{DRIFT} = \begin{pmatrix} 1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_{DIPOLE} = \begin{pmatrix} \cos {}^{s}/_{\rho} & \rho \sin {}^{s}/_{\rho} & 0 & 0 \\ -{}^{s}/_{\rho} \sin {}^{s}/_{\rho} & \cos {}^{s}/_{\rho} & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thin lens approximation

- In many practical situations we have that the focal length of the lens is much bigger than the length of the magnet

$$f = \frac{1}{kl_q} \gg l_q$$

- In this case we can make the following approximation

$$l_q \to 0 \; while \; kl_q = cte$$

- The quadrupole matrix elements can then be reduced to

FOCUSING
k<0

$$\begin{pmatrix} \cos\sqrt{|k|}s & {}^{1}/_{\sqrt{|k|}} \sin\sqrt{|k|}s \\ -\sqrt{|k|}\sin\sqrt{|k|}s & \cos\sqrt{|k|}s \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

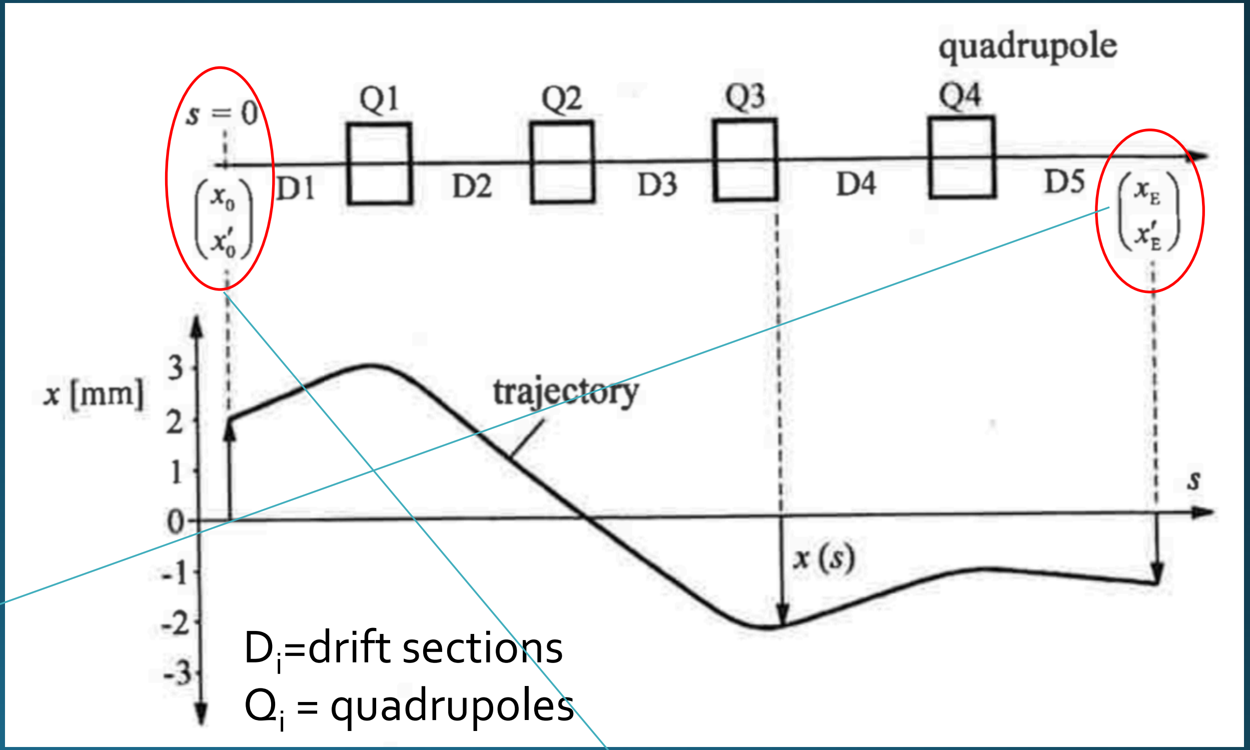
DEFOCUSING
k>0

$$\begin{pmatrix} \cosh\sqrt{k}s & {}^{1}/_{\sqrt{k}} \sinh\sqrt{k}s \\ \sqrt{k}\sinh\sqrt{k}s & \cosh\sqrt{k}s \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

Particle trajectory calculation through a system of many beam steering magnets

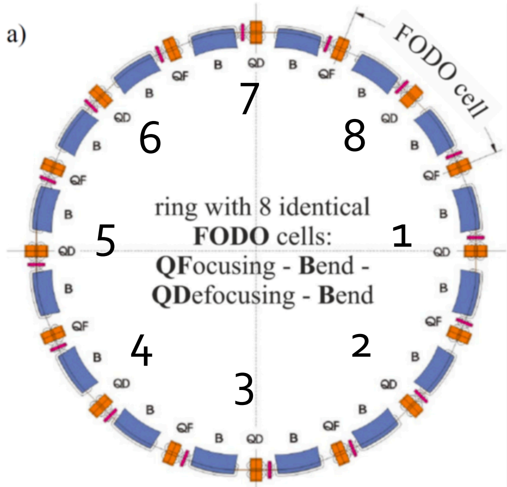
- So far we have only considered particle motion within one magnet
- But a storage ring or transfer line is made of many magnets

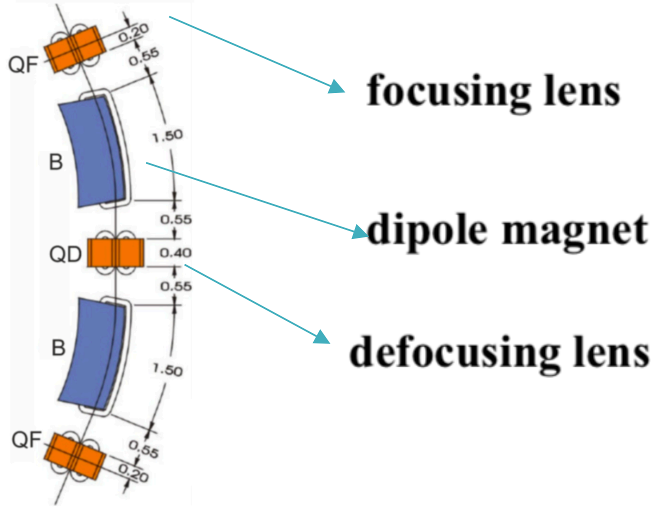
- The first approach would be:
    - Take the first element, calculate the position and angle after the beam crosses the element
    - Take the resulting position and angle and use it as initial position and angle through the next magnet
    - And so on so forth …

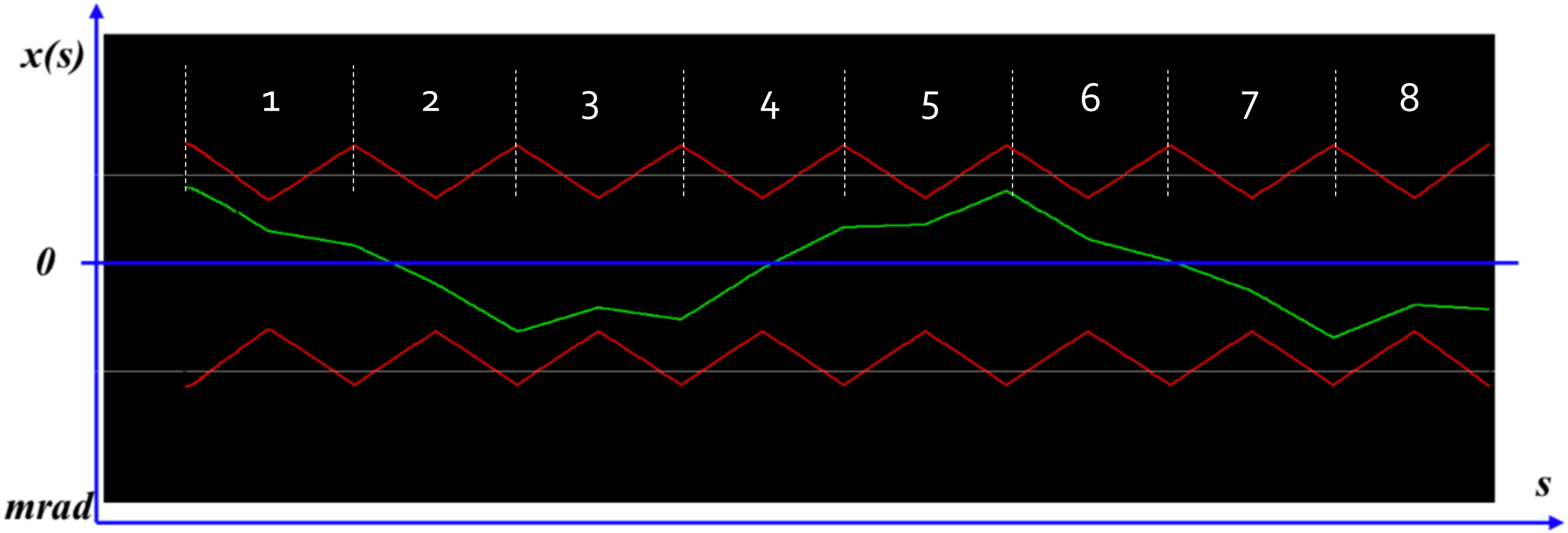


$$\mathbf{X}_{\mathrm{E}} = \mathbf{M}_{\mathrm{D5}} \cdot \mathbf{M}_{\mathrm{Q4}} \cdot \mathbf{M}_{\mathrm{D4}} \cdot \mathbf{M}_{\mathrm{Q3}} \cdot \mathbf{M}_{\mathrm{D3}} \cdot \mathbf{M}_{\mathrm{Q2}} \cdot \mathbf{M}_{\mathrm{D2}} \cdot \mathbf{M}_{\mathrm{Q1}} \cdot \mathbf{M}_{\mathrm{D1}} \cdot \mathbf{X}_0$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s2,s1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$



