Spinning Black Holes from Scattering Amplitudes

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Based on

Classical Observables from coherent-spin amplitudes

Rafael Aoude and Alexander Ochirov

[hep-th/2108.01649]

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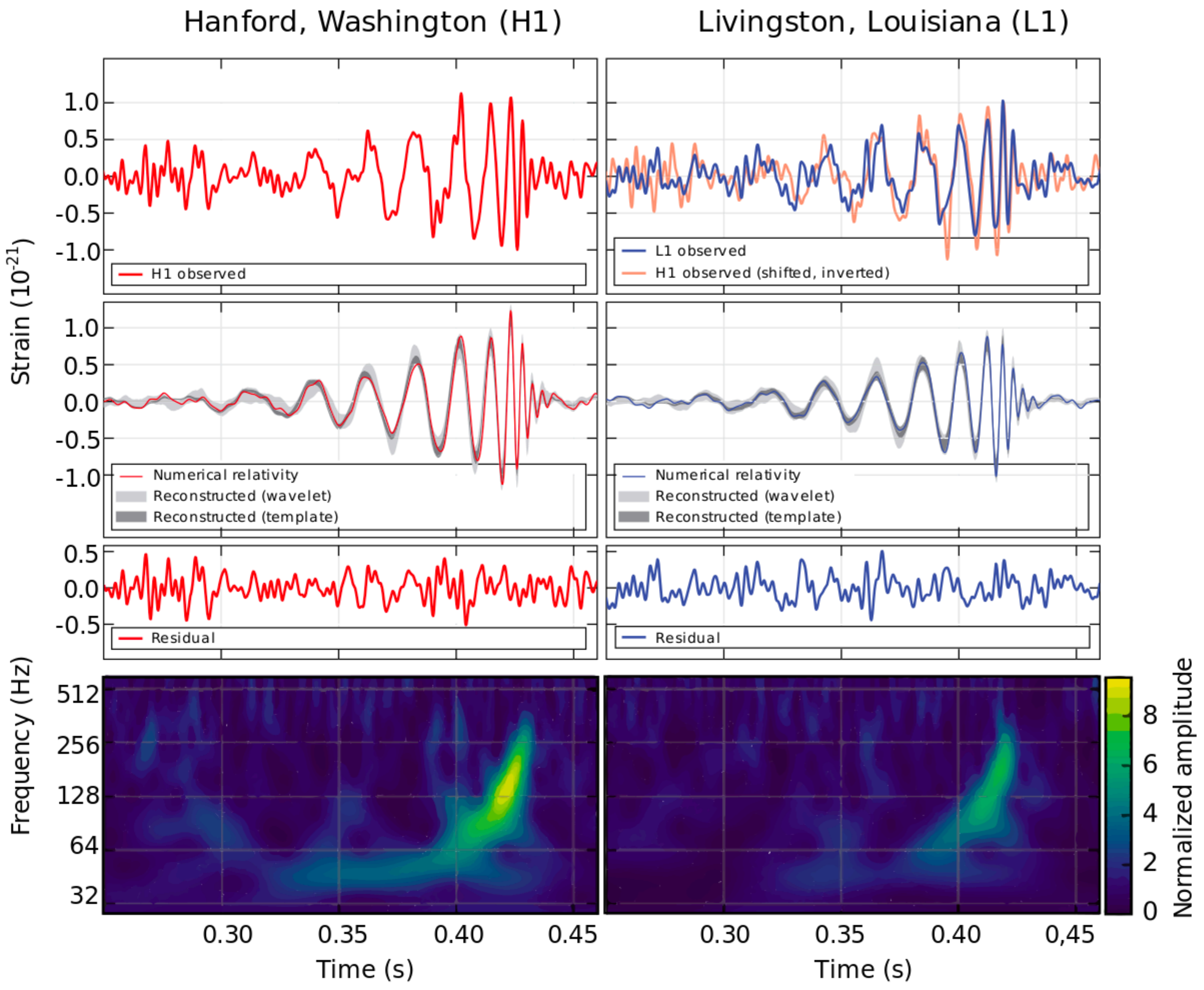
Outline

- Motivation
- Definite-spin amplitudes
- Coherent spin-states
- Coherent scattering amplitudes
- KMOC formalism
- Classical Observables / Hamiltonian from amplitudes
- Conclusion

Motivation

Burst in Gravitational Waves physics…



LIGO/Virgo have accumulated on

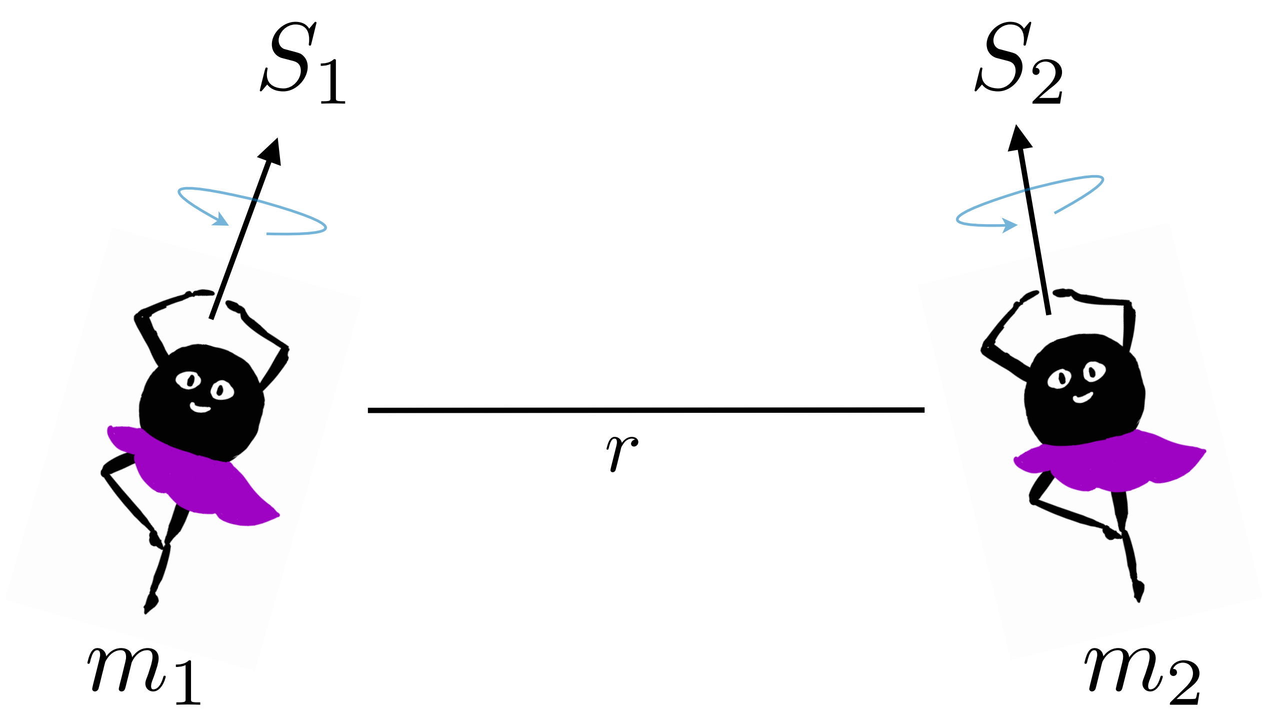
BH - BH merger
BH - NS merger
NS - NS merger

