Introduction to longitudinal beam dynamics I

Course objectives

Course objectives:

- Give an overview of the longitudinal beam dynamics in accelerators
- Understand the issue of synchronization between the particles and the accelerating cavity
- The course will focused on synchrotrons and the synchrotron motion
- We will discuss radio-frequency resonators and the transit time factor (?)

Part I covers:

1. SYNCRHONOUS PARTICLE or ON-MOMENTUM PARTICLE or IDEAL PARTICLE
2. OFF-MOMENTUM PARTICLE or REAL PARTICLE
3. MOMENTUM COMPACTION FACTOR
4. RELATIVISTIC AND NON-RELATIVISTIC REGIME
5. SLIP FACTOR
6. GAMMA TRANSITION
7. PHASE FOCUSING: NON-RELATIVISTIC REGIME
8. PHASE FOCUSING: RELATIVISTIC REGIME
9. SYNCHROTRON OSCILLATIONS
10. SYNCHROTRON MOTION & EQUATIONS OF MOTION
11. STABILITY CONDITION FOR THE CASE OF SMALL DEVIATIONS

Part 2 covers:

1. LONGITUDINAL PHASE SPACE AND SEPARATRIX

2. CASE 1: NO ACCELERATION (ABOVE TRANSITION):
   - STATIONARY BUCKET
   - SEPARATRIX

3. CASE 2: ACCELERATION (ABOVE TRANSITION)

4. RF BUCKET PARAMETERS
   - PHASE SPACE AREA
   - BUCKET AREA OR LONGITUDINAL ACCEPTANCE
   - BUCKET WIDTH
   - BUCKET HIGHT
   - SINGLE PARTICLE LONGITUDINAL EMITTANCE

Part 1

INTRODUCTION

- In general particles gain energy from the electric field in the longitudinal direction.
- The electric field can be generated by an electrostatic accelerator, but it is limited by the field breakdown and by the length of the accelerating column.
- Electrostatic accelerators have been mainly used for low energy.
- Alternatively, a radio-frequency (RF) cavity operating in a resonance condition can be used to provide the accelerating voltage:

$$V(t) = V_{max} \sin \varphi(t) \qquad \text{Eq. (1)}$$

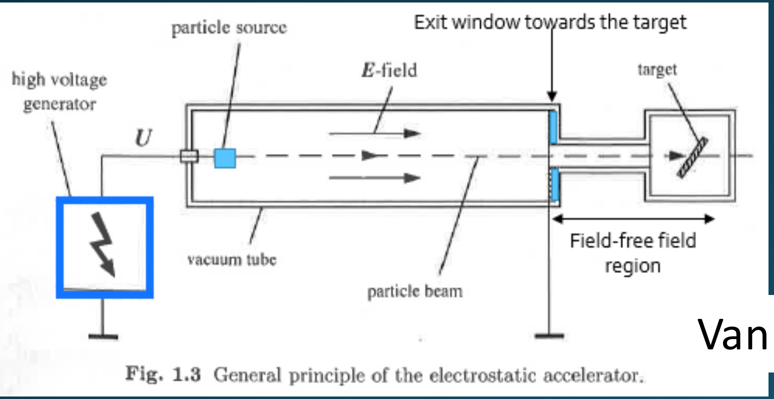
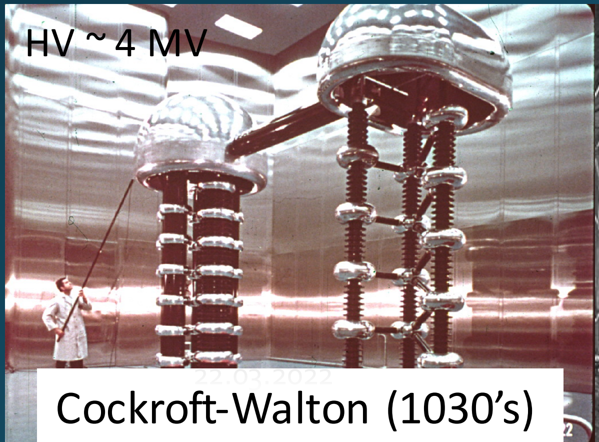
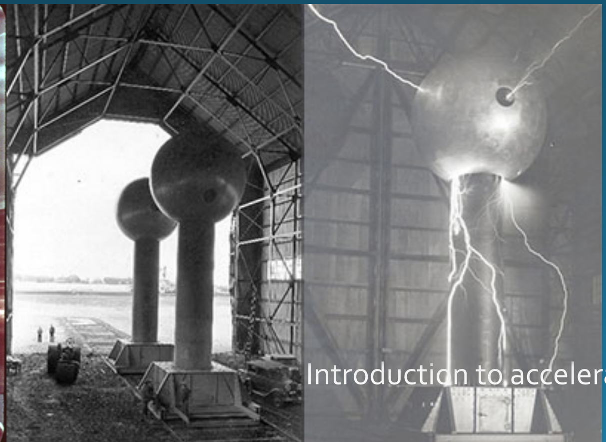


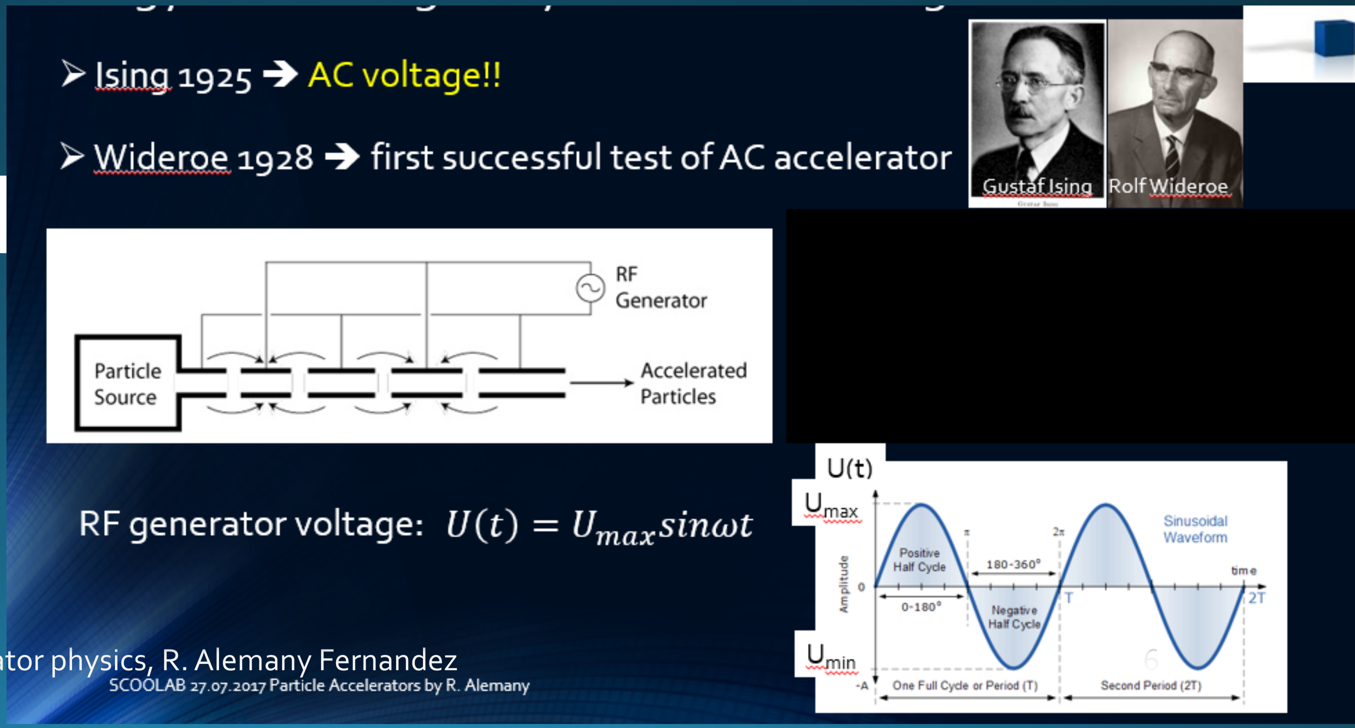
Fig. 1.3 General principle of the electrostatic accelerator.