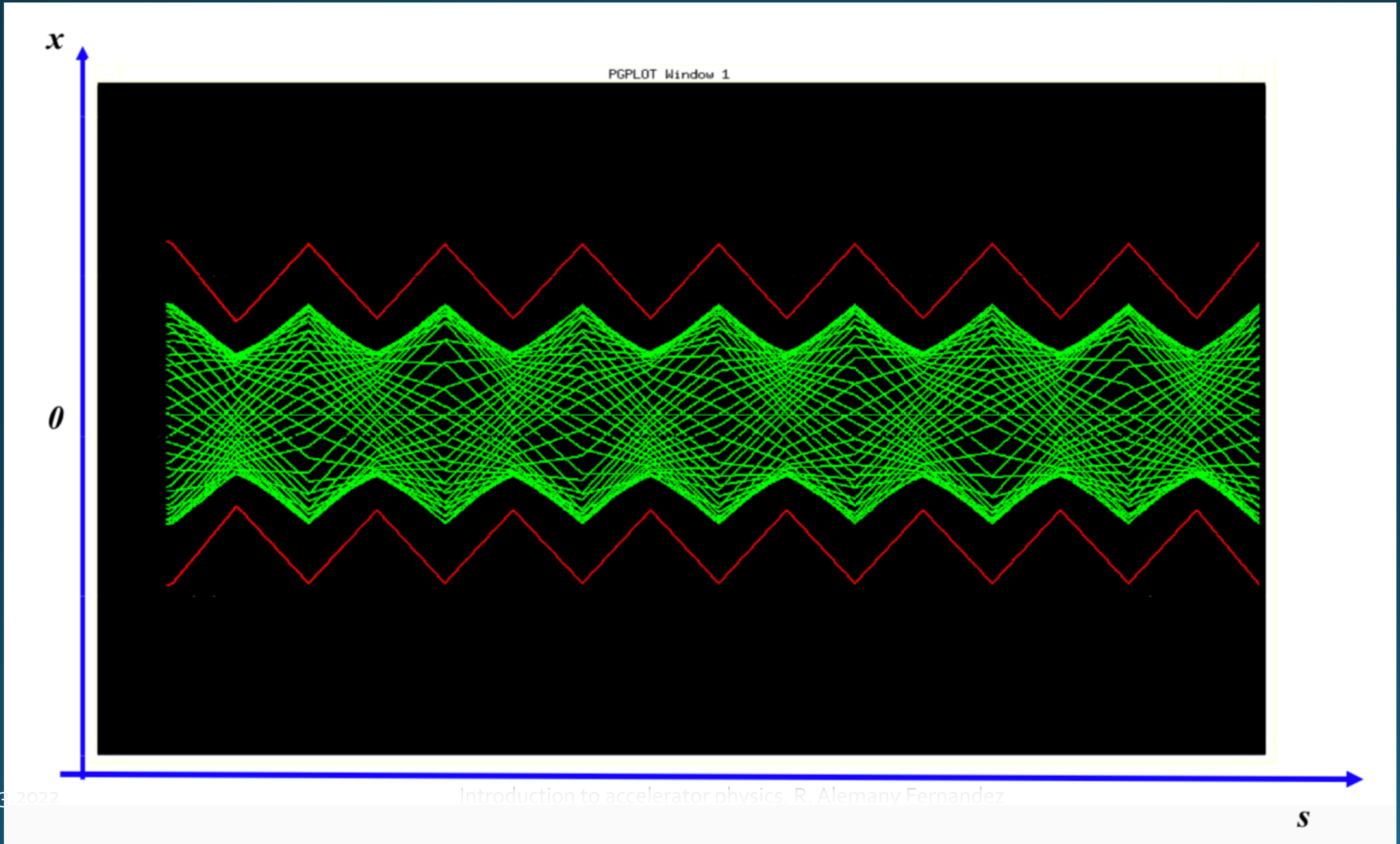




typical values in a strong foc. machine:

$x \approx mm,\ x' \leq mrad$

What will happen if the particle performs a second turn … and a third turn … and 10^{10} turns?



Hill's equation

- Up to now we have simplified our calculations assuming that k = cte within the quadrupole magnet
- But when we have to use a set of magnets in an storage ring, each quadrupole can have a different strength → the restoring force is not constant k=k(s)
- More over, the particle crosses the magnets over and over again, therefore, k(s)=k(s+L), i.e. k(s) is periodic (L: accelerator circumference)
- Therefore, the original equation of motion

$$x'' - kx = 0$$

(k=cte) (slide 9)

- Has to be generalized to k=k(s)

$$x(s)'' - k(s)x(s) = 0$$

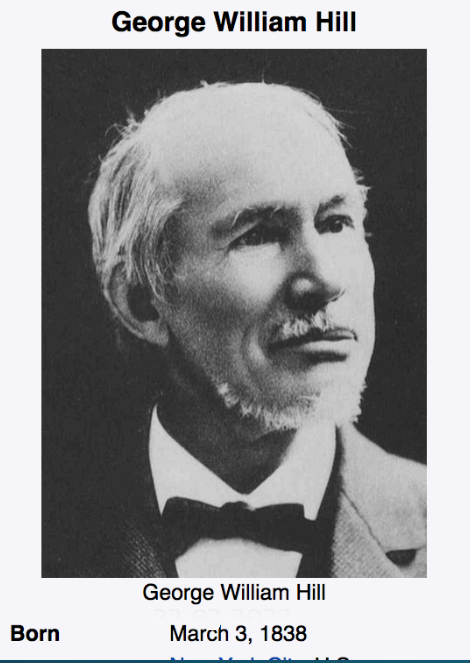
Hill's equation of motion

$$x(s)'' - k(s)x(s) = 0$$

Hill's equation of motion

- The solution, x(s), describes a transverse oscillation about the ideal orbit known as **betatron oscillation**, whose amplitude and phase depend on the position s along the orbit

- Hill's equation owns his name from the Astronomer G. W. Hill



In 1878, Hill provided the first complete mathematical solution to the problem of the apsidal precession of the Moon's orbit around the Earth, a difficult problem in lunar theory first raised in Isaac Newton's *Principia Mathematica* of 1687.[2] This same work also introduced what is now known in physics and mathematics as the "Hill differential equation", which describes the behavior of a parametric oscillator and which made an important contribution to the mathematical Floquet theory. (from Wikipedia)

- The solution to this equation has the usual cosine form

$$x(s) = Au(s)\cos(\psi(s) + \phi)$$

- The constant amplitude factor A and the phase ϕ are integration constants fixed by the initial conditions
- Inserting the solution x(s) and its derivative x'(s) into the Hill's equation we obtain

$$A[u'' - u\psi'^2 - k(s)u]cos(\psi + \phi) \; - A[2u'\psi' + \mathrm{u}\psi''] \sin(\psi + \phi) = 0$$

- Since $\psi(s)$ has a different value at every point in the orbit, and A≠0, the equation above can only be satisfied if

$$u'' - u\psi'^2 - k(s)u = 0$$

$$2u'\psi' + \mathrm{u}\psi'' = 0 \quad \Rightarrow \quad 2\frac{u'}{u} + \frac{\psi''}{\psi'} = 0 \quad \Rightarrow \quad \psi(s) = \int_0^s \frac{d\sigma}{u^2(\sigma)}$$

$$u'' - \frac{1}{u^3} - k(s)u = 0$$

Non-linear differential equation with no general analytic solution ➔ numerical methods
How do we do with complicated magnet structures with many individual magnets?