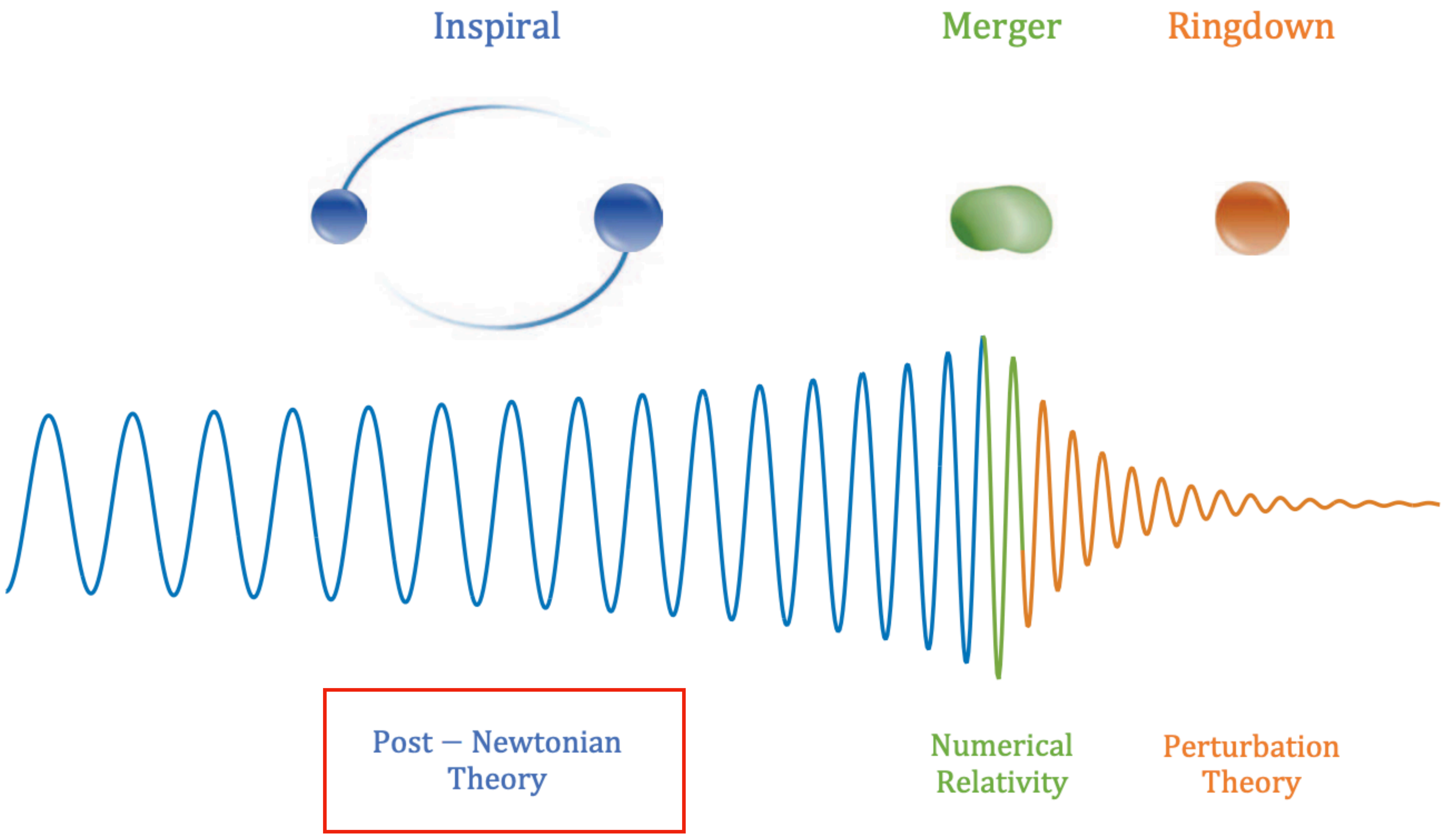
Accurate description of Binary Inspiral dynamics

How can we use QFT methods to describe the binary inspiral problem?

Motivation

[Figure from Antelis and Moreno, [1610.03567]]





Tradicional methods: EOB formalism [Buonanno Damour 99']

Post-Newtonian (PN): $1 \gg \frac{Gm}{r} \sim v^2$

QFT approach:

Post-Minkowskian (PM): $1 \gg \frac{Gm}{r}, \qquad v^2 \sim 1$

Motivation

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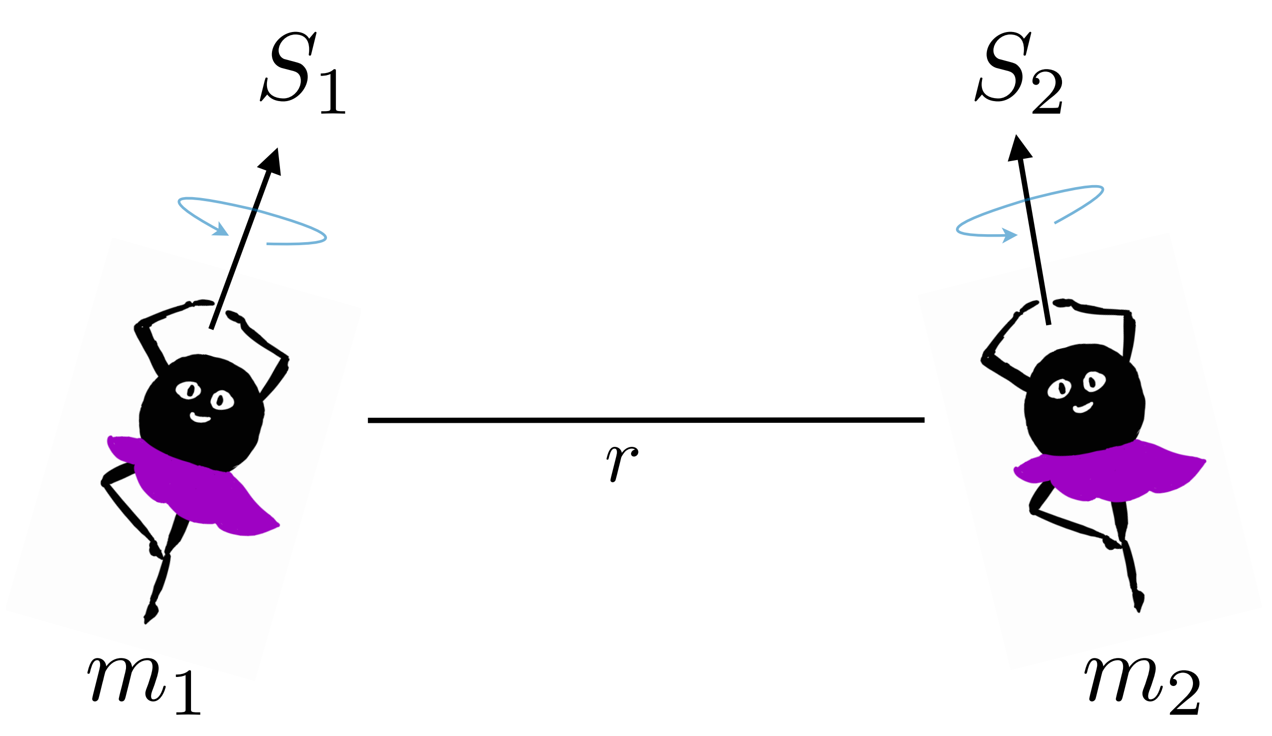