Van de Graaff (1030's)
HV ~ 2 MV – 10 MV



Cockroft-Walton (1030's)



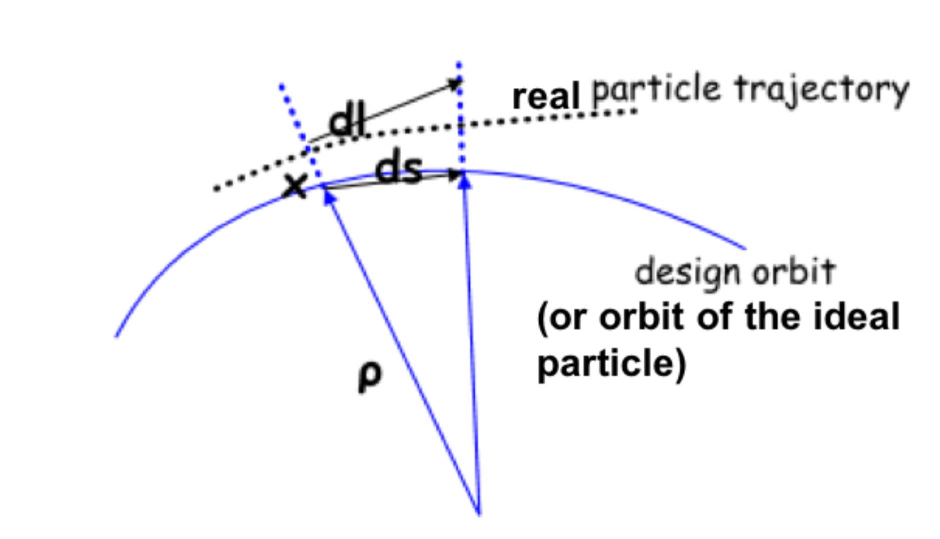


$$V(t) = V_{max} \sin \varphi(t) \qquad \text{Eq. (1)}$$

When is the best time or phase at which the particle has to enter the cavity to get the best accelerating performance???

Let’s try to answer the question by concentrating our argumentation on the behavior of the IDEAL PARTICLE

- The IDEAL particle has the design momentum which perfectly follows the magnetic field variation ➔ $p_s = qB\rho$
- The IDEAL particle goes through the center of the quadrupoles, and does not see any restoring force
- The IDEAL particle has the design momentum and therefore does not have a dispersive orbit
- The IDEAL particle runs on the design orbit defined by the dipole magnets and travel a distance ds with bending radius ρ at a given moment in time



SYNCRHONOUS PARTICLE or ON-MOMENTUM PARTICLE or IDEAL PARTICLE

- The IDEAL particle has the design momentum, therefore, it enters the RF cavity at the right time or phase

- The IDEAL particle is synchronized with the RF phase ➔ has the so called synchronous phase, φ_s, at a revolution period $T_s = \delta t_s = T_0$(*) and its momentum is $p_s = p_0$
- The IDEAL particle is the synchronous particle of charge q that gains energy every time it crosses the accelerating gap by this amount:

$$\Delta E = \int_{s_1}^{s_2} F_{elec} ds = \int_{s_1}^{s_2} qE_{elec} ds = qE\Delta s = qV = qV_{max} sin\varphi_s$$

(*) unfortunately, every author uses a different notation

But life is everything except IDEAL

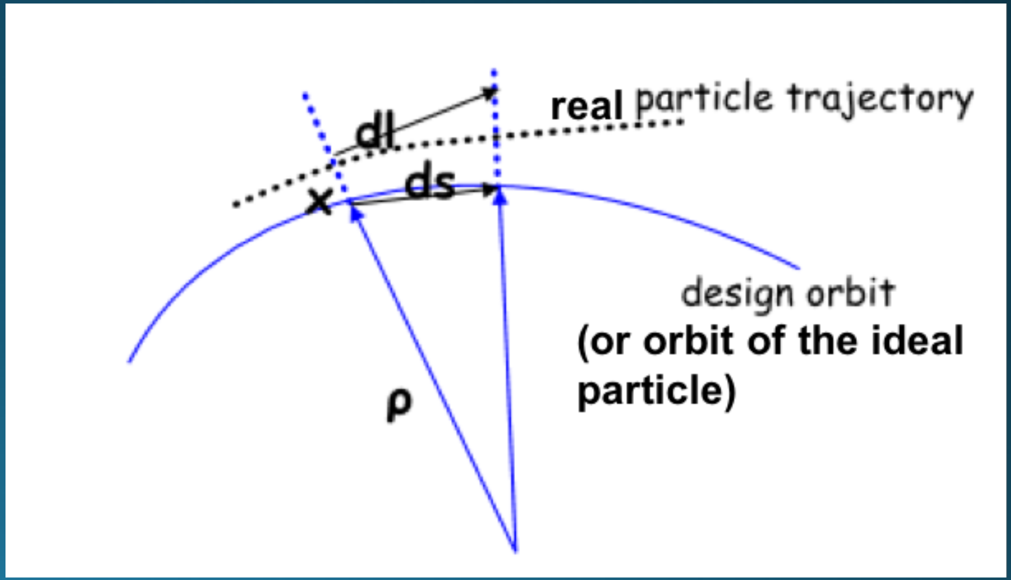
OFF-MOMENTUM PARTICLE or REAL PARTICLE

- A beam is composed of many particles, each one with a slightly different momentum which is surely different from the ideal momentum.
- Such a real particle with momentum p has its own off-momentum closed orbit given by $x_{\Delta E}(s) = D(s)\frac{\Delta p}{p_s}$, where $\frac{\Delta p}{p_s} = \frac{p - p_s}{p_s}$ is the fractional momentum deviation w.r.t. synchronous particle and D(s) is the dispersion function.

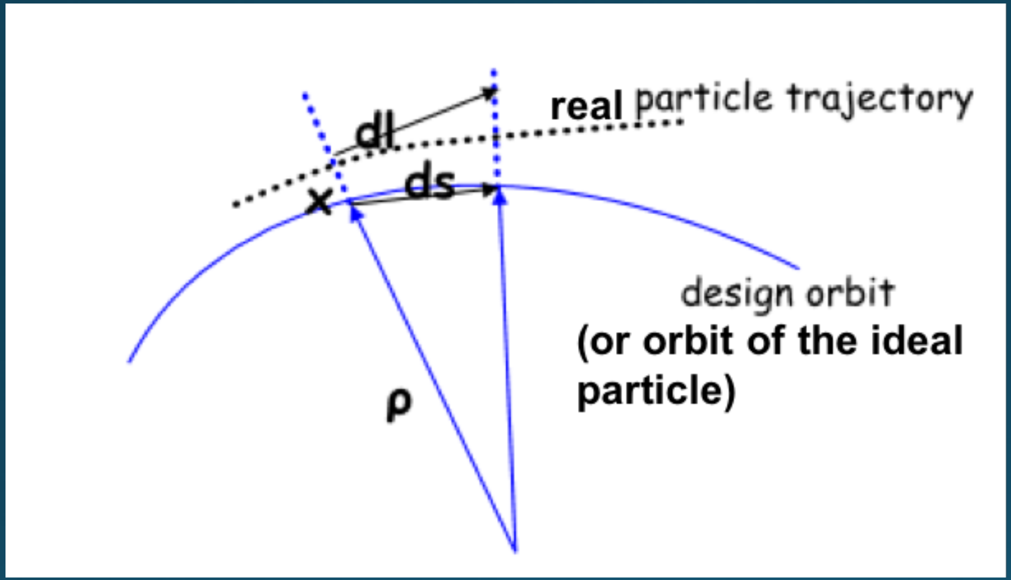
“s” of synchronous

“s” position in the accelerator

- The real particle will enter the RF cavity with a phase, φ , at a revolution period $T \quad = \delta t$
- The real particle will run on a displaced orbit (displaced to the outer side of the ring in the right figure) and travel a distance dl with bending radius $\rho + \mathrm{x}$.



If the real particle trajectory is too wrong, it will take the particle longer (or shorter) time to get to the RF cavity, and therefore, it will see a different phase of the voltage and therefore a different voltage and will gain a … wrong energy



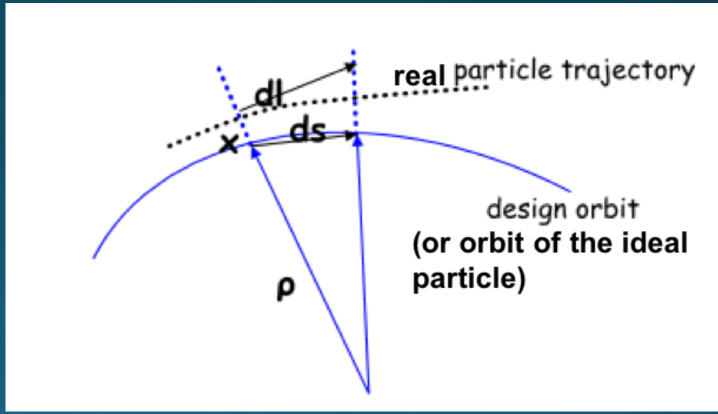
The synchronization of the motion in a synchrotron depends critically on the total path length

MOMENTUM COMPACTION FACTOR

MOMENTUM COMPACTION FACTOR

- The synchronization of the motion in a synchrotron depends critically on the total path length.
- The path length can change because of the betatron motion and it is proportional to $\sqrt{\beta}$, therefore is not a big change.
- It is interesting to understand the deviation of the total path length for an off-momentum particle from that of the on-momentum closed orbit, i.e.