Forget about this, we'll use the same approach as for a single particle trajectory, i.e. we'll develop a matrix method to calculate the FULL BEAM OPTICS in the same simple way

- First we introduce the beta function (also known as amplitude function)

$$\beta(s) \equiv u^2(s)$$

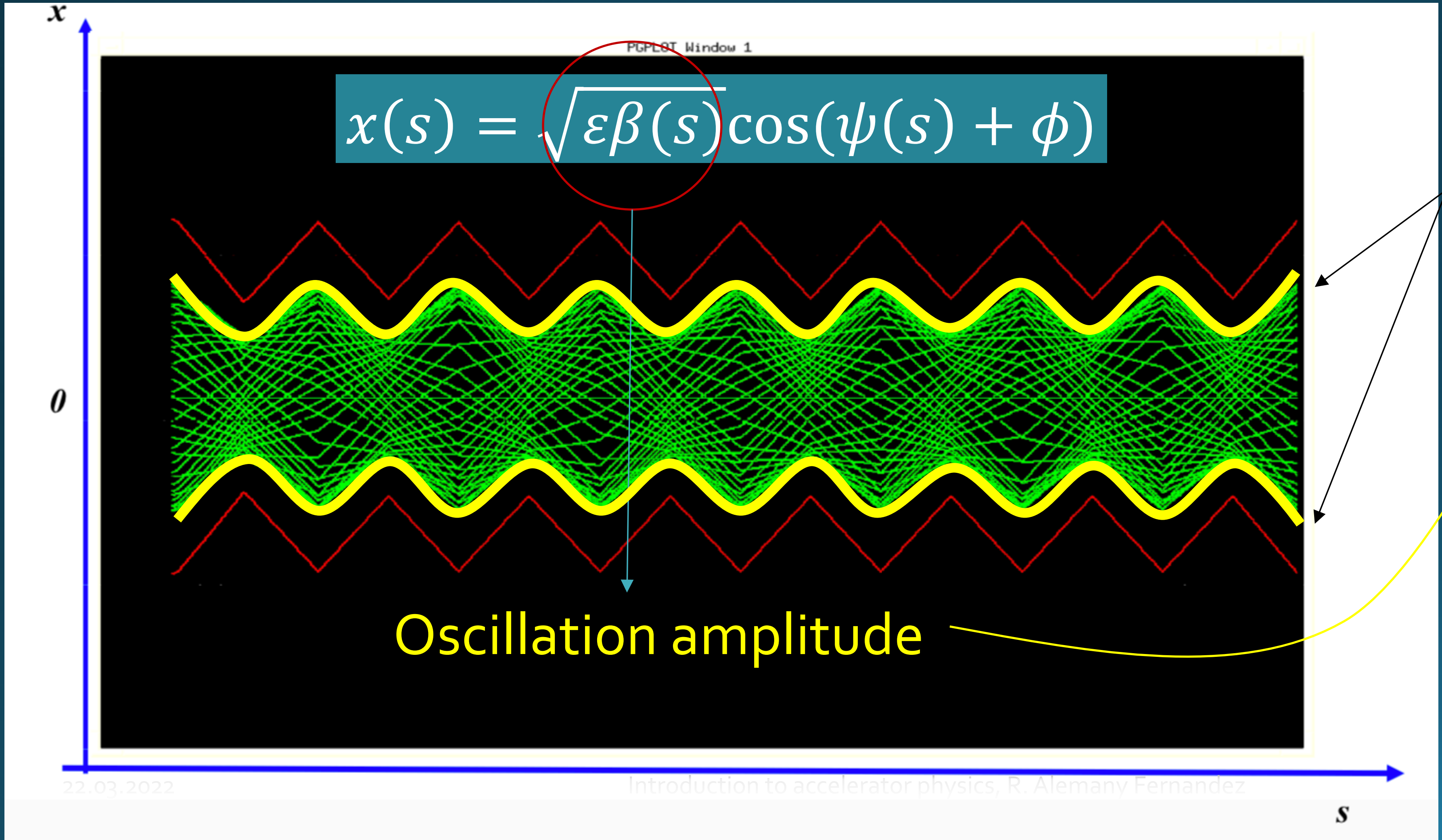
- Second we replace the amplitude factor A by $\sqrt{\varepsilon}$ a constant term called emittance

$$x(s) = \sqrt{\varepsilon\beta(s)}\cos(\psi(s) + \phi)$$

➔

$$\psi(s) = \int_0^s \frac{d\sigma}{\beta\,(\sigma)}$$

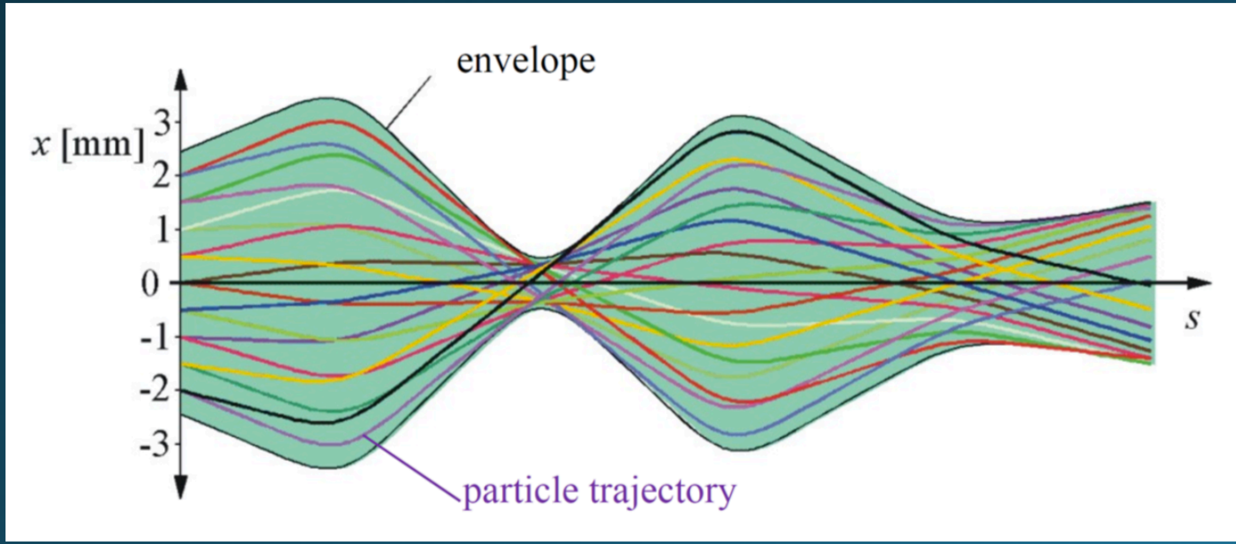
We saw before that turn after turn the particles perform betatron oscillations around the orbit amplitude-position-dependent under the action of the quadrupoles. We see that this action has a net FOCUSING effect



Beam envelope

Since all particle trajectories lie inside this envelope, it defines the <u>BEAM SIZE</u>

- If we know how $\beta(s)$ evolves step by step through the magnet structure, in the same way as the particle trajectory and we know the value of the EMITTANCE ➔ we can know the transverse beam size at any point in the accelerator



(We'll come back to this)

Betatron tune

- Let's come back to the relation $$\psi(s) = \int_0^s \frac{d\sigma}{\beta\,(\sigma)}$$ PHASE ADVANCE

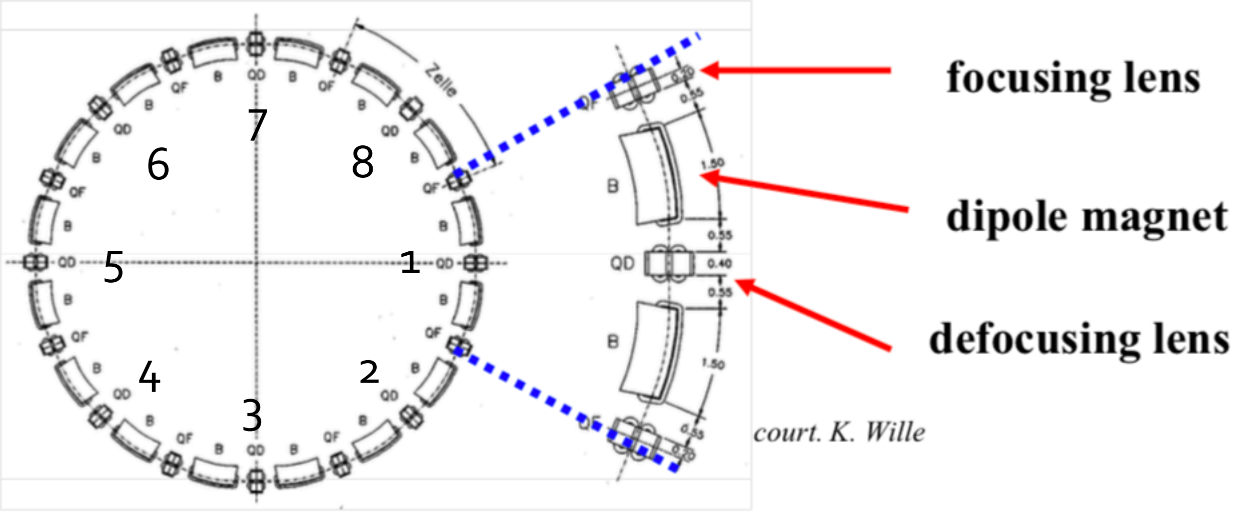
- In a circular accelerator the integral "$0 \longrightarrow s$" is equivalent to a complete revolution