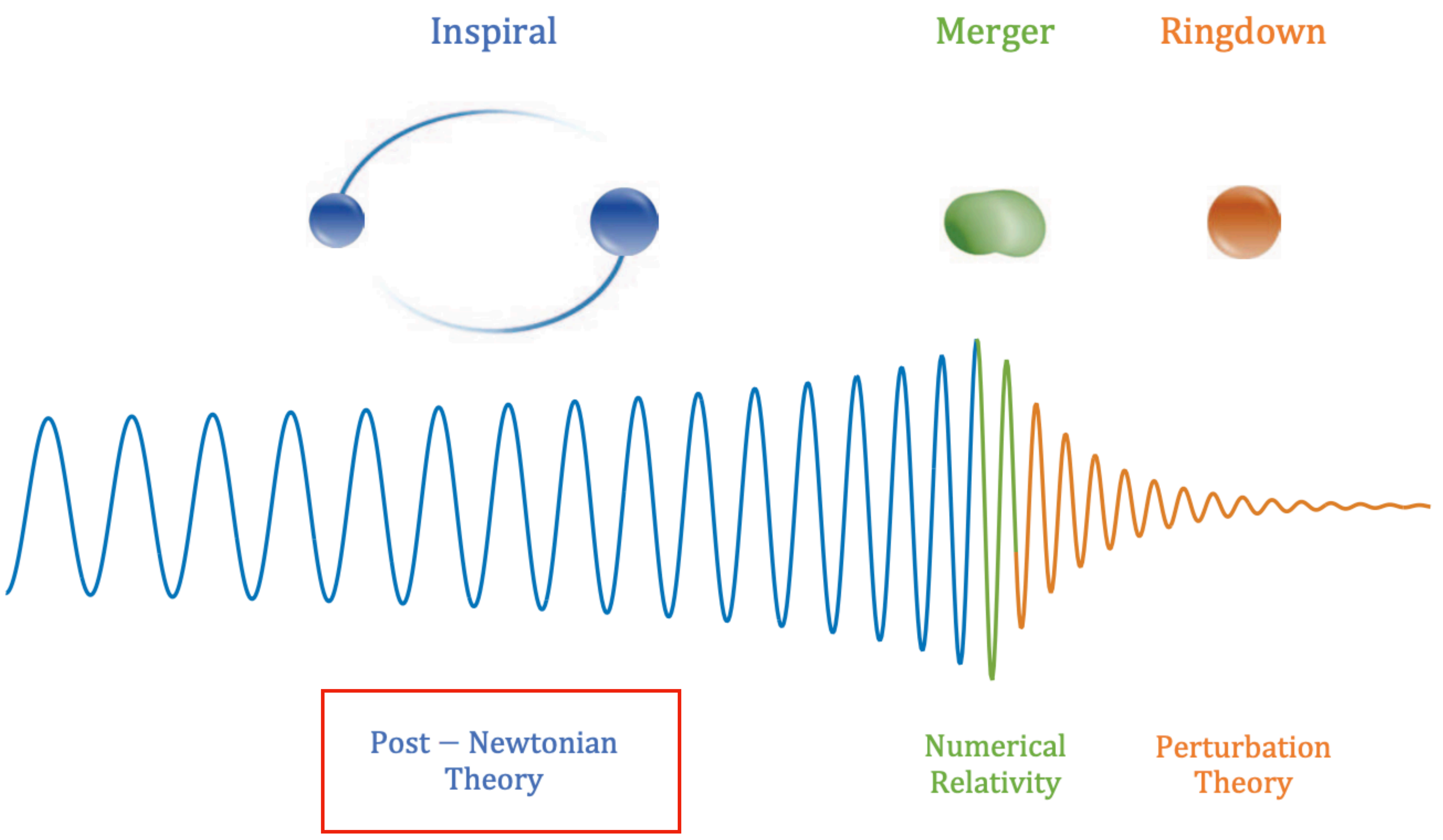

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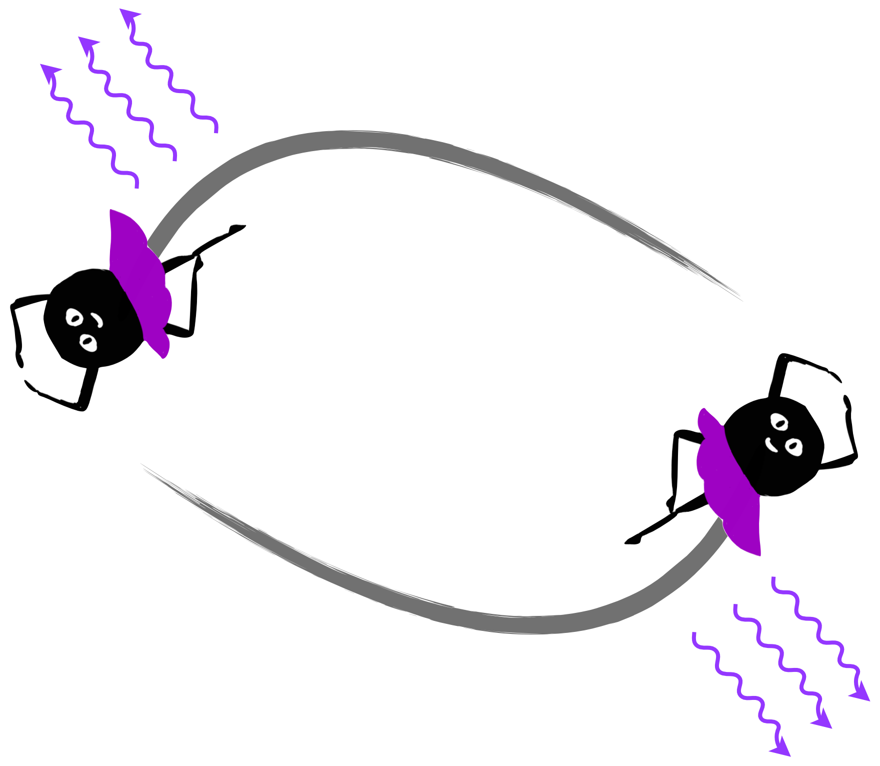
We focus on the scattering at 1PM (tree-level) all-order in spin

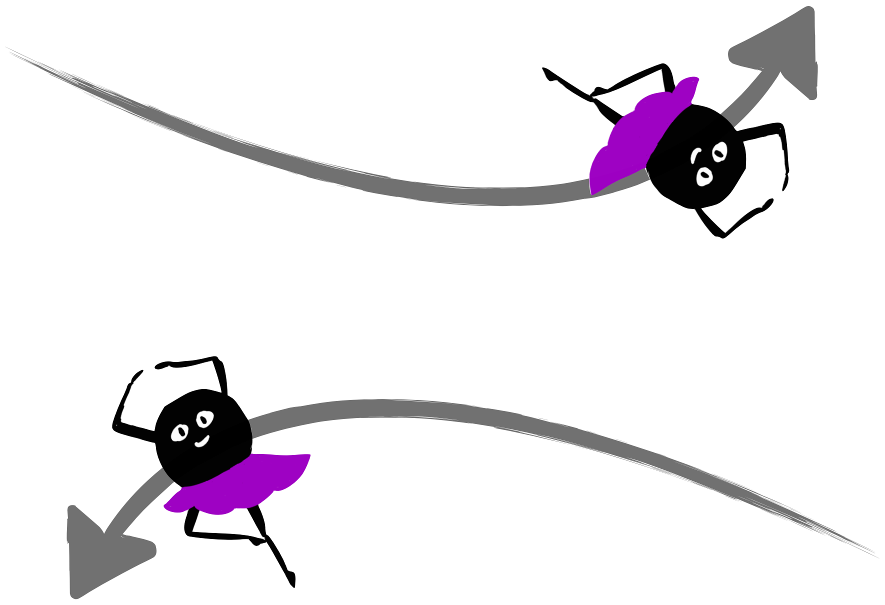
[Cheung, Rothstein, Solon,19’]

From Amplitudes to Hamiltonians (or potentials)

Two-body bounded problem



Scattering problem



Effective theory

Full theory

$$V(p,q)$$

$$A_{\text{full}}$$

$$\hbar \to 0$$

Matching

$$A_{\text{EFT}}(p,q) = A(p,q)$$

Buonanno's slide at Gravitational scattering, inspiral and radiation 2021

Comparison between PMs and NR binding energies

- 2-body non-spinning (local-in-time) **Hamiltonian at 4PM order** computed using scattering-amplitude methods.

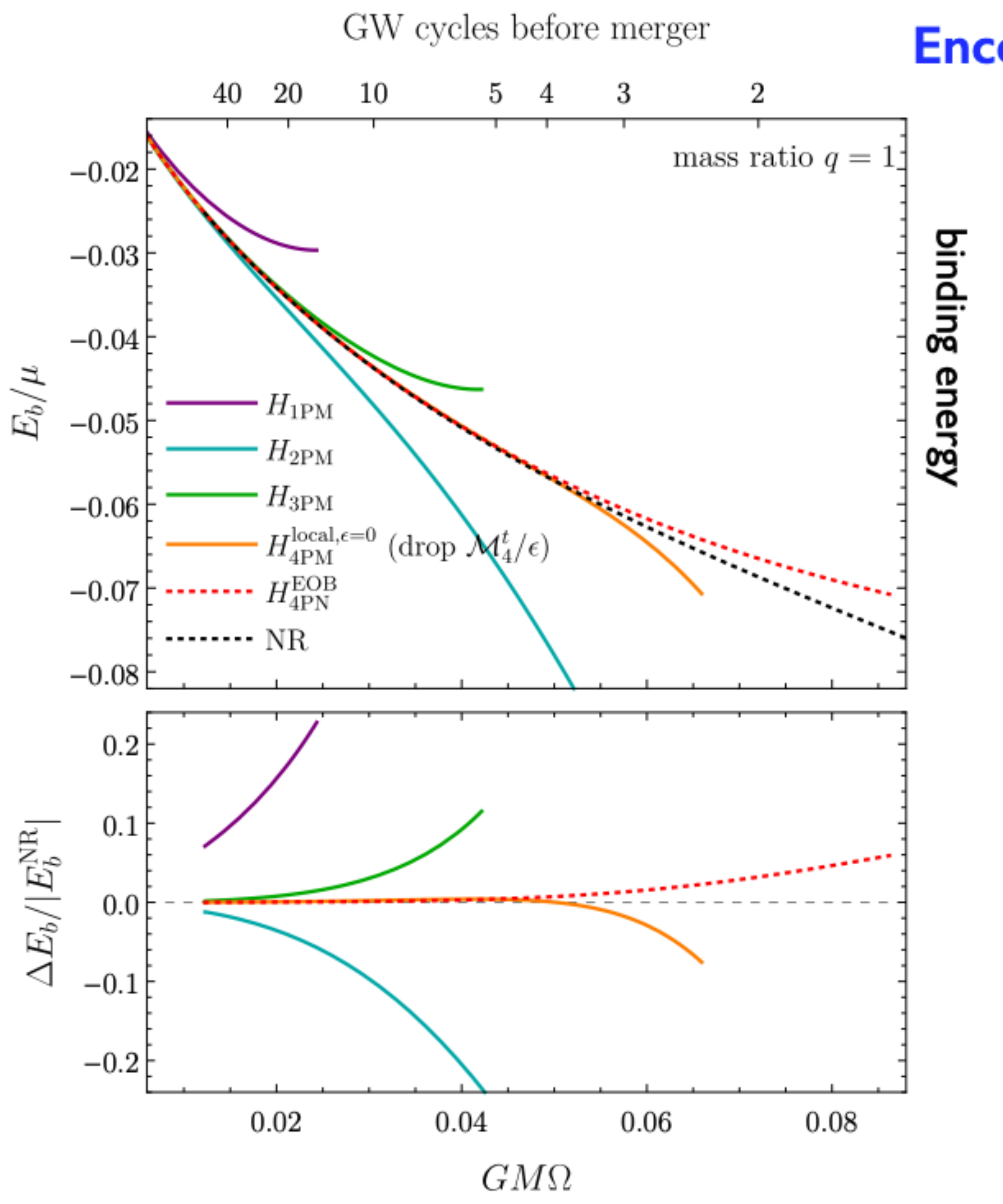
  (Cheung et al. 18, Bern et al. 19, Bern et al. 21)