$$\frac{dl}{ds} = \frac{\rho+x}{\rho}$$



- Solving for dl:

$$dl = \left(1 + \frac{x}{\rho(s)}\right) ds$$

Eq. 2

- Integrating around the machine we get the orbit length of the off-momentum particle, where x is the radial displacement caused by the momentum error and the dispersion function, i.e. $x_{\Delta E}(s) = D(s)\frac{\Delta p}{p_s}$.

$$l_{\Delta E} = \oint dl = \oint \left(1 + \frac{x_{\Delta E}(s)}{\rho(s)}\right) ds = l_s + \frac{\Delta p}{p_s} \oint \frac{D(s)}{\rho(s)} ds \qquad \text{Eq. 3}$$

- Finally we obtain the expression for the **difference in orbit length** between the ideal particle and the real particle which is determined by the **amount of momentum error** and **dispersion** in the storage ring:

$$\Delta l_{\Delta E} = l_{\Delta E} - l_s = \frac{\Delta p}{p_s} \oint \frac{D(s)}{\rho(s)} ds \qquad \text{Eq. 4}$$

- From Eq. 4 we get a very important relationship between the relative orbit difference $\frac{\Delta l_{\Delta E}}{l_s}$ and the relative momentum error $\frac{\Delta p}{p_s}$ called **MOMENTUM COMPACTION FACTOR (α or α_c or α_p, depending on the author)**:

$$\boxed{\frac{\Delta l_{\Delta E}}{l_s} = \alpha \frac{\Delta p}{p_s}} \qquad \text{Eq. 5}$$

$$\alpha = \frac{1}{l_s} \oint \frac{D(s)}{\rho(s)} ds \qquad \text{Eq. 6}$$

The synchronization of the motion in a synchrotron depends critically on the total path length

MOMENTUM COMPACTION FACTOR

$$\frac{\Delta l_{\Delta E}}{l_s} = \alpha \frac{\Delta p}{p_s} \qquad \text{Eq. 5}$$

$$\alpha = \frac{1}{l_s} \oint \frac{D(s)}{\rho(s)} ds \qquad \text{Eq. 6}$$

- How much the momentum compaction factor is?
- For first estimates we assume equal bending radii in all dipoles such $\frac{1}{\rho}$ is constant, and we replace the integral of the dispersion around the ring by a sum over the average dispersion in the dipole magnets (outside the dipoles $\frac{1}{\rho} = 0$)

$$\oint_{dipoles} D(s)ds \approx l_{\Sigma(dipoles)} \langle D \rangle_{dipoles}$$ Eq. 7

$$\alpha \approx \frac{1}{l_s}\frac{1}{\rho} l_{\Sigma(dipoles)} \langle D \rangle_{dipoles} = \frac{1}{l_s}\frac{1}{\rho} 2\pi\rho \ \langle D \rangle_{dipoles}$$
$$= \frac{2\pi}{l_s} \langle D \rangle_{dipoles} = \frac{\langle D \rangle_{dipoles}}{r_s}$$ Eq. 8

$$\boxed{\alpha \approx \frac{\langle D \rangle_{dipoles}}{r_s}}$$ (r_s = geometrical radius) Eq. 9

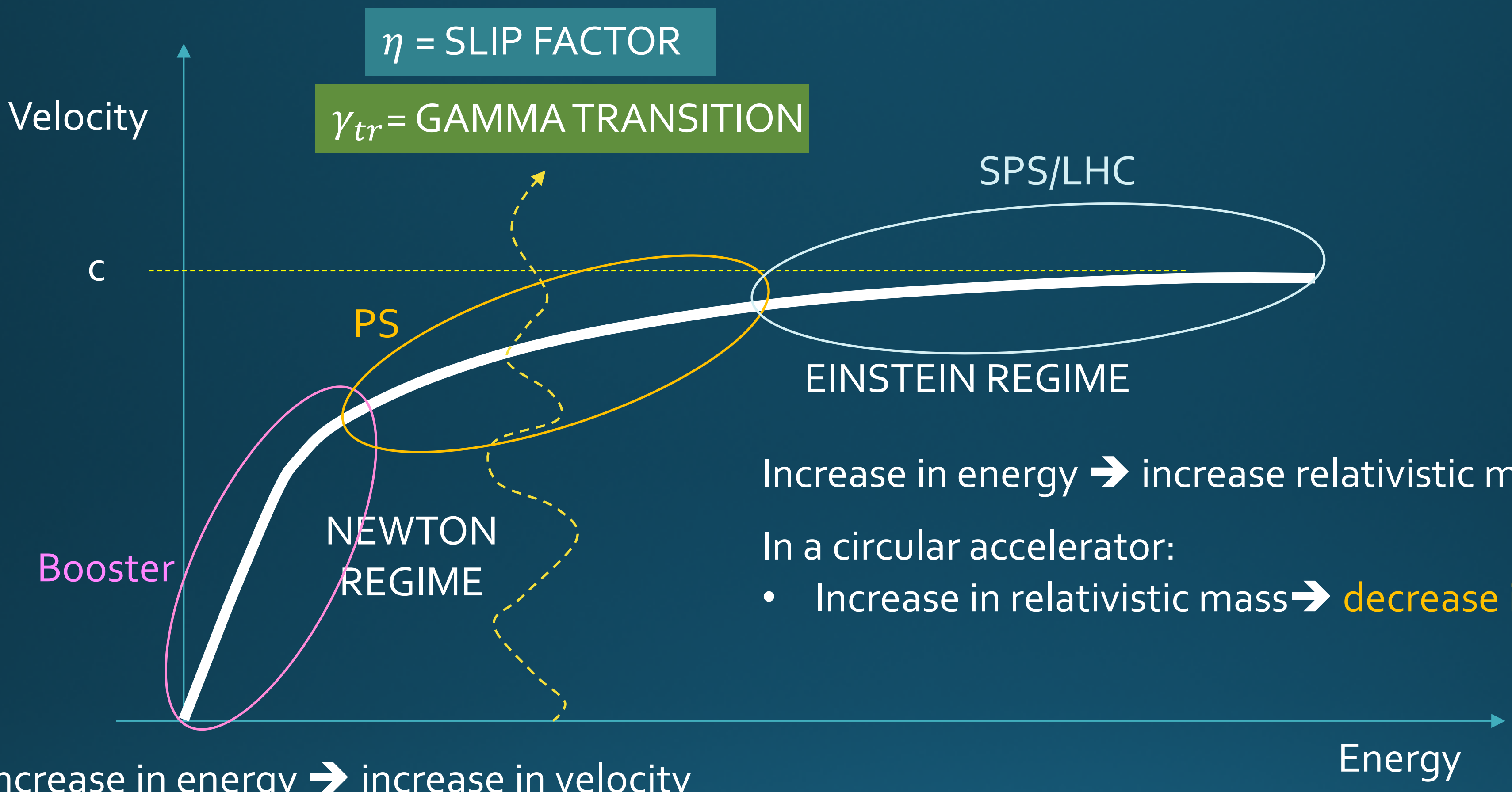
- It can be demonstrated that for well behaving optics:

$$\boxed{\alpha \approx \frac{1}{Q_x^2}}$$ Eq. 9'

SPS Q_x = 26 → $\alpha \sim 2 \cdot 10^{-3}$

PS Q_x =6.2 → $\alpha \sim 2.7 \cdot 10^{-2}$

RELATIVISTIC AND NON-RELATIVISTIC REGIME



Increase in energy ➔ increase relativistic mass

In a circular accelerator:

- Increase in relativistic mass➔ decrease in revolution frequency

Increase in energy ➔ increase in velocity

In a circular accelerator:

- Increase in velocity ➔ increase in revolution frequency

RELATIVISTIC AND NON-RELATIVISTIC REGIME

- The revolution period of a particle is $T = \frac{L}{v}$.
- The relative difference between the revolution period of the real and the ideal particle is:

$$\frac{\Delta T}{T_s} = \frac{\Delta l_{\Delta E}}{l_s} - \frac{\Delta v}{v_s} \qquad \text{Eq. 10}$$

- For relativistic particles where $v \approx c$, the ideal and real particle have the same velocity and therefore Eq. 10 simplifies to:

$$\frac{\Delta T}{T_s} \approx \frac{\Delta l_{\Delta E}}{l_s} \qquad \text{Eq. 11}$$

- Recalling Eq. 5

$$\frac{\Delta T}{T_s} \approx \alpha \frac{\Delta p}{p_s} \qquad \text{Eq. 12}$$