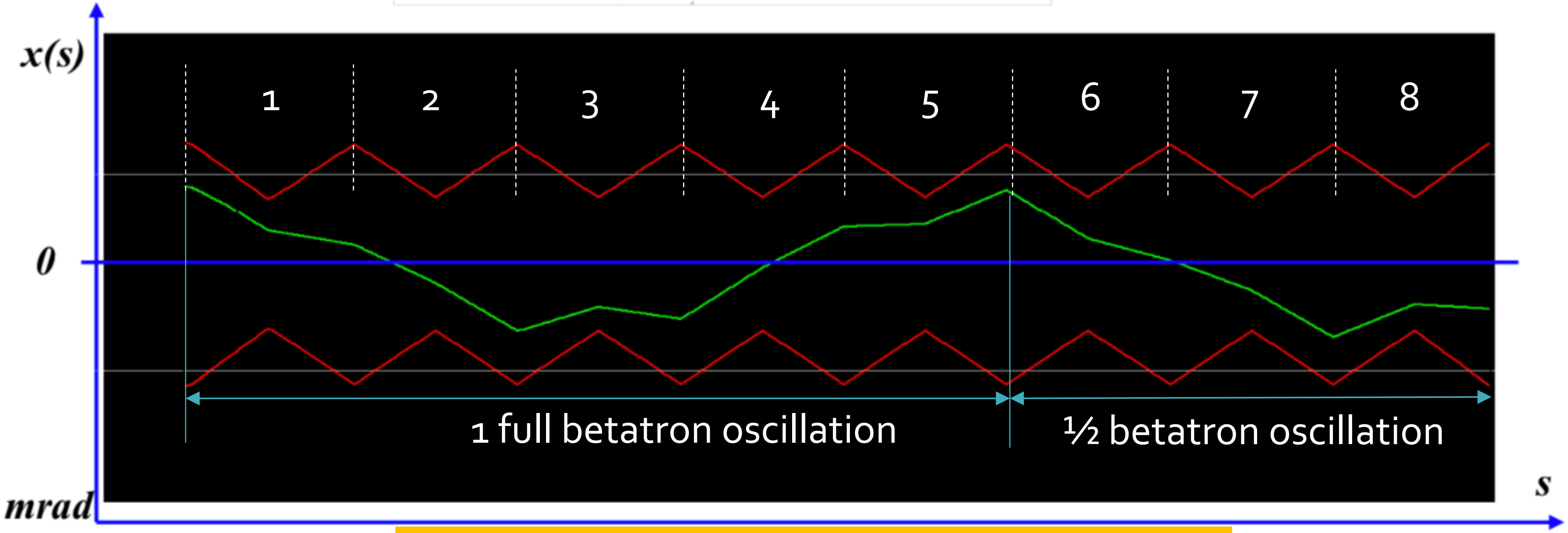
$$Q_{x,y} = \frac{1}{2\pi} \oint \frac{ds}{\beta_{x,y}\,(s)}$$

BETATRON TUNE

Number of betatron oscillations the beam (each particle) performs after one turn

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s2,s1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$





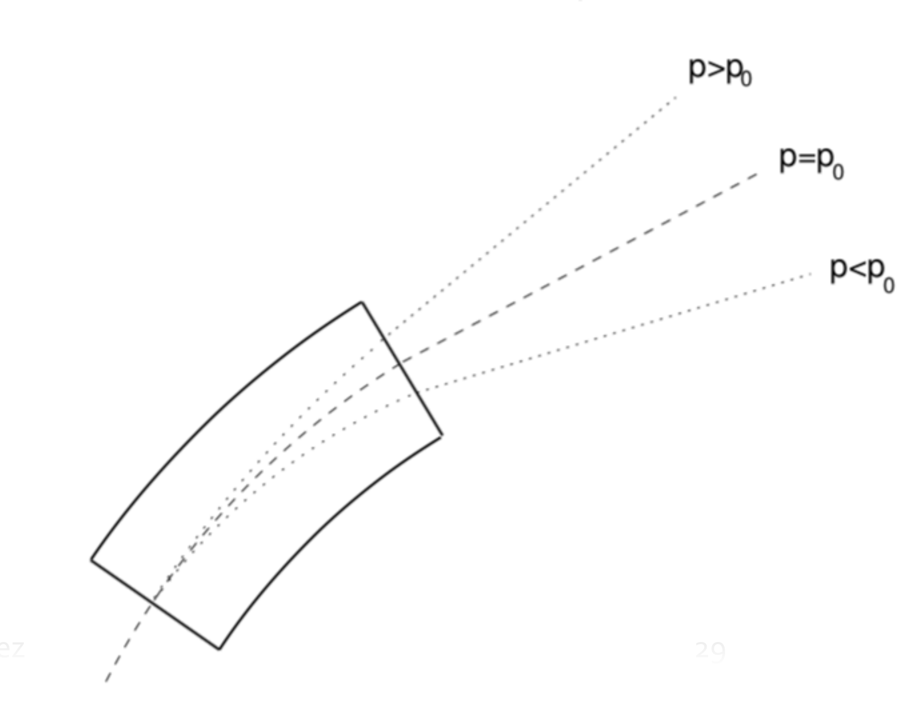
typical values in a strong foc. machine:

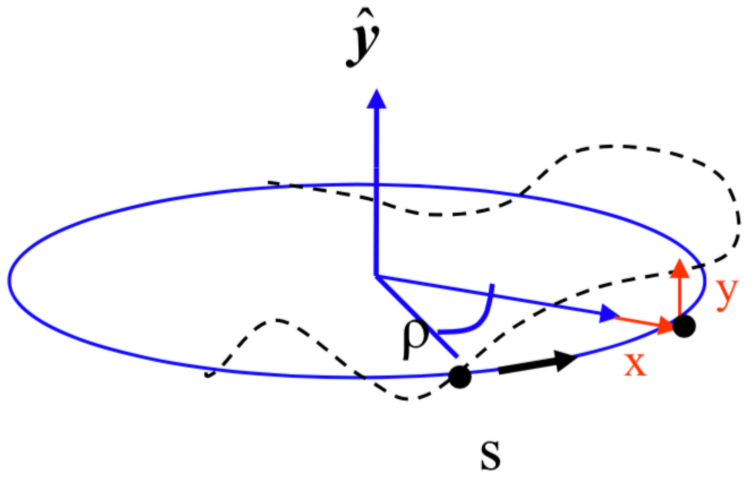
$x \approx mm, \; x' \leq mrad$

1.5 betatron oscillations in x ➔ Tune = Q_x = 1.5

Dispersion D(s)

- So far we have studied monochromatic beams of particles, but this is slightly unrealistic
- We always have some small momentum spread among all particles: $\Delta p = p - p_0 \neq 0$
- Consider three particles with *p* respectively: less than, greater than, and equal to p_o , traveling through a dipole
- Remember Bρ = p /q
- The system introduces a correlation of momentum with transverse position
- This correlation is known as **dispersion** (an intrinsic property of the dipole magnets)





Force acting on the particle

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

remember: ***$x \approx mm$, $\rho \approx m$ …*** **→** ***develop for small x***

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}\left(1-\frac{x}{\rho}\right) = eB_y v$$

consider only linear fields, and change independent variable: $t \rightarrow s$

$$B_y = B_0 + x\frac{\partial B_y}{\partial x}$$

$$x'' - \frac{1}{\rho}\left(1-\frac{x}{\rho}\right) = \frac{e\;B_0}{mv} + \frac{e\;x\;g}{mv}$$

$p=p_0+\Delta p$

… but now take a small momentum error into account !!!