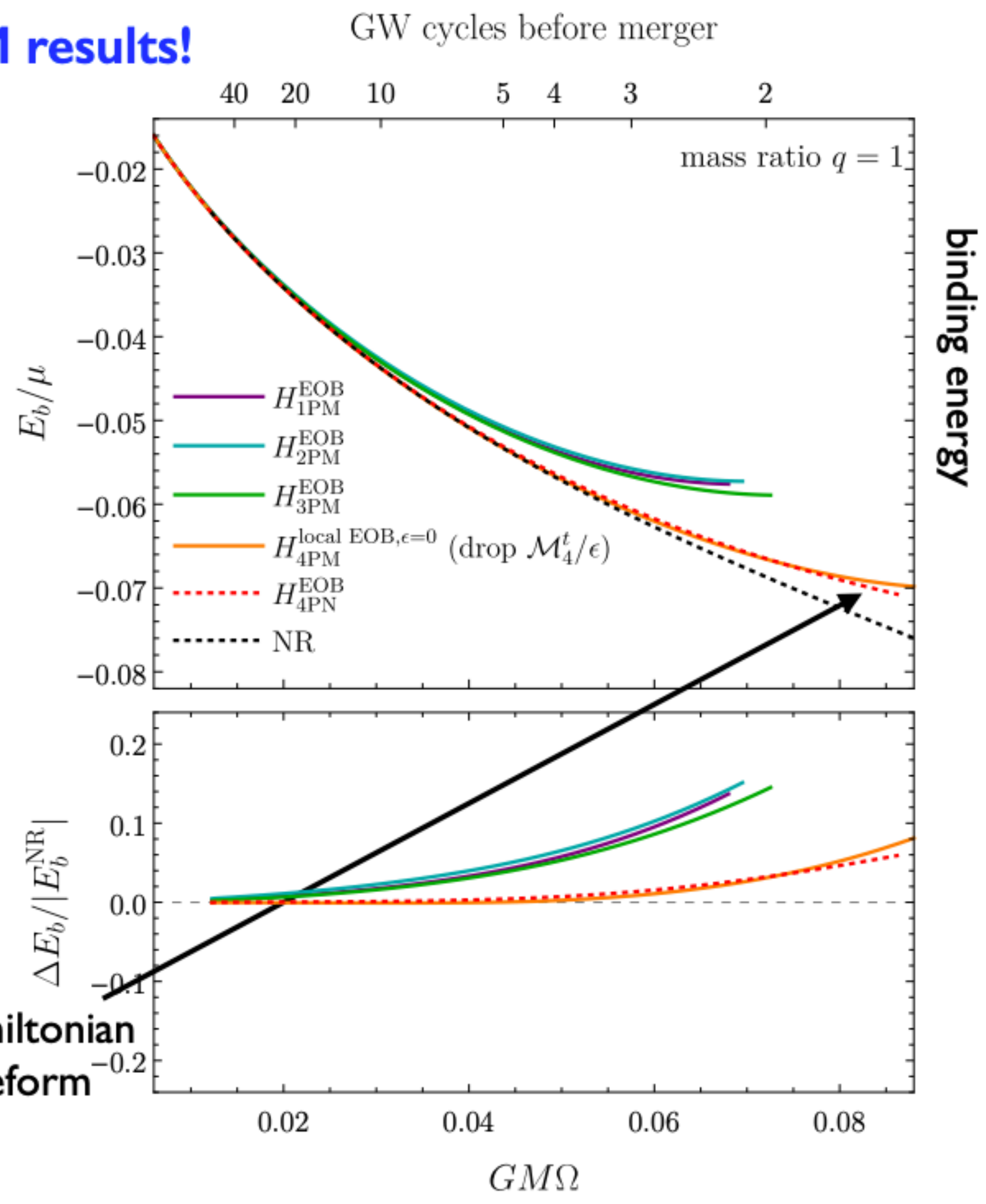
- **Crucial** to **push PM** calculations at **higher order,** and **resum them** in EOB formalism.

  (Damour 19, Antonelli, AB, Steinhoff, van de Meent & Vines 19, Khalil, AB, Steinhoff & Vines in prep 21)

Encouraging (local-in-time) **4PM results!**

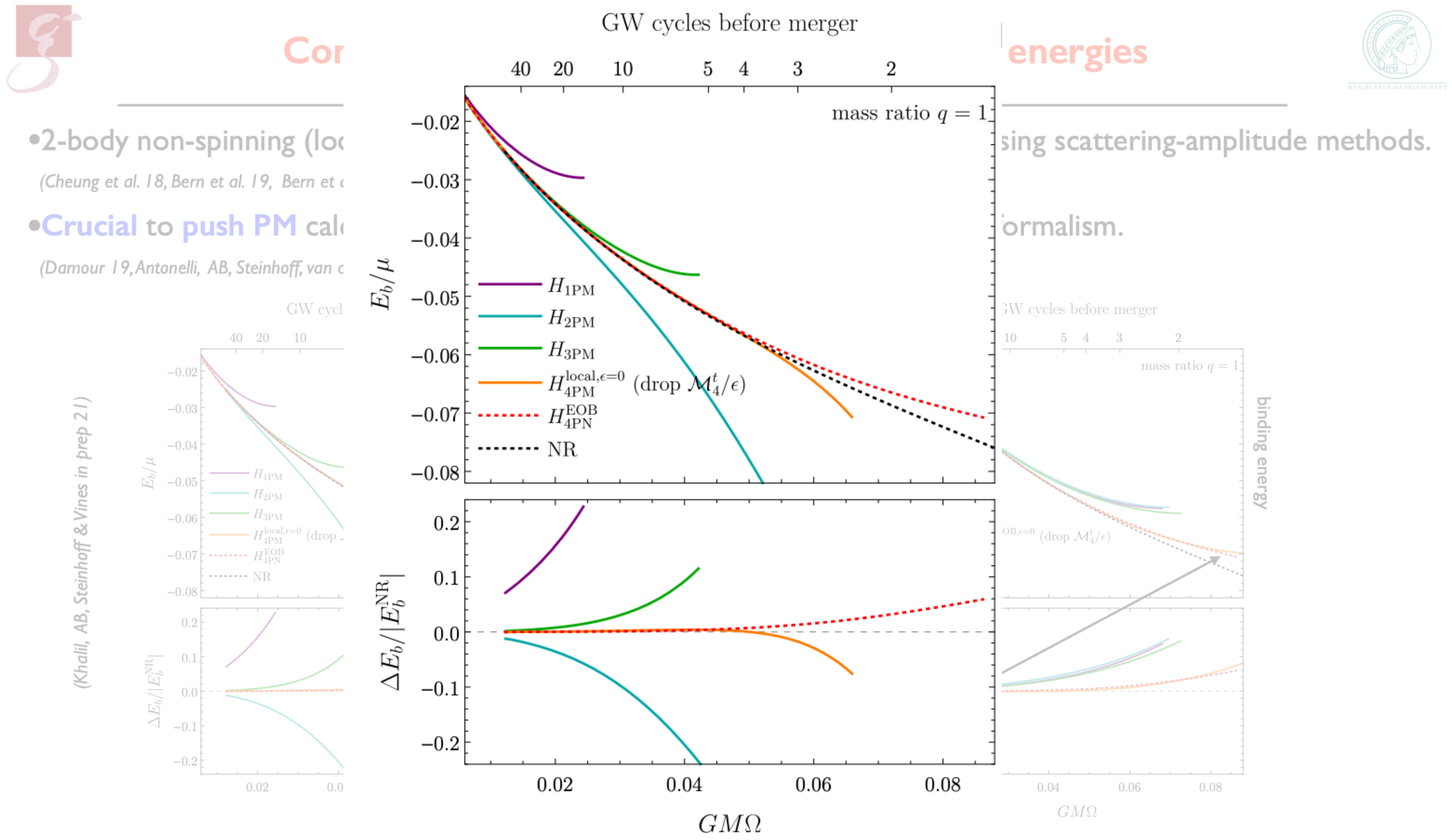
(Khalil, AB, Steinhoff & Vines in prep 21)





current (uncalibrated) Hamiltonian used to build EOBNR waveform models for LIGO/Virgo