- Therefore the problem of synchronization can be solved by dealing with the relative momentum error.
- For non-relativistic beams, the problem of synchronization requires a more detailed treatment. The parameter of interest is the ratio between the relative momentum error and the relative frequency deviation of the particle:

$$\boxed{\frac{\Delta f}{f} = \eta \frac{\Delta p}{p_s}} \qquad \text{Eq. 13}$$

- The ratio is called the **slip factor η**

η = SLIP FACTOR

SLIP FACTOR

- Let's express the revolution frequency as a function of the machine radius and particle velocity:

$$\Omega = 2\pi f \;\&\; \Omega = \frac{v}{r} = \frac{\beta c}{r} \rightarrow f = \frac{\beta c}{2\pi r} \qquad \text{Eq. 14}$$

- Let's calculate the derivative:

$$\mathrm{d}f = \frac{c}{2\pi} d\left(\frac{\beta}{r}\right) = \frac{c}{2\pi} \frac{r d\beta - \beta dr}{r^2} \qquad \text{Eq. 15}$$

- Let's calculate the relative frequency error:

$$\frac{\mathrm{d}f}{f} = \frac{d\beta}{\beta} - \frac{dr}{r} \qquad \text{Eq. 16}$$

- Remember Eq. 5, the equation of the momentum compaction factor that relates a change in radius or path length with the relative momentum error via the compaction factor, therefore, $\frac{\boldsymbol{dr}}{\boldsymbol{r}} = \alpha \frac{\Delta p}{p}$
- The particle momentum is related to the particle energy and velocity:

$$p = mv = \beta\gamma \frac{E_0}{c} \Longrightarrow \frac{\boldsymbol{dp}}{\boldsymbol{p}} = \frac{d\beta}{\beta} + \frac{d(1-\beta^2)^{-\frac{1}{2}}}{(1-\beta^2)^{-\frac{1}{2}}} = (1-\beta^2)^{-1}\frac{d\beta}{\beta} \qquad \text{Eq. 17}$$

- Introducing the last equations in Eq. 16 we get:

$$\underbrace{\frac{\mathrm{d}f}{f} = \frac{d\beta}{\beta} - \frac{dr}{r}}_{\text{Eq. 16}} = \left(\frac{\mathbf{1}}{\boldsymbol{\gamma}^2} - \boldsymbol{\alpha}\right)\frac{dp}{p} = -\left(\boldsymbol{\alpha} - \frac{\mathbf{1}}{\boldsymbol{\gamma}^2}\right)\frac{dp}{p} \rightarrow \eta = \boldsymbol{\alpha} - \frac{\mathbf{1}}{\boldsymbol{\gamma}^2}$$

η = SLIP FACTOR

Eq. 18

- Which for relativistic particles, reduces to Eq. 12 ($\boldsymbol{\gamma}^2$ >>>>>>) → $\frac{\Delta T}{T_s} \approx \alpha \frac{\Delta p}{p_s}$ Eq. 12
- IMPORTANT REMARKS about Eq. 18:
  1. The change of revolution frequency depends on the particle energy, γ.
  2. It can change sign during acceleration.
  3. **Non-relativistic particles** will get faster at the beginning and will **arrive earlier** at the accelerating cavity.
  4. **Relativistic particles**, $v \approx c$, will not get faster anymore but rather get more massive and they will be pushed to dispersive orbits. Therefore the path length will be bigger and bigger and they will **arrive later** to the cavity.
  5. The boundary between the two regimes is defined for the case where there is no frequency dependence on the relative momentum error, i.e. $\eta = 0$. In this case the velocity variation is exactly compensated by the trajectory variation and the corresponding energy is called **gamma transition, $\boldsymbol{\gamma_{tr}}$**

$$\gamma_{tr} = \frac{1}{\sqrt{\alpha}}$$

Eq. 19

  6. In general accelerators are designed to avoid crossing transition as it involves RF phase changes, otherwise particles lose their longitudinal focusing and the bunch profile will dilute and get lost.

SPS Q_x = 26 → α ~ 2 · 10^{-3} → γ_{tr} = 22.36 → $E_{tr}(p+) = \gamma_{tr} * E_o/q$ ~ 21 GeV

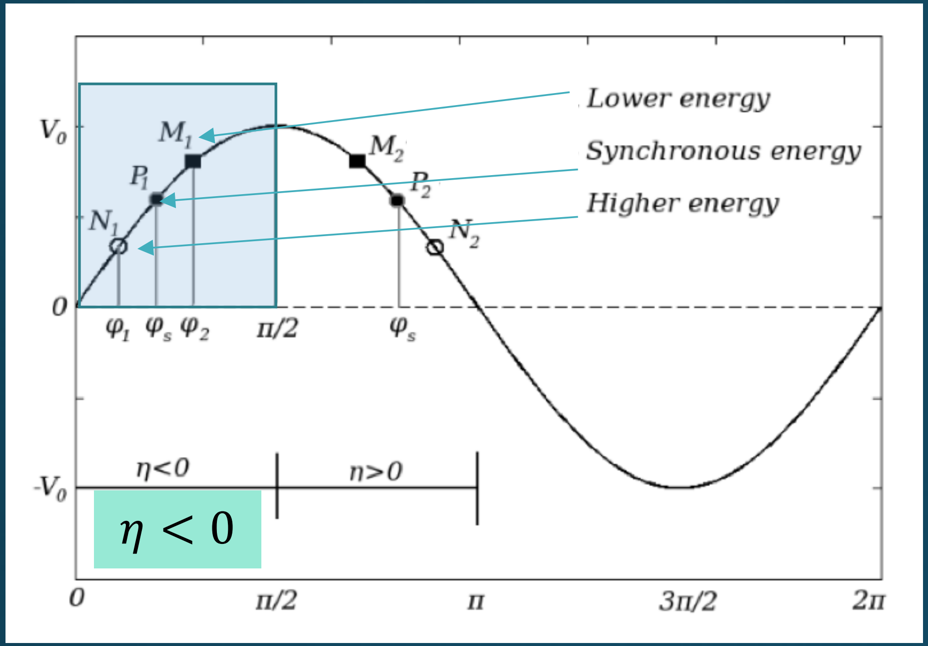
PS Q_x =6.2 → α ~ 2.7 · 10^{-2} → γ_{tr} = 6.1 → $E_{tr}(p+) = \gamma_{tr} * E_o/q$ ~ 5.8 GeV

The transition energy is determined by the choice of the lattice

PHASE FOCUSING:

- NON-RELATIVISTIC REGIME
- RELATIVISTIC REGIME

PHASE FOCUSING: NON-RELATIVISTIC REGIME

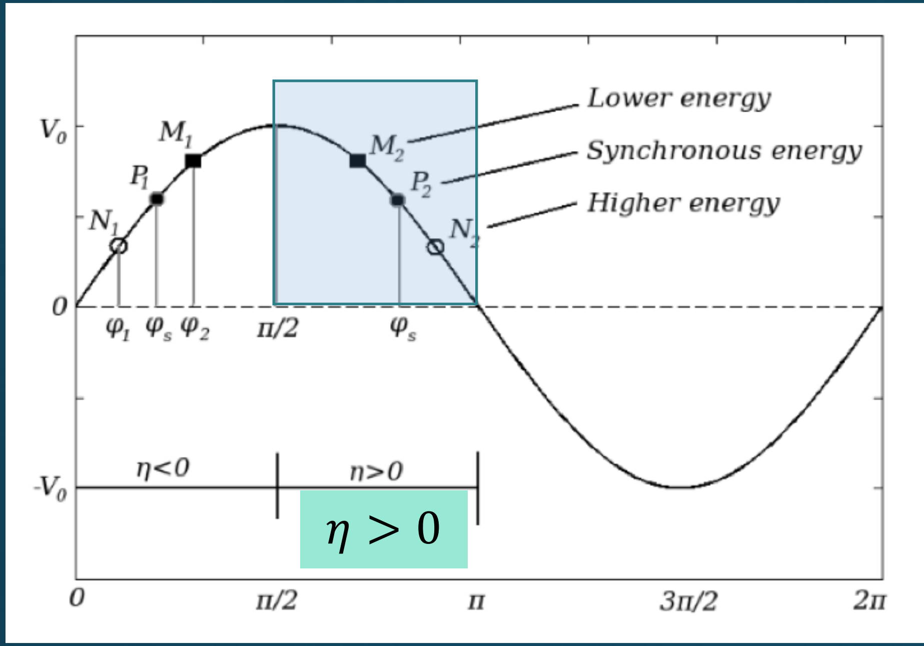


$$\eta = \alpha - \frac{1}{\gamma^2}$$

- The synchronous particles (P1) always arrives at the right phase to the cavity and therefore always sees the same voltage, $V_s = V_{max} sin\varphi_s$
- A particle that has smaller energy than the ideal particle will travel at a lower speed and will arrive later to the cavity and will see a higher voltage (M1). It will gain energy little by little until it reaches the ideal particle. But as it gains energy, it arrives earlier and earlier, and therefore at some point starts to see a lower voltage than the ideal particle (N1). Therefore will be slowly decelerated with respect to the ideal particle, arriving later and later. But this is compensated by an increasing voltage, therefore, the particle gains energy again.
- The real particle energy and phase will oscillate continuously around the ideal particle energy and phase