Dispersion:

develop for small momentum error $\quad \Delta p \ll p_0 \Rightarrow \dfrac{1}{p_0 + \Delta p} \approx \dfrac{1}{p_0} - \dfrac{\Delta p}{p_0^2}$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \underbrace{\frac{e\,B_0}{p_0}}_{-\frac{1}{\rho}} - \frac{\Delta p}{p_0^2}\,eB_0 + \underbrace{\frac{xeg}{p_0}}_{k\,*\,x} - \underbrace{xeg\,\frac{\Delta p}{p_0^2}}_{\approx\,0}$$

$$x'' + \frac{x}{\rho^2} \approx \frac{\Delta p}{p_0} * \underbrace{\frac{(-eB_0)}{p_0}}_{\frac{1}{\rho}} + k * x = \frac{\Delta p}{p_0} * \frac{1}{\rho} + k * x$$

$$x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0}\frac{1}{\rho} \quad \longrightarrow \quad \boxed{x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0}\frac{1}{\rho}}$$

***Momentum spread** of the beam adds a term on the r.h.s. of the equation of motion.*

*→ **inhomogeneous differential equation.***

Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \frac{1}{\rho} \cdot \frac{\Delta p}{p} \end{cases}$$

Normalise with respect to $\Delta p/p$:

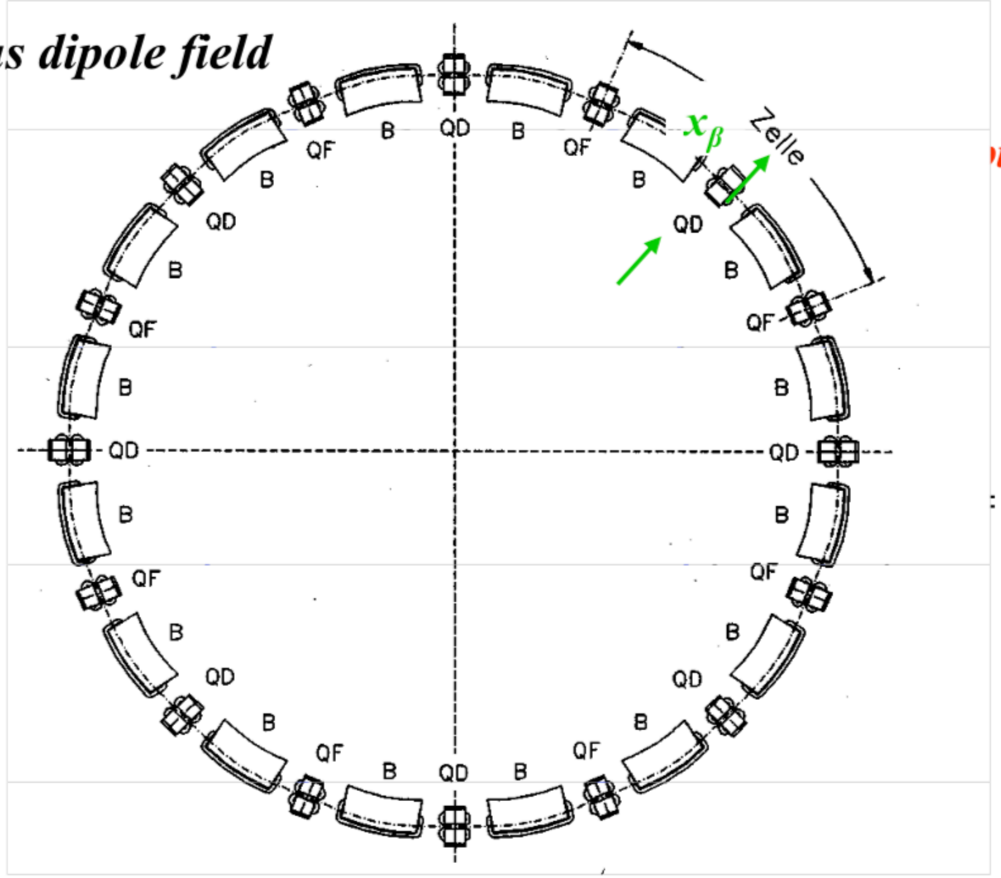
$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

* is that special orbit, an ideal particle would have for $\Delta p/p = 1$
* the orbit of any particle is the sum of the well known x_β and the dispersion
* as D(s) is just another orbit it will be subject to the focusing properties of the lattice

Dispersion

Example: homogeneous dipole field



…it for Δp/p > 0

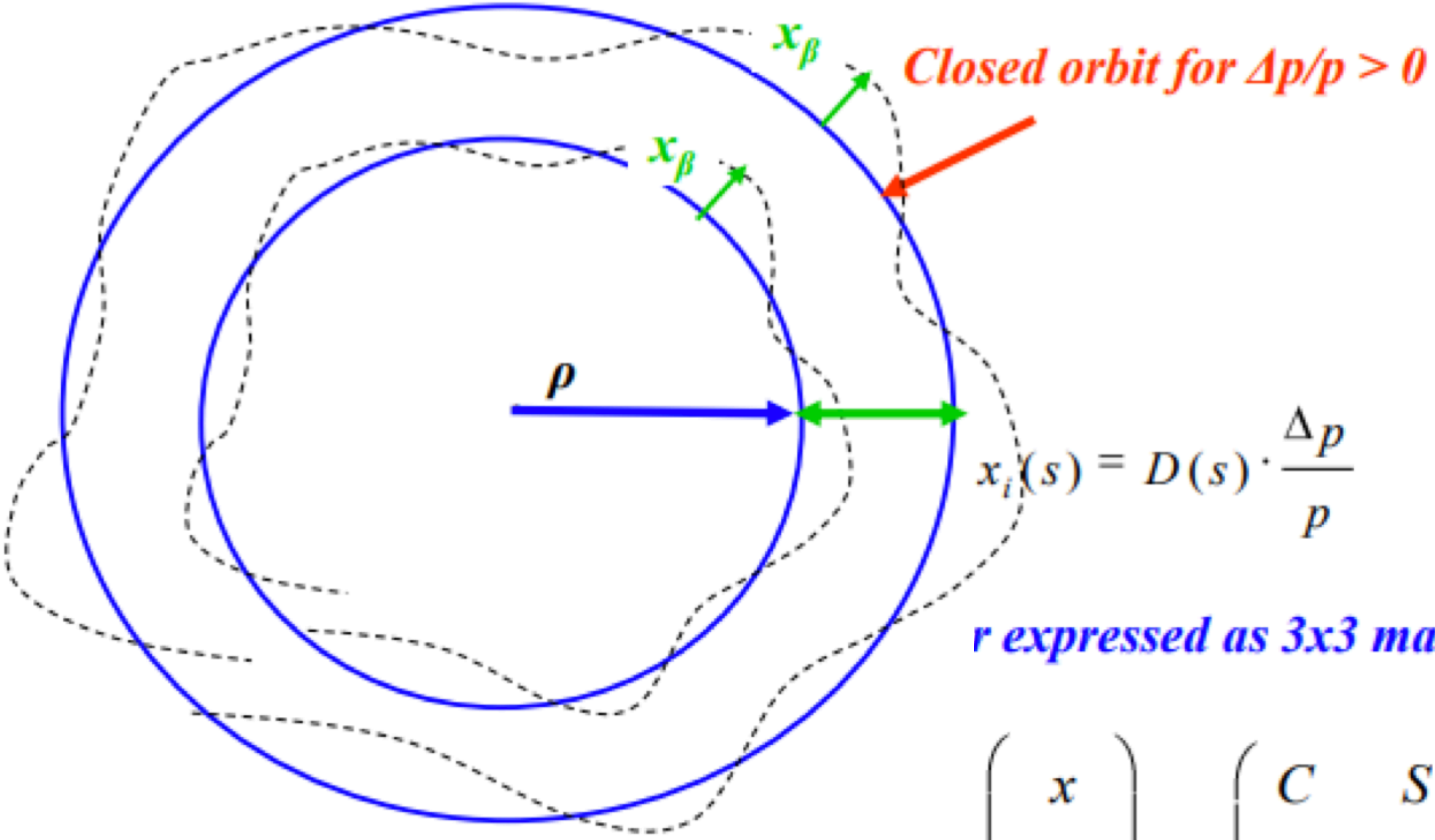
$$\ldots D(s) \cdot \frac{\Delta p}{p}$$

Matrix formalism:

$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x_0' + D(s) \cdot \frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}$$



$$x_i(s) = D(s) \cdot \frac{\Delta p}{p}$$

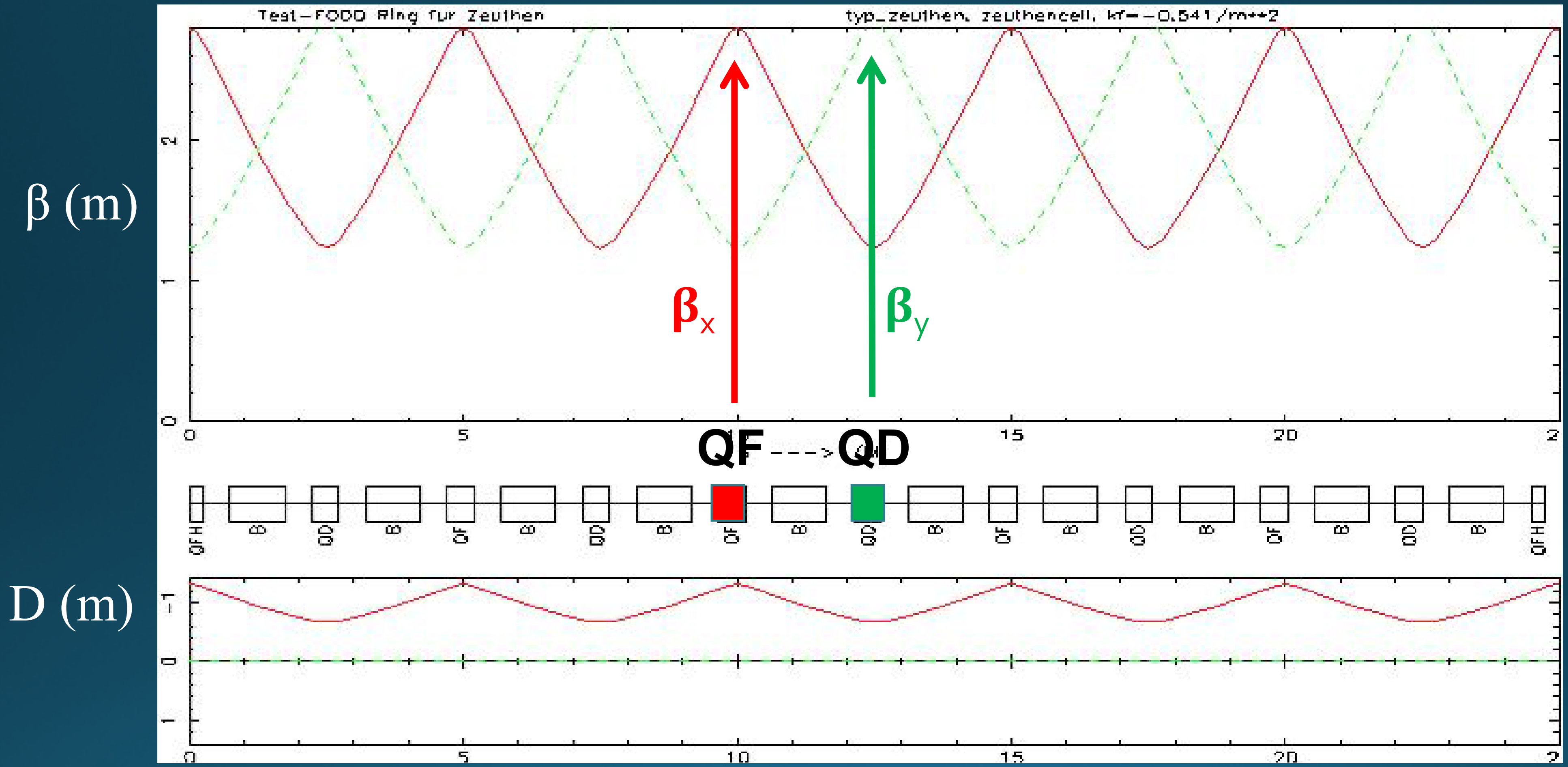
r expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$

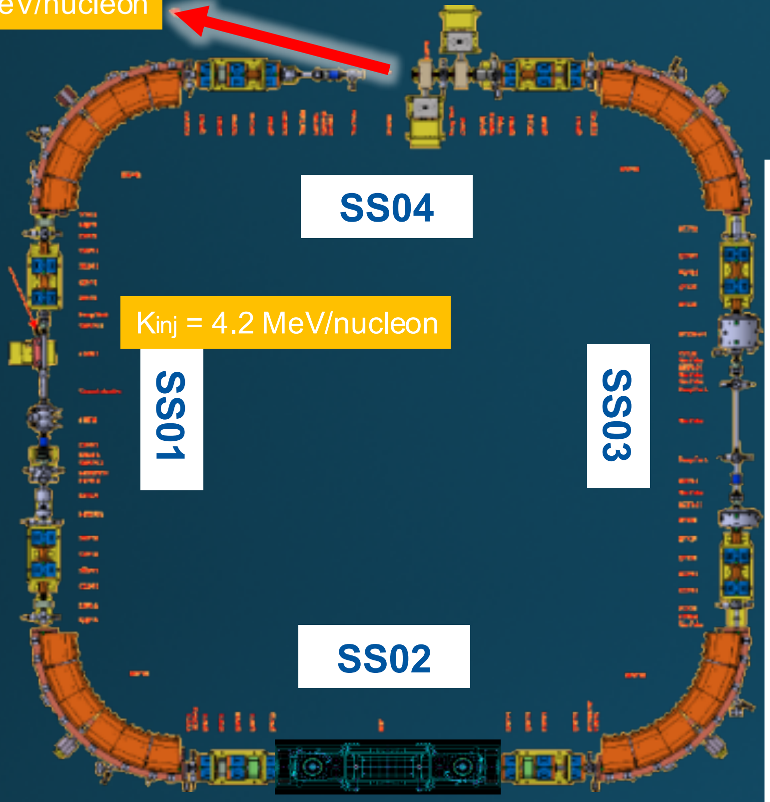
Examples of Twiss and dispersion functions