Buonanno’s slide at Gravitational scattering, inspiral and radiation 2021

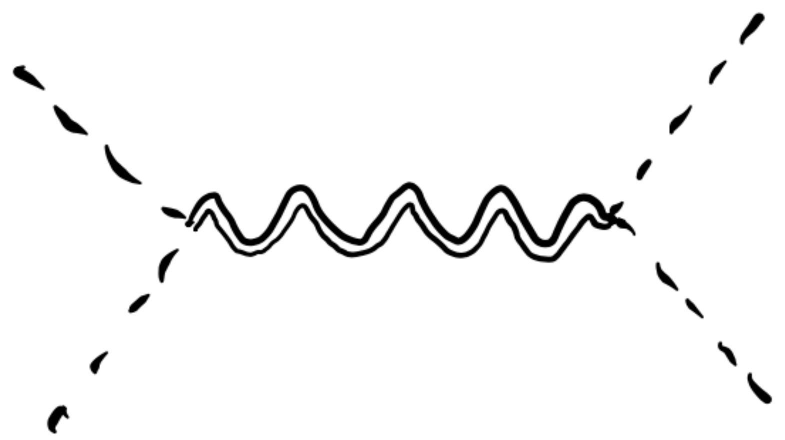


Long Range Gravitational Scattering

[Holstein, Ross 08']

$$V_G^{(1)}(\vec{r}) = -\int \frac{d^3q}{(2\pi)^3}\mathcal{M}(\vec{q})e^{-i\vec{q}\cdot\vec{r}}$$

spin-0 x spin-0 scattering:



$$^0V_G^{(1)}(\vec{r}) \;=\; -\frac{Gm_am_b}{r}\left[1+\frac{\vec{p}^2}{m_am_b}\left(1+\frac{3(m_a+m_b)^2}{2m_am_b}\right)+\dots\right]$$

(monopole)

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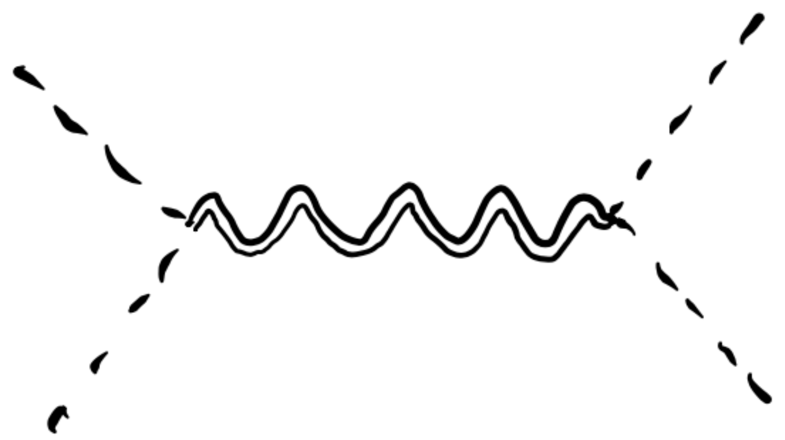

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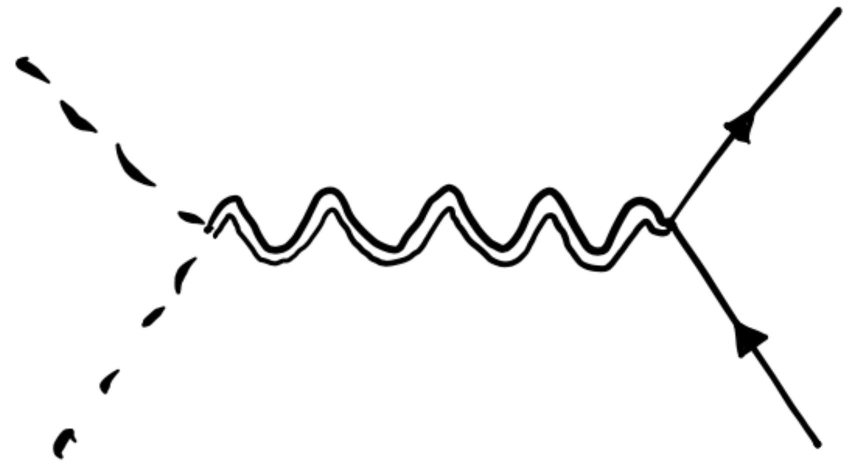
Spin-0 x spin-1/2 scattering:



$${}^{\frac{1}{2}}V_G^{(1)}(\vec{r}) \;=\; -\frac{Gm_am_b}{r}\chi_f^{b\dagger}\chi_i^b+\frac{G}{r^3}\frac{3m_a+4m_b}{2m_b}\vec{L}\cdot\vec{S}_b$$

(dipole/
spin-orbit)

Long Range Gravitational Scattering

[Holstein, Ross 08']

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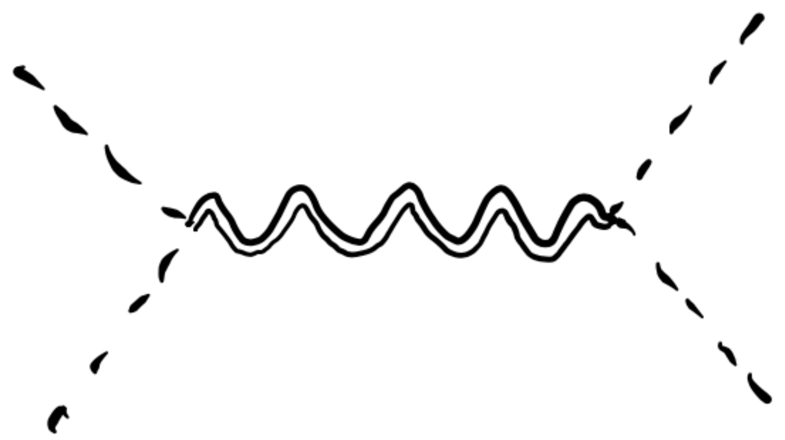

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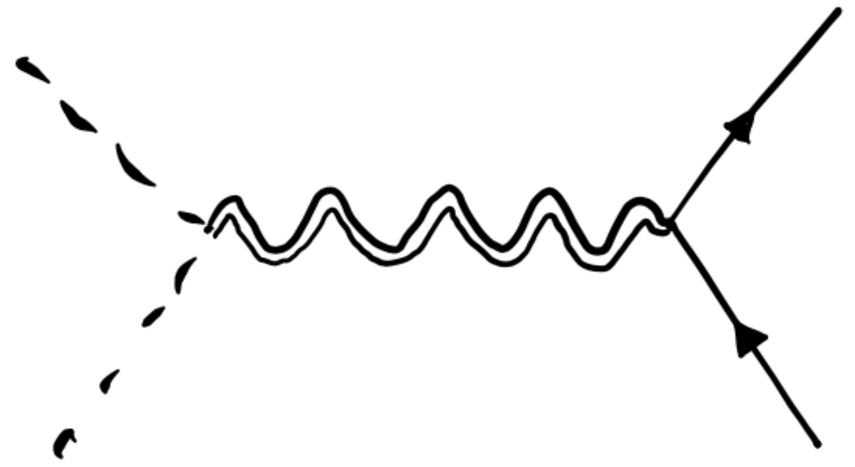