  → **SYNCHROTRON OSCILLATIONS**
- The net effect is a focusing effect

PHASE FOCUSING: RELATIVISTIC REGIME



$$\eta = \alpha - \frac{1}{\gamma^2}$$

- For relativistic particles, $v \approx c$, the same effect exists but based now on the relativistic mass increase with energy.
- The high energy particle (N2) will, due to its higher mass than the ideal particle, run on a longer orbit and it will arrive later to the cavity and will see a smaller voltage than the ideal one.
- Little by little the particle will catch the ideal particle relativistic mass and, little by little will arrive sooner and sooner until overpass the ideal particle (M2), but meaning it has less energy than the ideal one. However, this will be compensated by the fact that the real particle will see a higher voltage.
- The real particle energy and phase will oscillate continuously around the ideal particle energy and phase

  ➔ SYNCHROTRON OSCILLATIONS
- The net effect is a focusing effect
- As a consequence the synchronous phase has to be chosen depending on whether we are running the accelerator below or above transition.
- Accelerators crossing transition will have to perform a phase jump to keep the particles bunched.

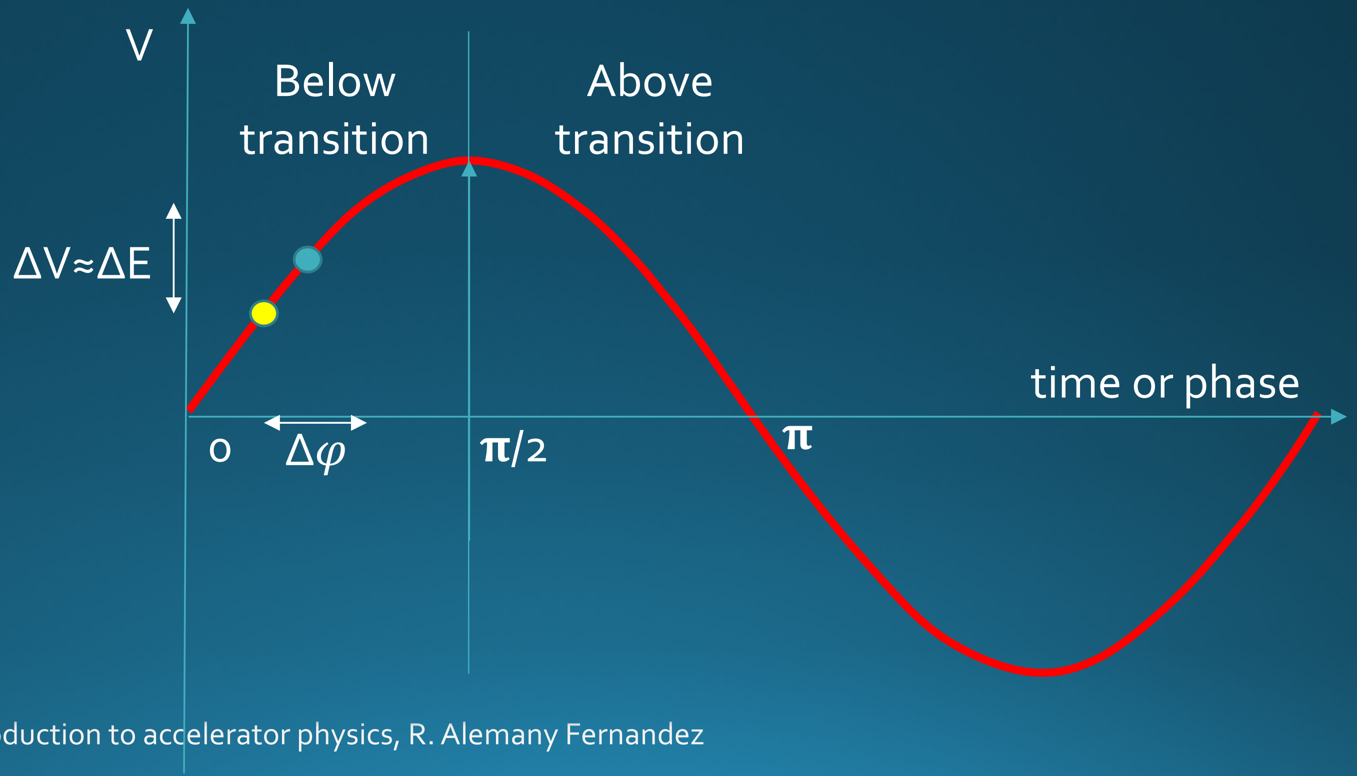
$$V(t) = V_{max} \sin\varphi(t) \qquad \text{Eq. (1)}$$

When is the best time or phase at which the particle has to enter the cavity to get the best accelerating performance???

SYNCHROTRON MOTION

- Every time particles cross the cavity they gain or lose energy performing synchrotron oscillations or ENERGY-PHASE oscillations around the ideal particle or synchronous particle.

The energy and phase variations can be described by differential equations of motion



SYNCHROTRON MOTION

- Every time particles cross the cavity they gain or lose energy performing synchrotron oscillations or phase oscillations around the ideal particle or synchronous particle.
- We will call the relative energy difference:

$$w = \frac{\Delta W}{W_s} = \frac{W - W_s}{W_s} \quad \text{Eq. 20}$$

- And the phase difference:

$$\Delta\varphi = \varphi - \varphi_s \quad \text{Eq. 21}$$

- Where as before, the index s refers to the synchronous particle.
- In practice the energy fluctuation is very small, $w \ll 1$, however is not the same for the phase oscillations.
- We are going to stablish the differential equations of motion relating the energy variations and the phase variations.
- We assume a single accelerating gap, which does not remove validity to the exercise since we could always replace a set of accelerating cavities by a single one of the same effect.
- The RF generator plays a given voltage function given in Eq. 1 ($V(t) = V_{max} \sin\varphi$)
- When the particle of charge q crosses the accelerating gap from s_1 to s_2 the **energy gain (per one turn!)** is:

$$\boldsymbol{W} = \int_{s_1}^{s_2} \vec{F} d\vec{s} = \int_{s_1}^{s_2} q\vec{E}_{field} d\vec{s} = \mathrm{q}E_{field}\Delta s = qV = qV_{max} sin\varphi \quad \text{Eq. 22}$$

$$W - W_s = qV_{max}(sin\varphi - sin\varphi_s) \quad \text{Eq. 23}$$

ADIABATIC PROCESS

- In practice, the particle energy varies slowly in comparison with the synchrotron oscillations period, it is a smooth function of t
- The relative energy difference is:

$$w = \frac{\Delta W}{W_s} = \frac{qV_{max}}{W_s}\,(sin\varphi - sin\varphi_s) \qquad \text{Eq. 24}$$

- Since $\mathrm{T} = \delta t = \frac{2\pi}{\Omega}$ (period=time takes to give one turn) and $\dot{w} \approx \frac{\delta w}{\delta t}$ we can write (Note 1):

FIRST EQUATION OF MOTION

$$\boxed{\dot{w} = \frac{qV_{max}\Omega_s}{2\pi W_s}\,(\sin(\Delta\varphi + \varphi_s) - sin\varphi_s)} \qquad \text{Eq. 25}$$

- where $\sin(\Delta\varphi + \varphi_s) = \sin\varphi$

Note 1: We are ignoring higher order terms to keep the discussion in the linear regime and assuming $\Omega \approx \Omega_s$, i.e.:

$$\dot{w} = \frac{d}{dt}\left(\frac{\Delta W}{W_s}\right) = \frac{\Delta\dot{W} W_s - \Delta W \dot{W}_s}{{W_s}^2} = \frac{\Delta\dot{W}}{W_s} - \frac{\Delta W \dot{W}_s}{{W_s}^2} \cong \frac{\Delta\dot{W}}{W_s}$$

Note 2: in literature (e.g. S.Y. Lee) you'll find the equation written in this mode:

$$\frac{d}{dt}\left(\frac{\Delta W}{\Omega_s}\right) = \frac{1}{2\pi} qV_{max}(\sin(\Delta\varphi + \varphi_s) - sin\varphi_s) \; or \; \frac{d}{dt}\left(\frac{\Delta p}{p_s}\right) = \frac{\Omega_s}{2\pi\beta_s^2 W_s} qV_{max}(\sin(\Delta\varphi + \varphi_s) - sin\varphi_s)$$

Where $(\Delta\varphi, \frac{\Delta W}{\Omega_s})$ or $(\Delta\varphi, \frac{\Delta p}{p_s})$ are pairs of conjugate phase-space coordinates.