- LEIR
- SPS
- LHC

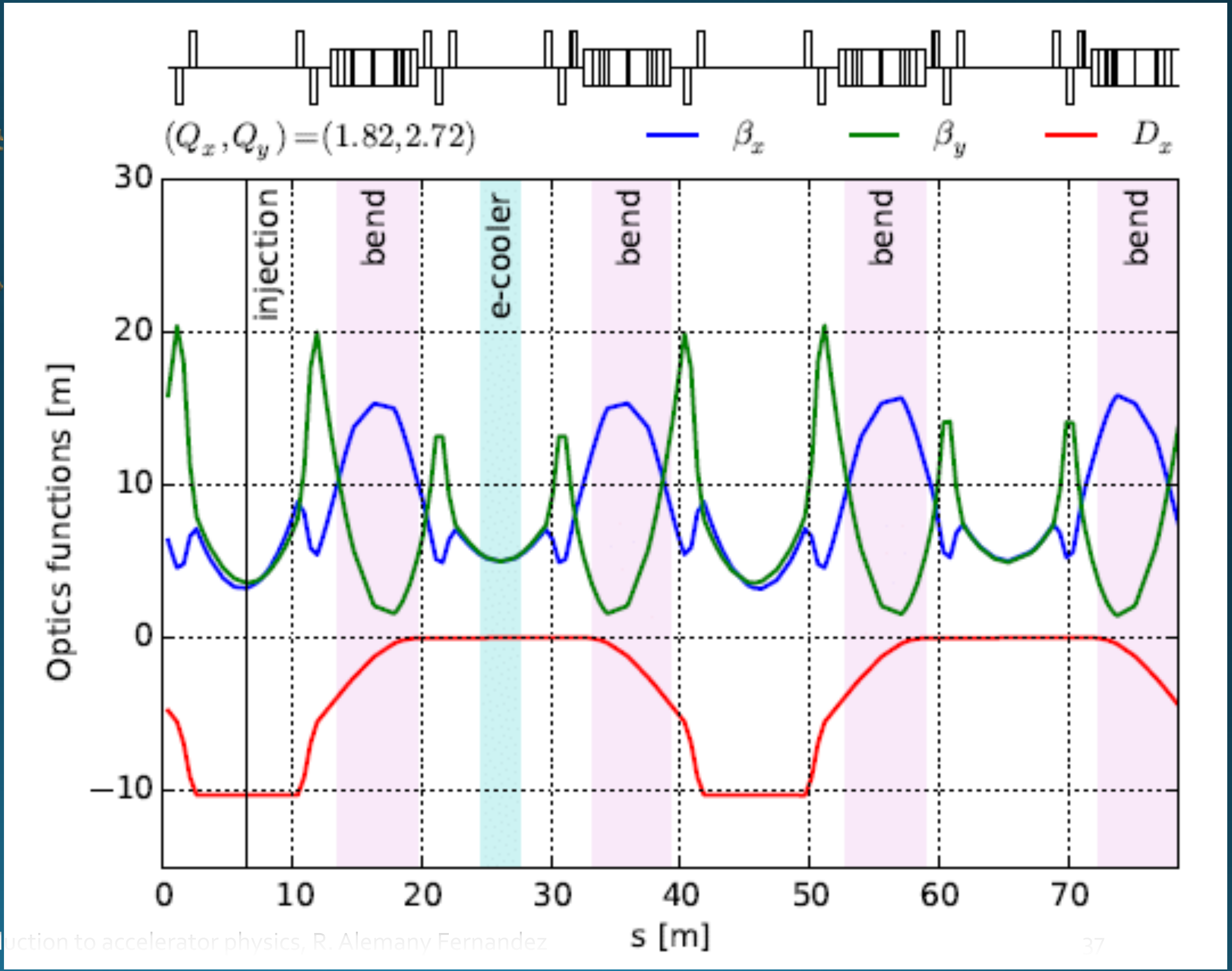
HERA beta function in the arcs



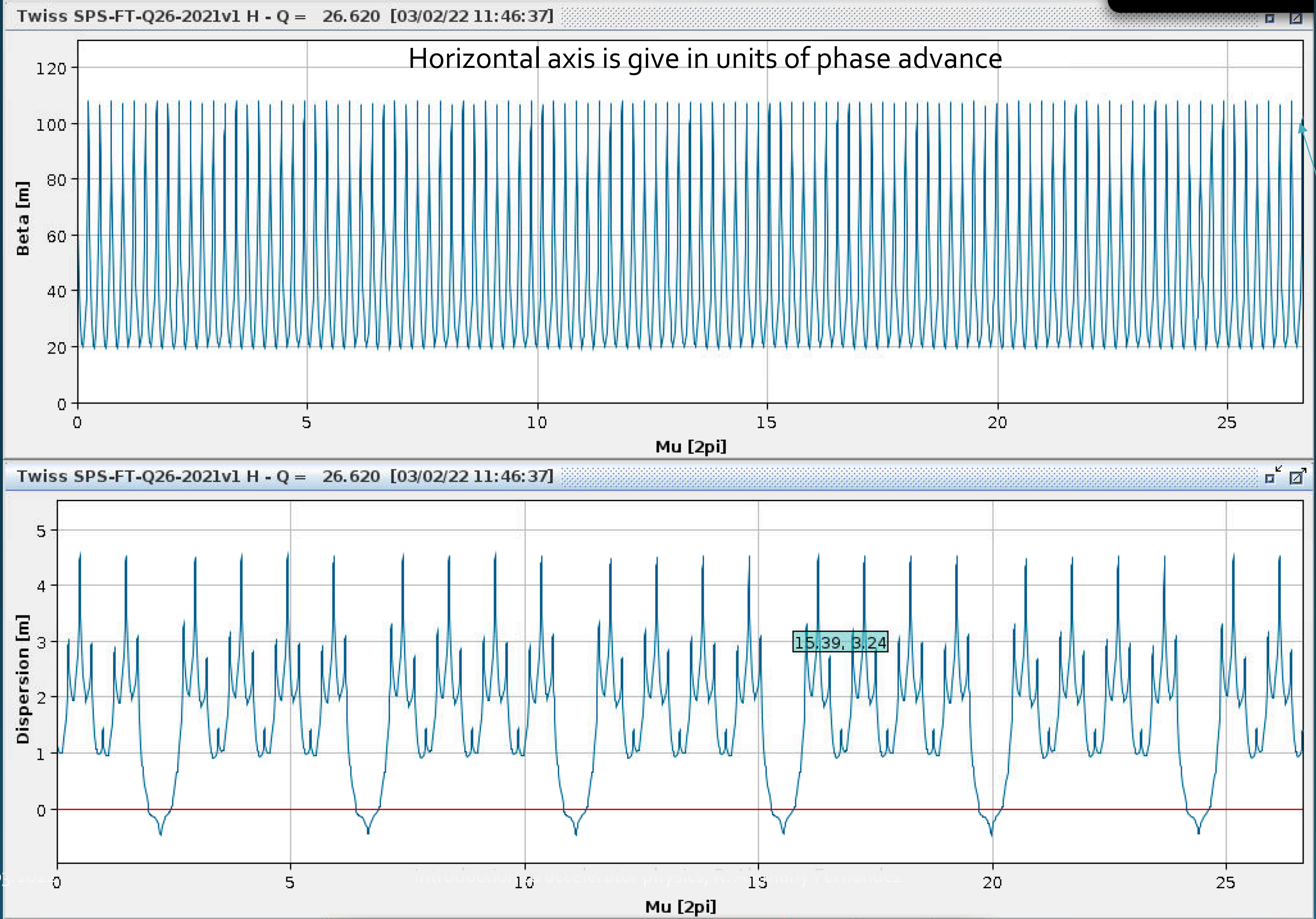
LEIR



Kinj: kinetic energy at injection
Kext: kinetic energy at extraction

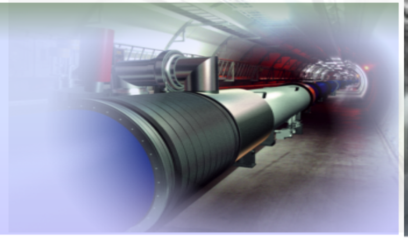


SPS

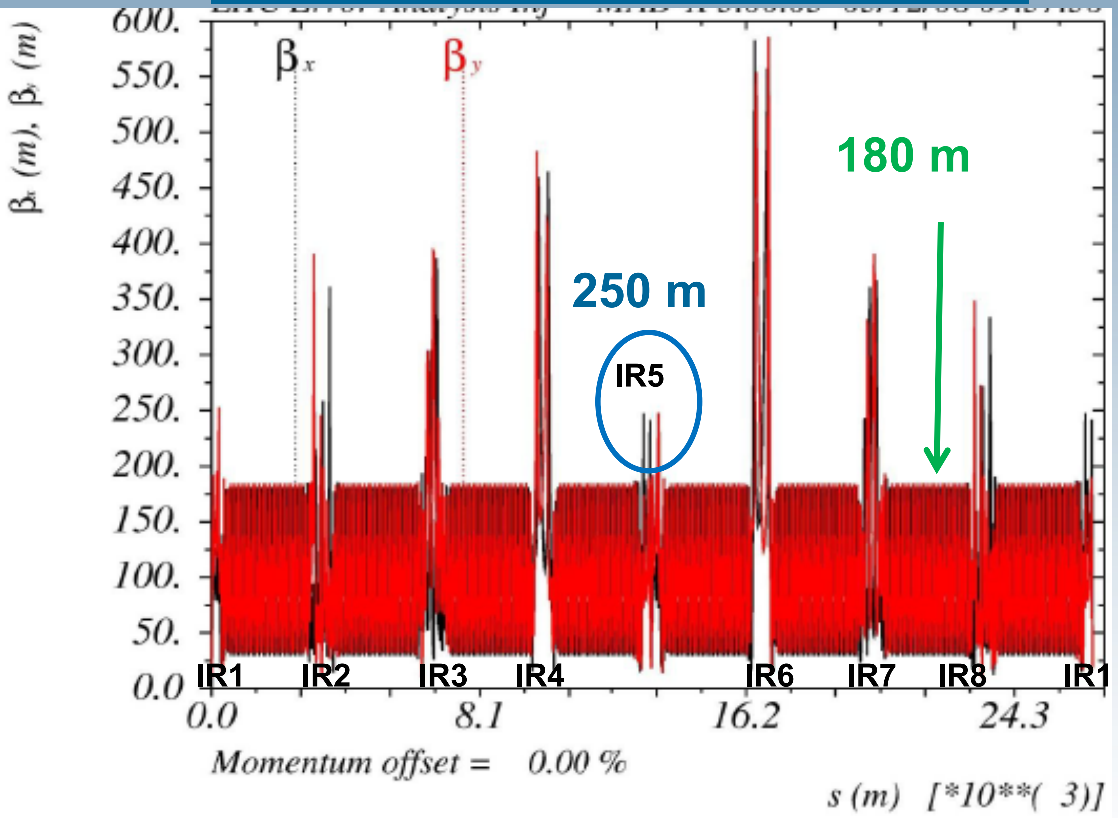


26.62 x 2π

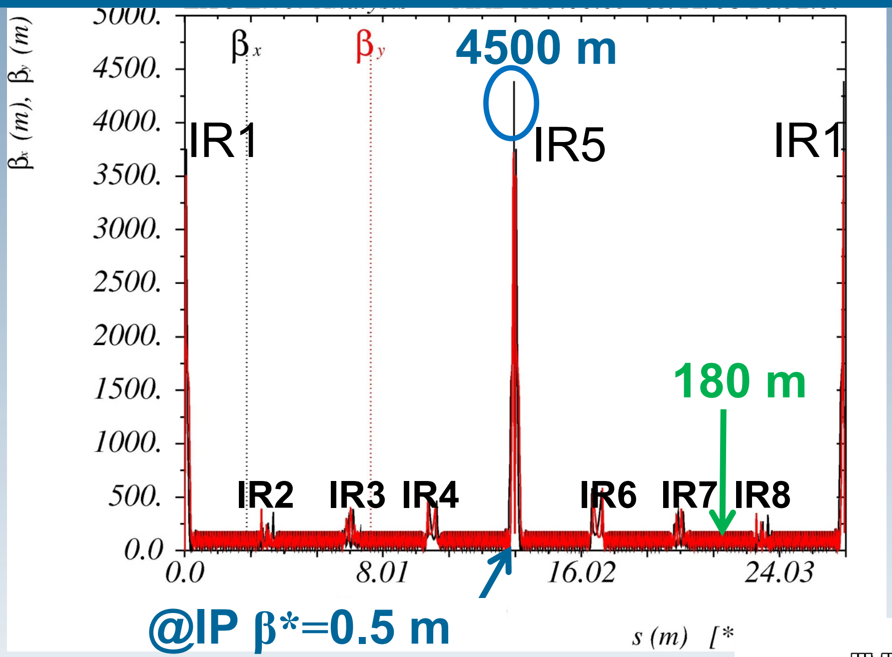
LHC Operational cycle:
Squeeze ➔ reduce β* (β @IP)