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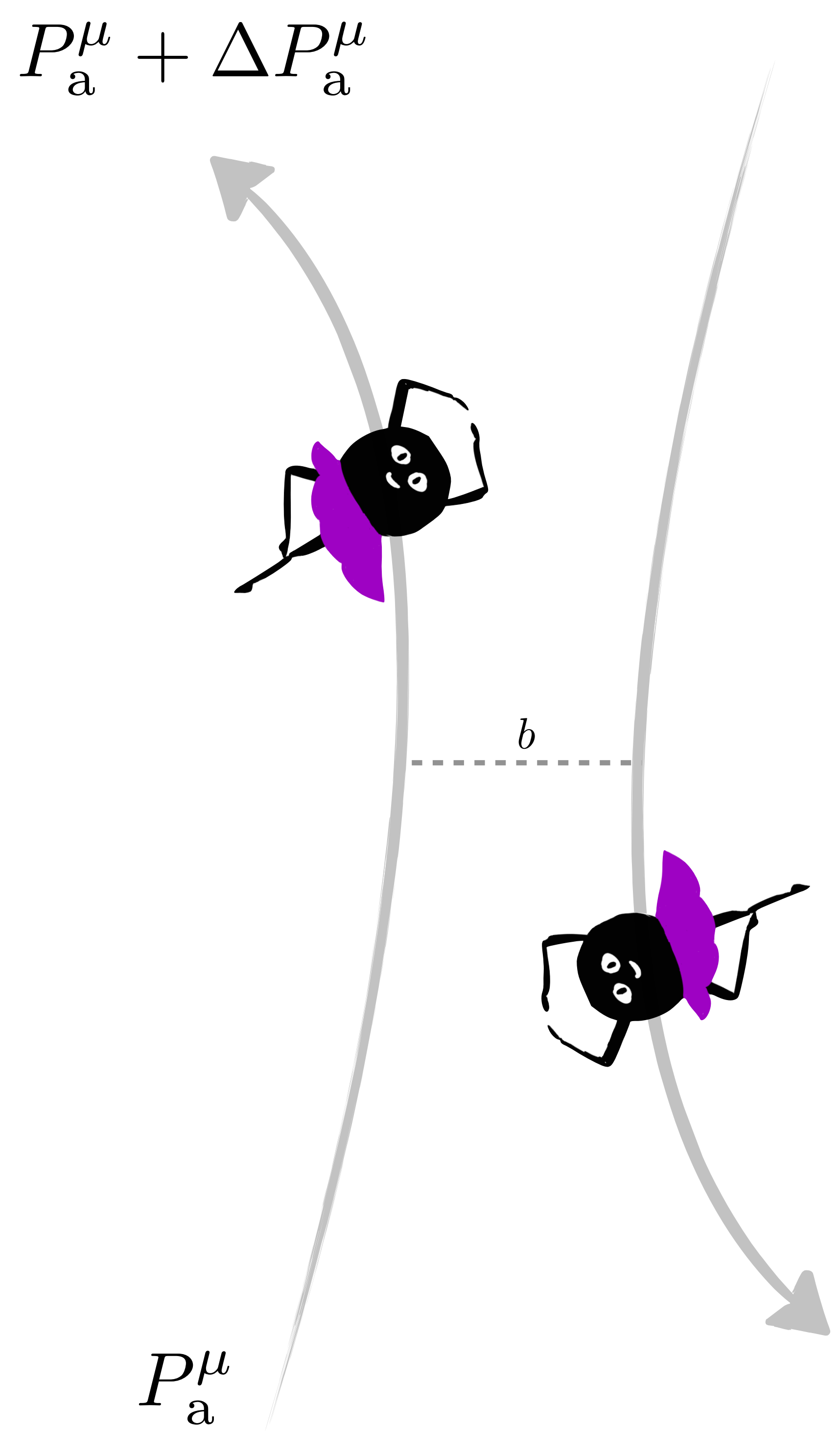
(dipole/ spin-orbit)

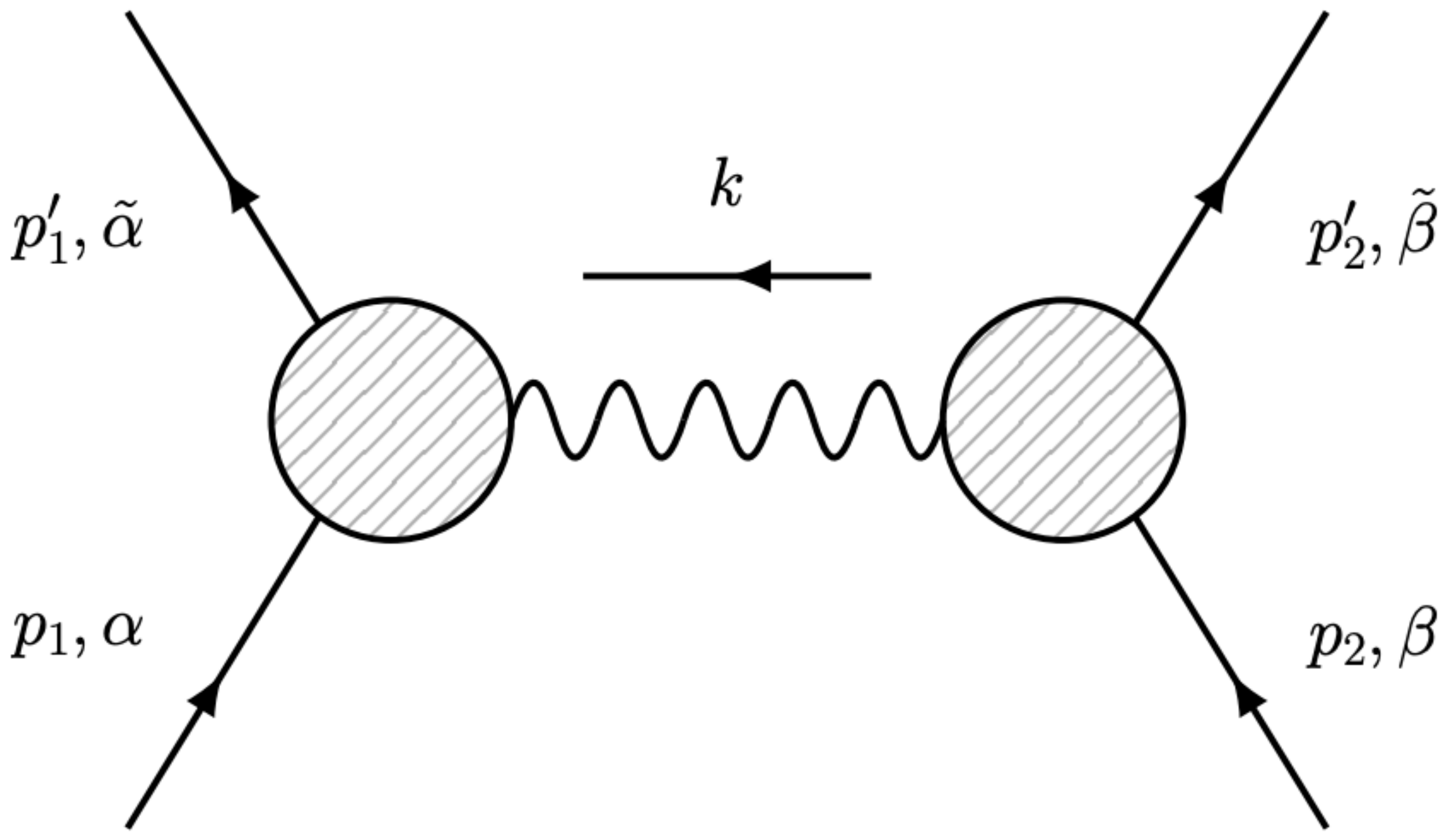
How do we obtain all the multipoles? Scattering observables…

Pictorically…

Classical limit: $\hbar \to 0$



$$\Delta P_{\rm a}^\mu = -\hbar \frac{\partial}{\partial b_\mu} \int_{p_{\rm a}, p_{\rm b}} |\psi_{\rm a}(p_{\rm a})|^2 |\psi_{\rm b}(p_{\rm b})|^2$$



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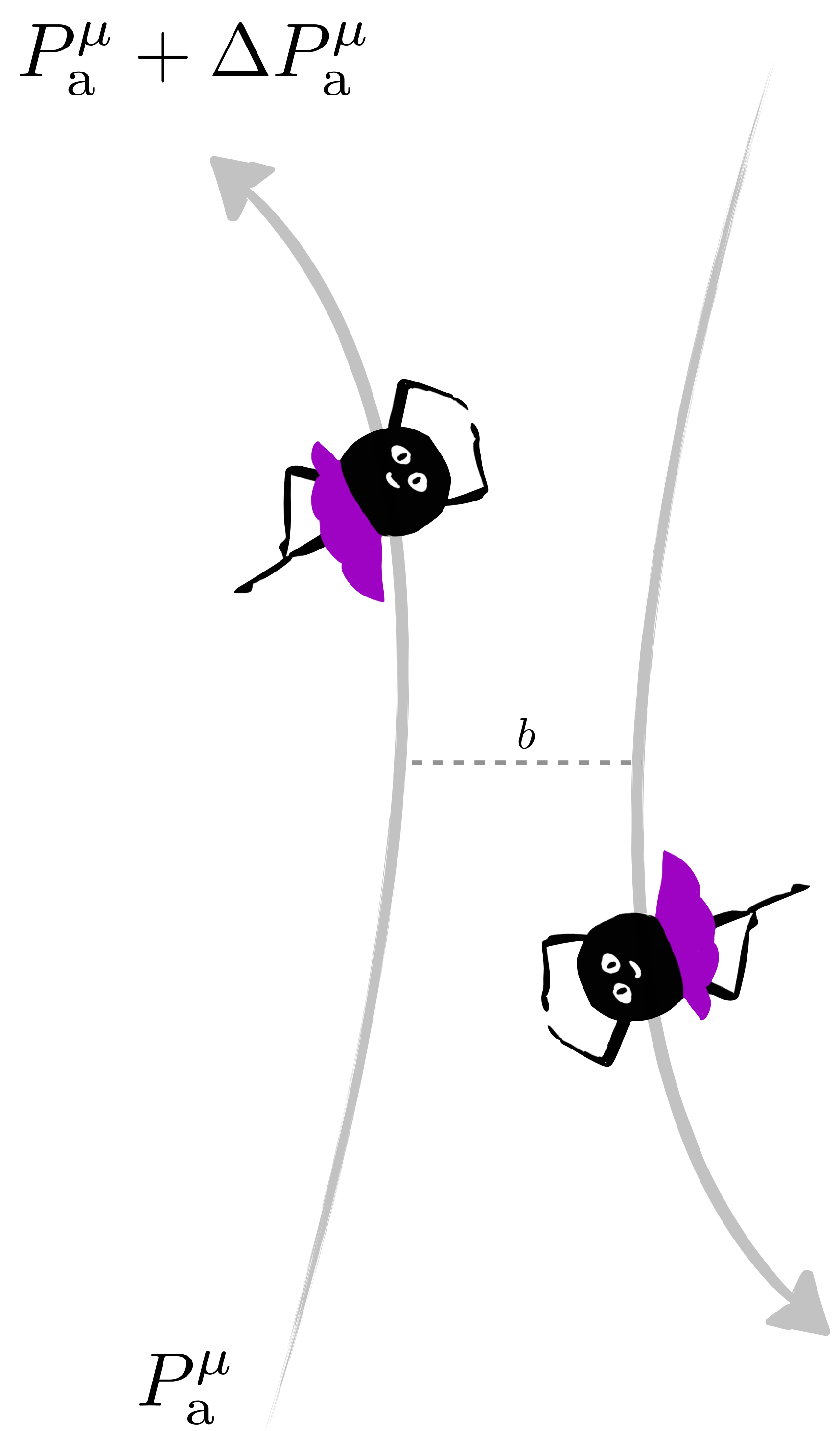