Synchrotrons

- Particles follow **a constant orbit** with mean radius $\boldsymbol{R}$ ($\boldsymbol{\rho}$ is the dipole bending radius)

- **Acceleration** by longitudinal electric field provided by the **RF cavities**. The RF cavities are placed in a specific point in the ring. In every turn, after each passage through the RF cavities, the particles gain energy:

$$(\Delta E)_{\text{turn}} = q\hat{V}_{\mathbf{RF}}\boldsymbol{sin\varphi}$$

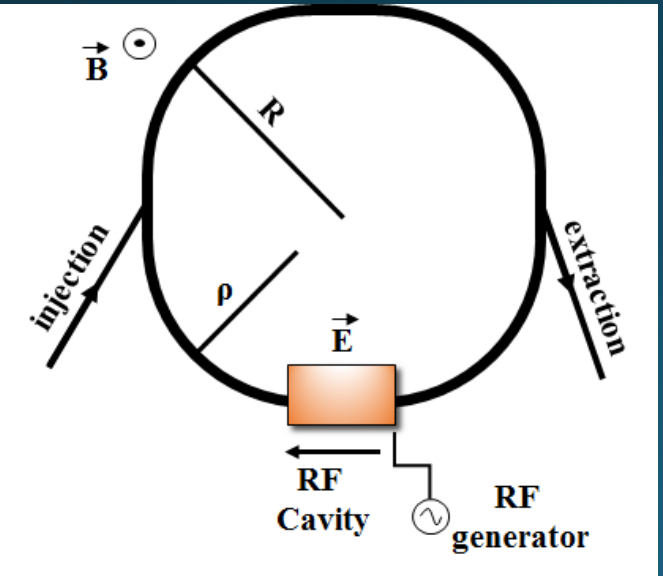
$\hat{V}_{\mathbf{RF}}$=Vmax: RF voltage amplitude
$\boldsymbol{\omega}_{\mathbf{RF}}$: angular RF frequency
$\boldsymbol{\varphi_0} = \boldsymbol{\varphi_s}$: RF synchronous/stable phase
$\boldsymbol{p}$: particle momentum
q : charge of particle

- The magnetic field $\vec{\boldsymbol{B}}$ follows the energy change to keep particles in orbit:

$$B(t)\rho = \frac{p(t)}{q} \Rightarrow \frac{dp}{dt} = q\rho\frac{dB}{dt}$$

- After one turn: dp ➔ *Δp* & *dt* ➔ *Δt=T*

$$\Delta p = q\rho\dot{B}T \text{ with T the revolution period: } T = \frac{2\pi R}{v}$$



- Using the basic relation: $v\dot{p} = \dot{E}$, with *v* the particle velocity, after one turn we have:

$$\boldsymbol{v}\frac{\boldsymbol{\Delta p}}{\Delta t} = \frac{(\Delta E)_{turn}}{\Delta t}$$

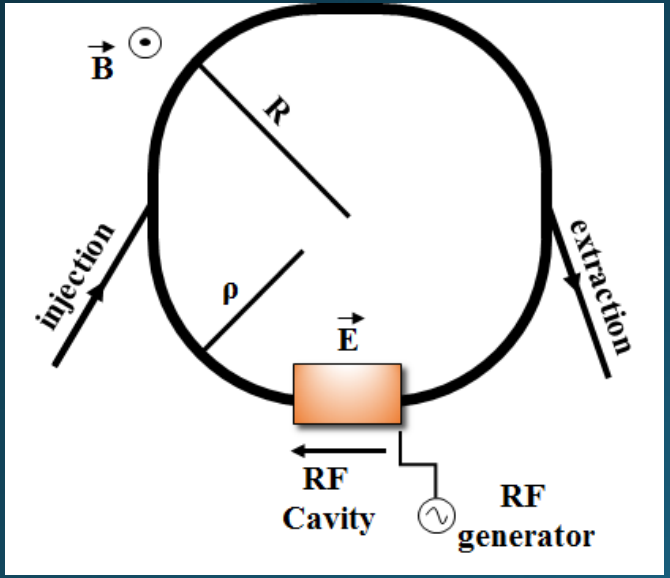
$$(\Delta E)_{\text{turn}} = q\hat{V}_{\mathbf{RF}}\boldsymbol{sin\varphi} = \boldsymbol{v\Delta p} = vq\rho\dot{B}\frac{2\pi R}{v}$$

$$(\Delta E)_{\text{turn}} = q\hat{V}_{\text{RF}} \sin\varphi = \text{v}\Delta p = 2\pi R q \rho \dot{B}$$

$$\sin\varphi_s = \frac{2\pi\rho R\dot{B}}{\hat{V}_{\text{RF}}} \text{ or } \sin\varphi_s = \frac{(\Delta E)_{\text{turn}}}{q\hat{V}_{\text{RF}}} \rightarrow \boxed{\varphi_s = \arcsin\frac{2\pi\rho R\dot{B}}{\hat{V}_{\text{RF}}}} \quad \boxed{\varphi_s = \arcsin\frac{(\Delta E)_{\text{turn}}}{q\hat{V}_{\text{RF}}}}$$

As particles gain energy their angular revolution frequency $\boldsymbol{\Omega} = \boldsymbol{\omega}_{\text{rev}}$ changes → $\boldsymbol{\Omega_{RF}} = \boldsymbol{\omega}_{\text{RF}}$ should be **synchronised** with $\boldsymbol{\omega}_{\text{rev}} = \boldsymbol{\Omega_s}$

$$\boxed{\textbf{Synchronism condition} \rightarrow \boldsymbol{\Omega_{RF} = h\Omega_s}}$$



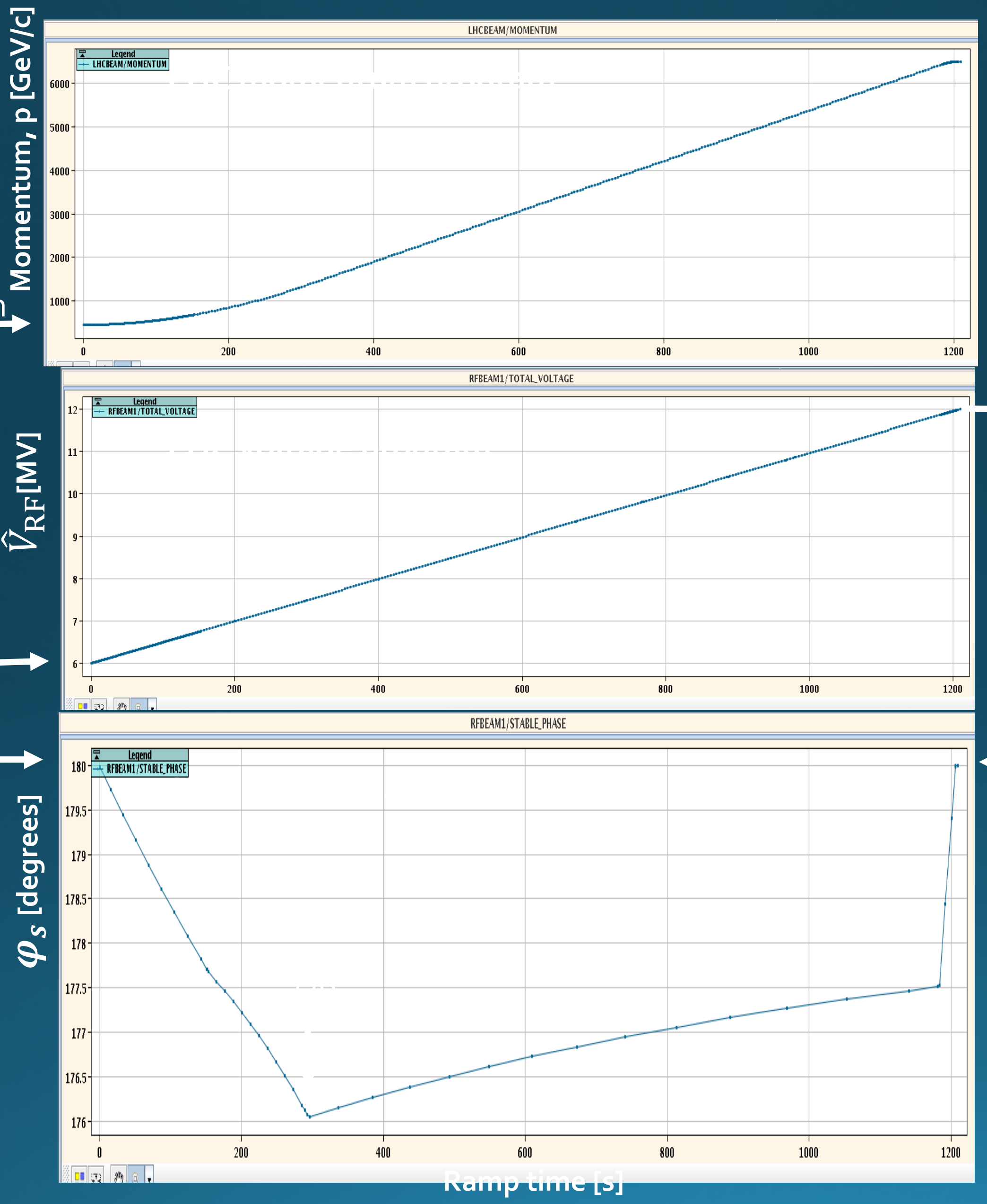
h **- harmonic number**:

- an integer number indicating the RF cycles per revolution period
- ***h*** **synchronous particles** equally spaced along the ring following the nominal (designed) orbit and having the nominal (designed) energy

Synchronous phase

$$\varphi_s = arcsin\frac{(\Delta E)_{\rm turn}}{q\hat{V}_{\rm RF}}$$

$$\varphi_s = arcsin\frac{2\pi\rho R\dot{B}}{\hat{V}_{\rm RF}}$$



On average, in the LHC, synchronous particles gain around **0.5 MeV/turn** during acceleration.