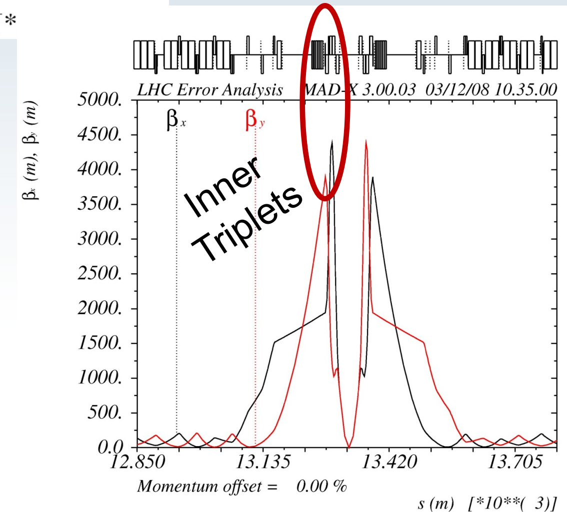


Beta function at Injection



Beta function at top energy and after squeeze

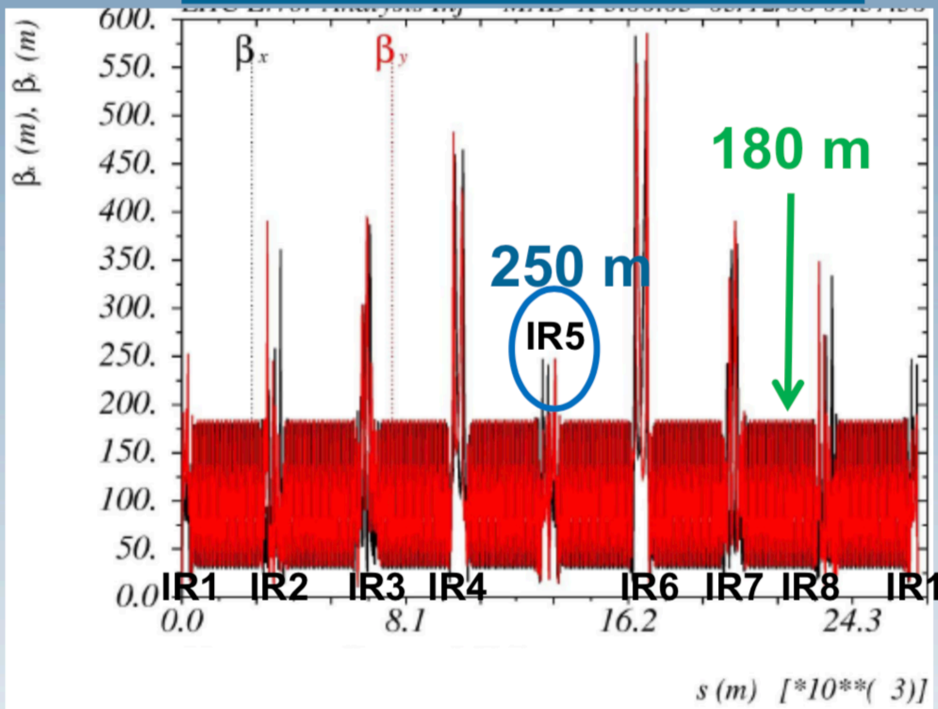




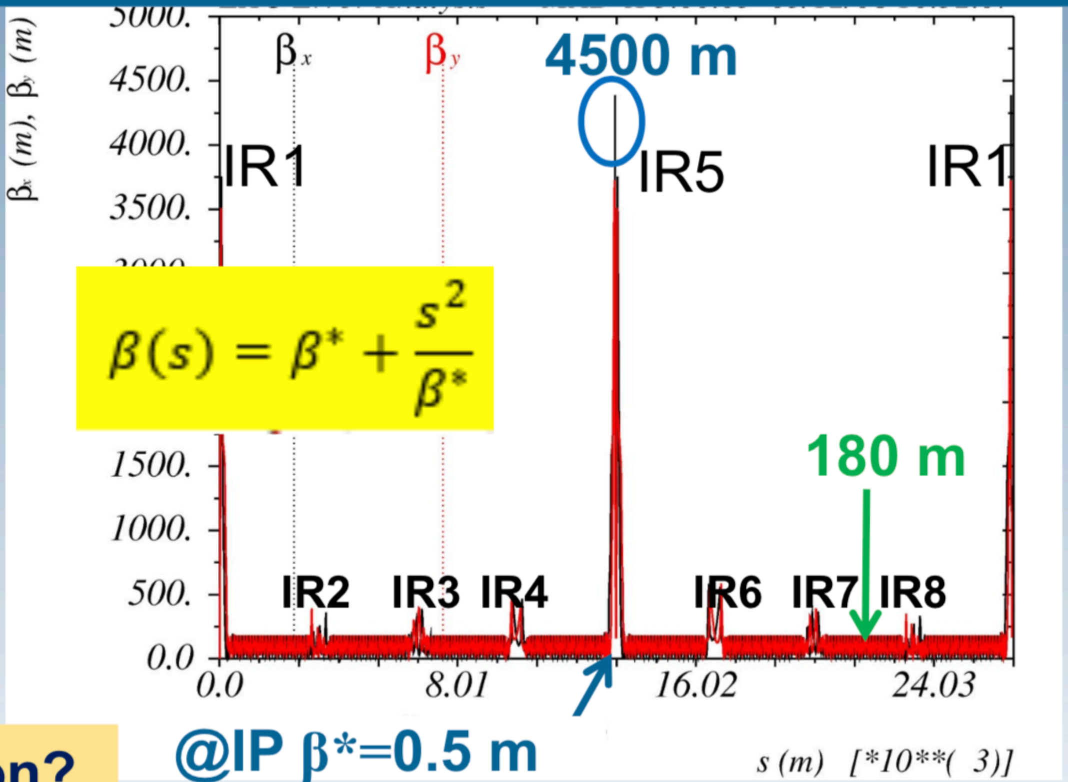
22.03.2022

Spares

Beta function at Injection



Beta function at top energy and after squeeze



$$\beta(s) = \beta^* + \frac{s^2}{\beta^*}$$

@IP β*=0.5 m

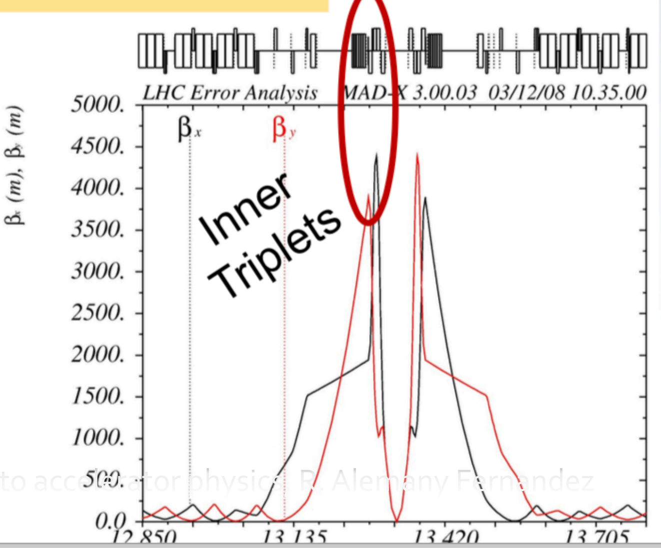
Why we cannot have β*= 0.5 m at injection?

$$\sigma = \sqrt{\beta \frac{\varepsilon_n}{(\beta\gamma)_{rel}}}$$

β=4500 m
γ(@450 GeV)~480
εn=3.5 μm rad

Remember: there is no D(s) here

σ ~ **6 mm !!**



Rbeampipe~29/24 mm
we could only accommodate ~ 4 times the beam size
and **we need at least 7σ clearance**

@ 7 TeV
σ_{IT} ~ **1.2 mm**