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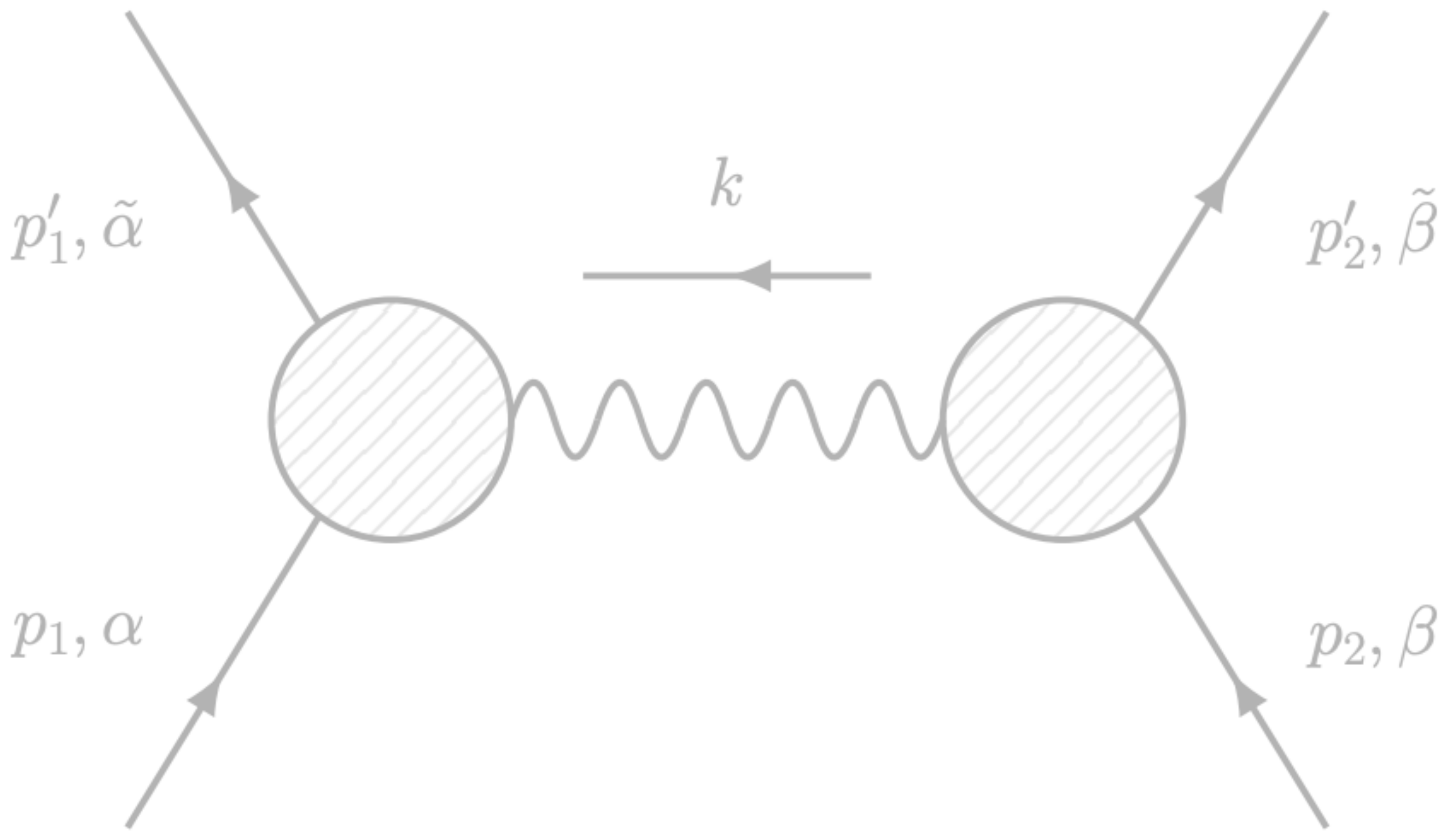
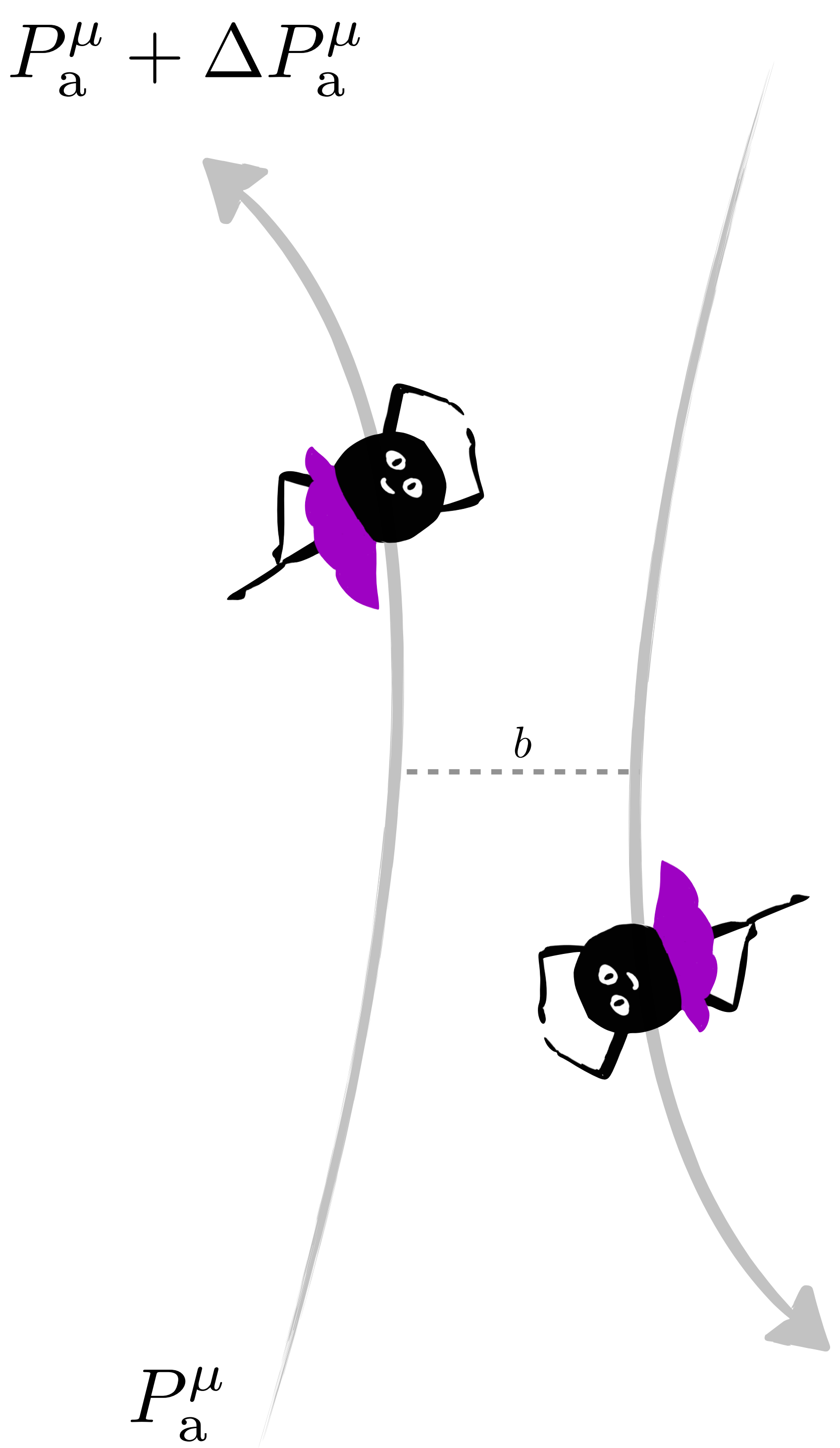