How did we obtain the first equation of motion?

- Just asking ourselves by **how much** the **energy gain difference** between the real and the ideal particle has **changed after one turn**
- We considered the problem very adiabatic: the particle energy varies slowly in comparison with the synchrotron oscillations period. We also made a small simplification ignoring higher order terms
- We also assumed that $\Omega \approx \Omega_s$

FIRST EQUATION OF MOTION

$$\dot{w} = \frac{qV_{max}\Omega_s}{2\pi W_s}\left(\sin(\Delta\varphi + \varphi_s) - sin\varphi_s\right) \qquad \text{Eq. 25}$$

Let's now obtain the second equation of motion

We will ask ourselves: **how much** the difference between the real particle phase and the ideal particle phase (**$\Delta\varphi = \varphi - \varphi_s$**) **has changed after one turn**

- To obtain the second equation of motion we need the concept of slip factor (Eq. 18) and that:

$$\frac{\Delta p}{p_s} = \frac{1}{{\beta_s}^2}\frac{\Delta W}{W_s} \qquad \text{Eq. 26 (*)}$$

- We start from the equations for the angular velocity:

$$\delta\varphi = \Omega_{RF}\delta t \;\&\; \delta\varphi_s = \Omega_{RF}\delta t_s \qquad \text{Eq. 27}$$

- After one turn, the RF angular velocity in Eq. 27 corresponds to the revolution frequency, and $\delta\varphi$ = $\delta\varphi_s = 2\pi$, i.e.

$$\delta\varphi = \delta\varphi_s = 2\pi = \Omega \quad \delta t = \Omega_s\delta t_s \qquad \text{Eq. 28}$$

- We ask ourselves the question by **how much** the difference between the real particle phase and the ideal particle phase ($\boldsymbol{\Delta\varphi = \varphi - \varphi_s}$) **has changed after one turn**?, i.e. how much is $\delta(\Delta\varphi)$?

$$\delta(\Delta\varphi) = \Omega_{RF}(\delta t - \delta t_s) \qquad \text{Eq. 29}$$

(*) $W^2 = p^2c^2 + E_0^2 \Rightarrow 2W\Delta W = 2p\Delta pc^2 \Rightarrow we\ divide\ by\ W^2 \Rightarrow \frac{\boldsymbol{\Delta W}}{\boldsymbol{W}} = \frac{p\Delta pc^2}{W^2} = \frac{m_0\gamma\beta c}{m_0^2\gamma^2c^4}\Delta pc^2 = \frac{\boldsymbol{\Delta p}}{\boldsymbol{p}}\beta^2$

- Where $\delta t - \delta t_s$ for one turn is $T - T_s$, i.e. the difference of the periods, and $\Omega_{RF} = h\Omega_s$, i.e. the acceleration **frequency** given by the **RF** system is a **multiple *h*** of the **synchronous** particle revolution **frequency**.

- Now we ask ourselves a more general question: what is the rate of change of $\delta(\Delta\varphi)$?, i.e.

$$\Delta\dot{\varphi} = \frac{\delta(\Delta\varphi)}{\delta t} = h\Omega_s\left(1 - \frac{\delta t_s}{\delta t}\right) \qquad \text{Eq. 30}$$

- From Eq. 28 we can express $\frac{\delta t_s}{\delta t}$ as the ratio of angular velocities:

$$\Delta\dot{\varphi} = \frac{\delta(\Delta\varphi)}{\delta t} = h\Omega_s\left(1 - \frac{\Omega}{\Omega_s}\right) \qquad \text{Eq. 31}$$

$$\Delta\dot{\varphi} = \frac{\delta(\Delta\varphi)}{\delta t} = h\Omega_s\left(1 - \frac{\Omega}{\Omega_s}\right)$$ Eq. 31

$$\Delta\dot{\varphi} = \frac{\delta(\Delta\varphi)}{\delta t} = -h\Omega_s\frac{\Delta\Omega}{\Omega_s}$$ Eq. 32

- Since the angular velocity and the energy of the particle are related, we can try to get an equation of motion with both variables present. Since the $2\pi f = \Omega$, we can straight away use the equation of the slip factor, Eq. 18, that relates the relative change of revolution frequency with the relative momentum error: $\frac{\Delta f}{f} = -\eta\frac{\Delta p}{p_s}$, with $\eta = \boldsymbol{\alpha} - \frac{\mathbf{1}}{\boldsymbol{\gamma}^2}$

- As second step we replace the relative momentum error by Eq. 26 to get:

$$\frac{\Delta p}{p_s} = \frac{1}{{\beta_s}^2}\frac{\Delta W}{W_s}$$ Eq. 26

$$\frac{\boldsymbol{\Delta\Omega}}{\boldsymbol{\Omega_s}} = -\left(\alpha - \frac{1}{\gamma^2}\right)\frac{dp}{p} = -\boxed{\left(\alpha - \frac{1}{\gamma^2}\right)\frac{1}{\beta_s^2}}\frac{\Delta W}{W_s} = -\boldsymbol{\Gamma_s w}$$

$$\Gamma_s = \frac{\eta}{\beta_s^2}$$ Eq. 33

SECOND EQUATION OF MOTION

$$\boxed{\Delta\dot{\varphi} = \frac{\delta(\Delta\varphi)}{\delta t} = h\Gamma_s\Omega_s w}$$ Eq. 34

$\Omega_{RF} = h\Omega_s$ ➔angular RF frequency is a multiple of the revolution frequency, h: harmonic number

Note: in literature (e.g. S.Y. Lee) you'll find the equation written like:

$$\Delta\dot{\varphi} = \frac{h\Omega_s^2\eta}{\beta_s^2 W_s}\left(\frac{\Delta W}{\Omega_s}\right) \quad \text{or} \quad \Delta\dot{\varphi} = \mathrm{h}\Omega_s\eta\frac{\Delta p}{p_s}$$

Where $\left(\Delta\varphi, \frac{\Delta W}{\Omega_s}\right)$ or $\left(\Delta\varphi, \frac{\Delta p}{p_s}\right)$ are pairs of conjugate phase-space coordinates. To change from one to another we use Eq. 26:

$$\frac{\Delta p}{p_s} = \frac{1}{\beta_s^2}\left(\frac{\Delta W}{W_s}\right)\frac{\Omega_s}{\Omega_s}$$

SECOND EQUATION OF MOTION

$$\Delta\dot{\varphi} = \frac{\delta(\Delta\varphi)}{\delta t} = h\Gamma_s\Omega_s w$$

Eq. 34

$\Omega_{RF} = h\Omega_s$ ➔angular RF frequency is a multiple of the revolution frequency, h: harmonic number

Example:
If h =1, 1 RF period is equal to 1 particle period
LHC harmonic h = 35640, while the particle gives 1 turn the RF system has oscillated 35640 times

The harmonic number indicates the number of available “buckets”, i.e. the number of available potential wells where the particles are stable ==> see next course

SYNCHROTRON EQUATIONS OF MOTION

FIRST EQUATION OF MOTION $$\dot{w} = \frac{qV_{max}\Omega_s}{2\pi W_s}\left(\sin(\Delta\varphi + \varphi_s) - sin\varphi_s\right) \quad \text{Eq. 25}$$

SECOND EQUATION OF MOTION $$\Delta\dot{\varphi} = \frac{\delta(\Delta\varphi)}{\delta t} = h\Gamma_s\Omega_s w \quad \text{Eq. 34}$$

$$\Gamma_s = \frac{\eta}{\beta_s^2}, \mathrm{h} = \text{harmonic number}$$

Differential equations system to describe the evolution as a function of time of the energy and phase of the real particle around the synchronous particle

Note: In order to obtain the first equation of motion, we have assumed that the beam energy can be change only by the applied RF field, and we have neglected any other energy variation due to interaction with the environment or the synchrotron radiation

PARTICULAR CASE → SMALL DEVIATIONS

CASE OF SMALL DEVIATIONS

- For this we need to take Eq. 25 and develop for the case $\Delta\varphi \ll$
- $\sin(\Delta\varphi + \varphi_s) = sin\Delta\varphi cos\varphi_s + cos\Delta\varphi sin\varphi_s \cong \Delta\varphi cos\varphi_s + sin\varphi_s$
- Plugging the above relation into the first equation of motion we get Eq. 35

FIRST EQUATION OF MOTION $$\dot{w} = \frac{qV_{max}\Omega_s}{2\pi W_s}\left(\sin(\Delta\varphi + \varphi_s) - sin\varphi_s\right)$$

FIRST EQUATION OF MOTION
FOR SMALL PHASE AMPLITUDE OSCILLATIONS

$$\boxed{\dot{w} = \frac{qV_{max}\Omega_s cos\varphi_s}{2\pi W_s}\Delta\varphi}$$

Eq. 35

- Now we derivate the second equation of motion to get:

SECOND EQUATION OF MOTION $$\Delta\dot{\varphi} = \frac{\delta(\Delta\varphi)}{\delta t} = h\Gamma_s\Omega_s w$$

$$\Delta\ddot{\varphi} = h\Gamma_s\Omega_s\dot{w}$$

- Replacing $\dot{w}$ by Eq. 35, and regrouping all the terms in this way:

$$\boxed{\Omega_{sy}^2 = -\frac{qh\Gamma_s V_{max}cos\varphi_s}{2\pi W_s}\Omega_s^2}$$

Eq. 36

HARMONIC OSCILLATOR EQUATION TO DESCRIBE THE SYNCHROTRON OSCILLATIONS FOR SMALL AMPLITUDES

$$\Delta\ddot{\varphi} + \Omega_{sy}^2 \Delta\varphi = 0 \qquad \text{Eq. 37}$$

Ω_{sy} = synchrotron oscillation frequency
➔ proportional to the revolution frequency

$$\Omega_{sy}^2 = -\frac{qh\Gamma_s V_{max} cos\varphi_s}{2\pi W_s}\,\Omega_s^2 \qquad \text{Eq. 36}$$

$$F = ma = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = m\ddot{x} = -kx.$$

Solving this differential equation, we find that the motion is described by the function

$$x(t) = A\cos(\omega t + \varphi),$$

where

$$\omega = \sqrt{\frac{k}{m}}.$$

Oscillation frequency:

$$\omega = 2\pi f$$

Mass-spring harmonic oscillator

An arbitrary potential can usually be approximated as a *harmonic* potential at the vicinity of a *stable* equilibrium point

HARMONIC OSCILLATOR EQUATION TO DESCRIBE THE SYNCHROTRON OSCILLATIONS FOR SMALL AMPLITUDES

$$\Delta\ddot{\varphi} + \Omega_{sy}^{2}\Delta\varphi = 0$$ Eq. 37

Ω_{sy} = synchrotron oscillations frequency
➔ proportional to the revolution frequency

$$\Omega_{sy} = \sqrt{-\frac{qh\Gamma_s V_{max} cos\varphi_s}{2\pi W_s}}\,\Omega_s$$

SYNCHROTRON TUNE ➔ number of synchrotron oscillations per turn

$$Q_{sy} = \frac{\Omega_{sy}}{\Omega_s} = \sqrt{-\frac{qh\Gamma_s V_{max} cos\varphi_s}{2\pi W_s}}$$ Eq. 38

$$Q_{sy} = \frac{\Omega_{sy}}{\Omega_s} = \sqrt{-\frac{qh\Gamma_s V_{max} cos\varphi_s}{2\pi W_s}} \quad \text{Eq. 38}$$

The synchrotron oscillation frequency cannot be an imaginary number, has to be real, otherwise the solution won't be the usual "A*cos(wt+phi)" function describing harmonic oscillations. Therefore, the ratio inside the square root has to be positive. For this to be positive, "$\Gamma_s \quad cos\varphi_s$" has to be negative.

$$\Gamma_s = \frac{\eta}{\beta_s^2}$$

STABILITY CONDITON FOR SMALL AMPLITUDE OSCILLATIONS IN PHASE

$$\Gamma_s cos\varphi_s < 0$$

Replacing Γ_s by its value:

$$\left(\alpha - \frac{1}{\gamma^2}\right)\frac{1}{\beta_s^2} cos\varphi_s < 0$$

Since $\beta_s^2 > 0$, the condition simplifies to:

$$\boxed{\left(\alpha - \frac{1}{\gamma^2}\right) cos\varphi_s < 0}$$

In **linear accelerators $\boldsymbol{\alpha = 0}$** !! therefore, the stability condition reduces to

$$\alpha = \frac{1}{l_s}\oint \frac{D(s)}{\rho(s)} ds \quad \text{Eq. 6}$$

$$cos\varphi_s > 0$$

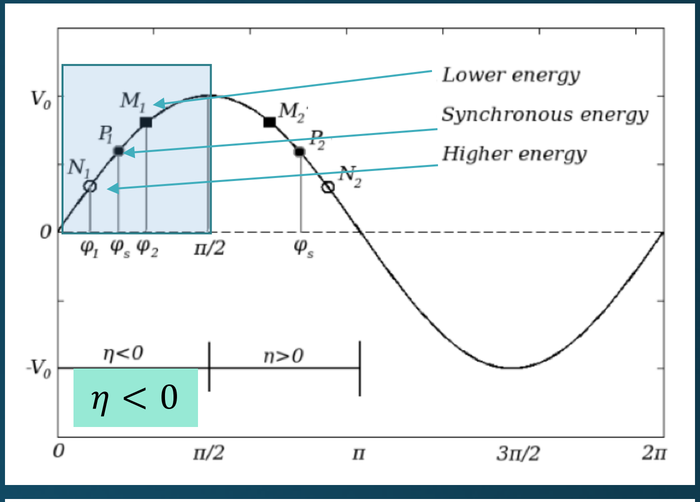
The longitudinal stability in a linear accelerator requires that φ_s is between 0 and π/2

$$\gamma_{tr} = \frac{1}{\sqrt{\alpha}} \quad \text{Eq. 19}$$

$$\left(\frac{1}{\gamma_{tr}^2} - \frac{1}{\gamma^2}\right) cos\varphi_s < 0 \quad \text{Eq. 42}$$

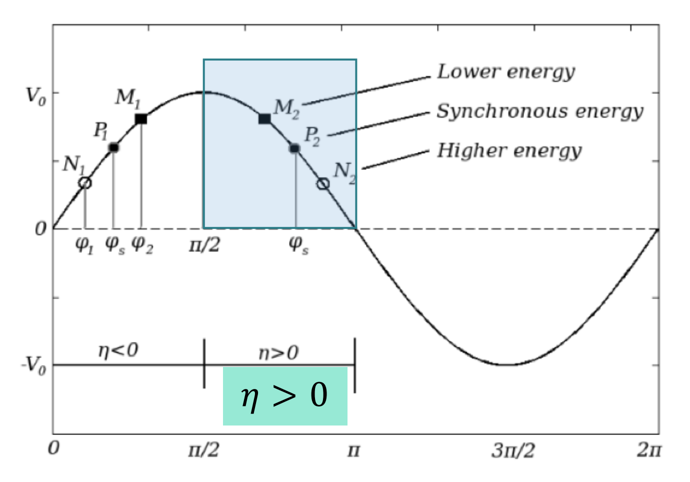
In circular accelerators the stability condition will depend on the energy of the particle.

At the beginning of the acceleration $\gamma < \gamma_{tr}$ ➔ $cos\varphi_s > 0$



The longitudinal stability in a circular accelerator below transition requires that φ_s is between 0 and π/2

At some point during the acceleration process $\gamma > \gamma_{tr}$ ➔ $cos\varphi_s < 0$



The longitudinal stability in a circular accelerator above transition requires that φ_s is between π/2 and π

$$V(t) = V_{max}\sin(\Omega_{RF}t + \varphi_s)$$

The principle of phase stability due to RF focusing assures that the ensemble of particles can be accelerated in synchrotrons even if they are not exactly at the synchronous phase and energy. Such particles just oscillate around the reference particle.

In the case where there is no acceleration applied to the synchronous particle, i.e.

$$V(\varphi_s) = V_{max}\sin\varphi_s$$
$$V(\varphi_s) = 0, sin\varphi_s = 0$$

The phase stability gives for the synchronous particle:

$$\varphi_s = 0, \text{ below transition}$$
$$\varphi_s = \pi, \text{ above transition}$$

Acceleration