The KMOC formalism:
- quantum expectation values
- chosen initial quantum states
- classical observables when $\hbar \to 0$

Pictorically…

Classical limit: $\hbar \to 0$



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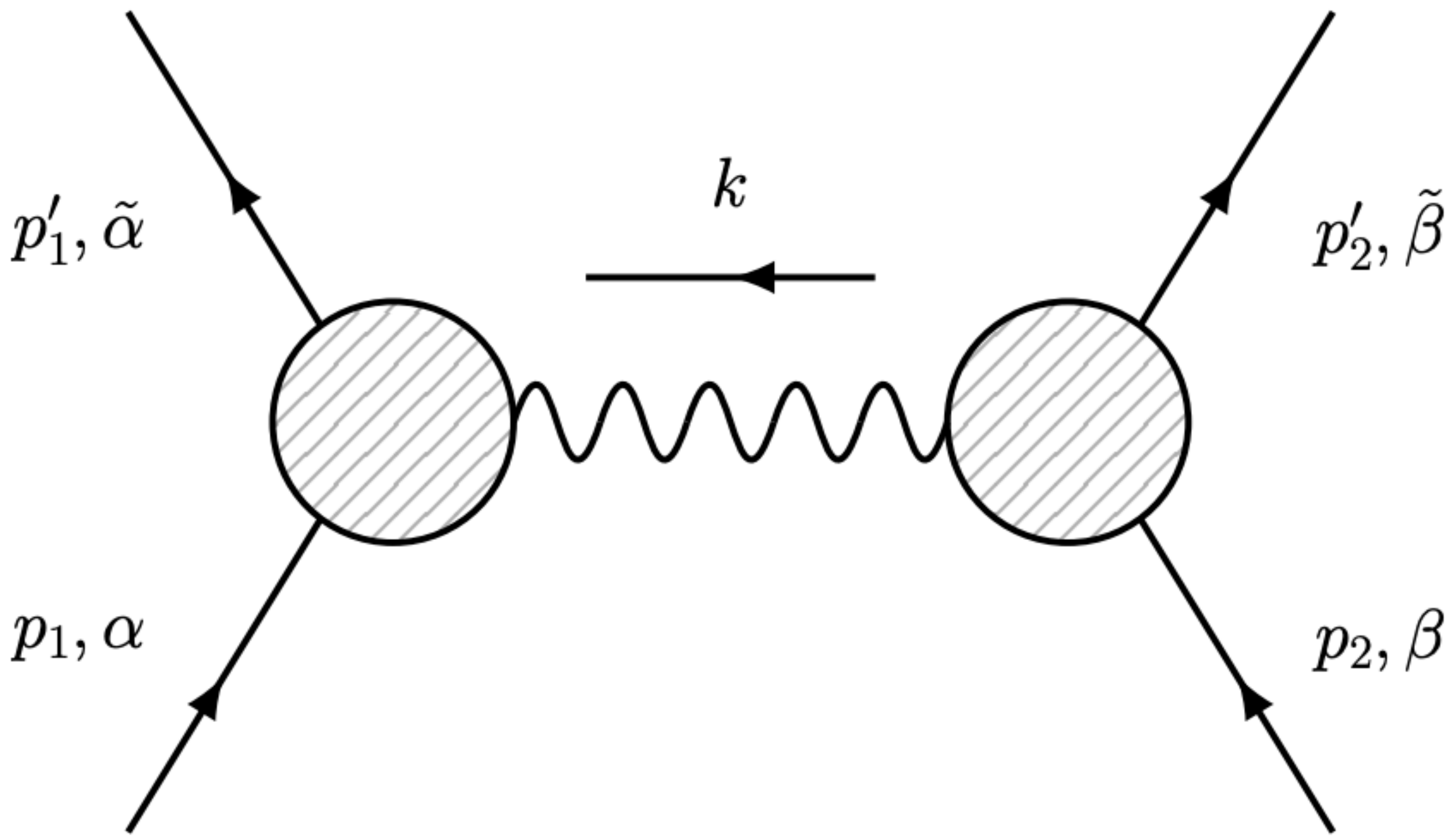
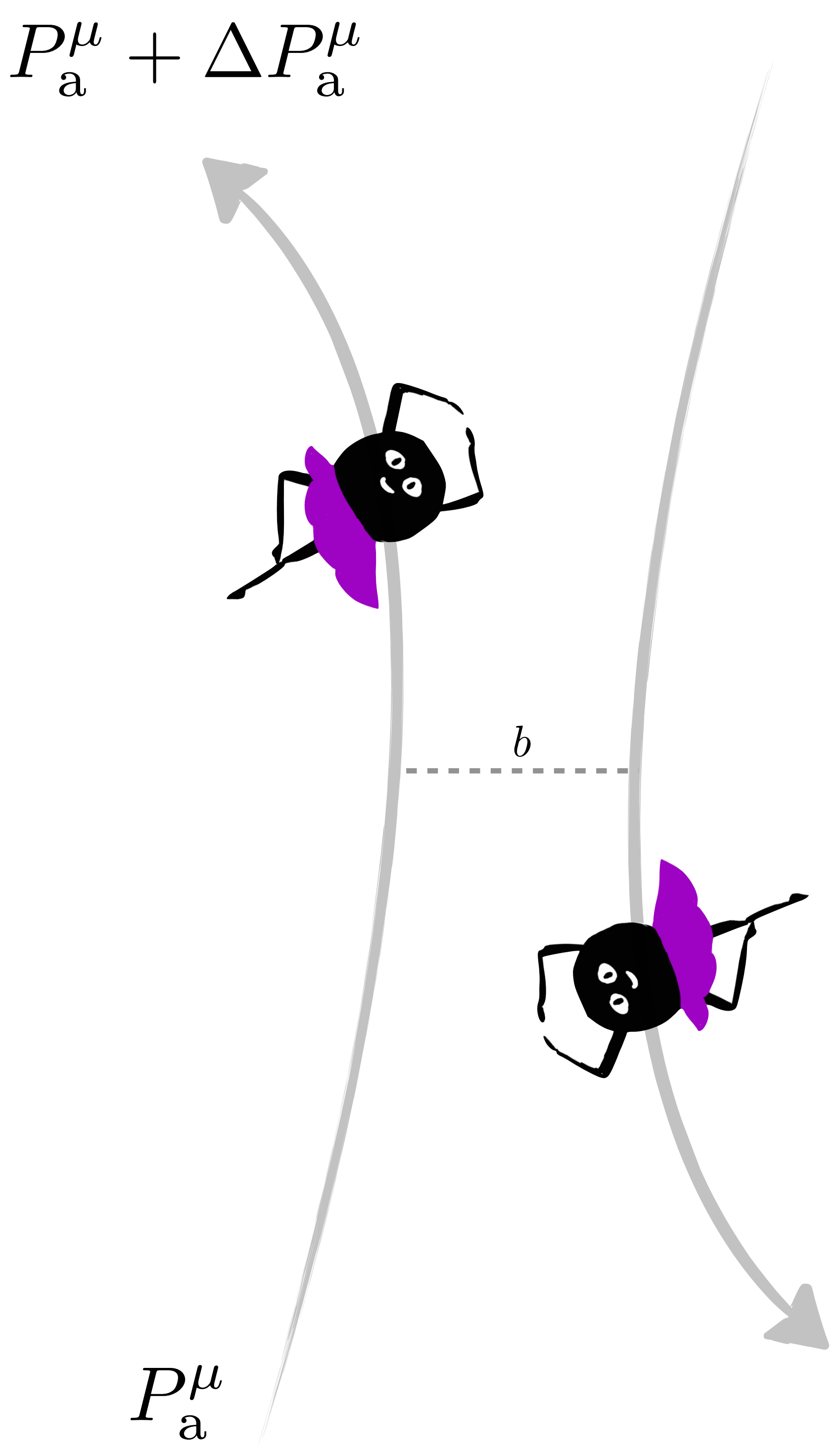


Scattering of two coherent-spin states mediated by a graviton

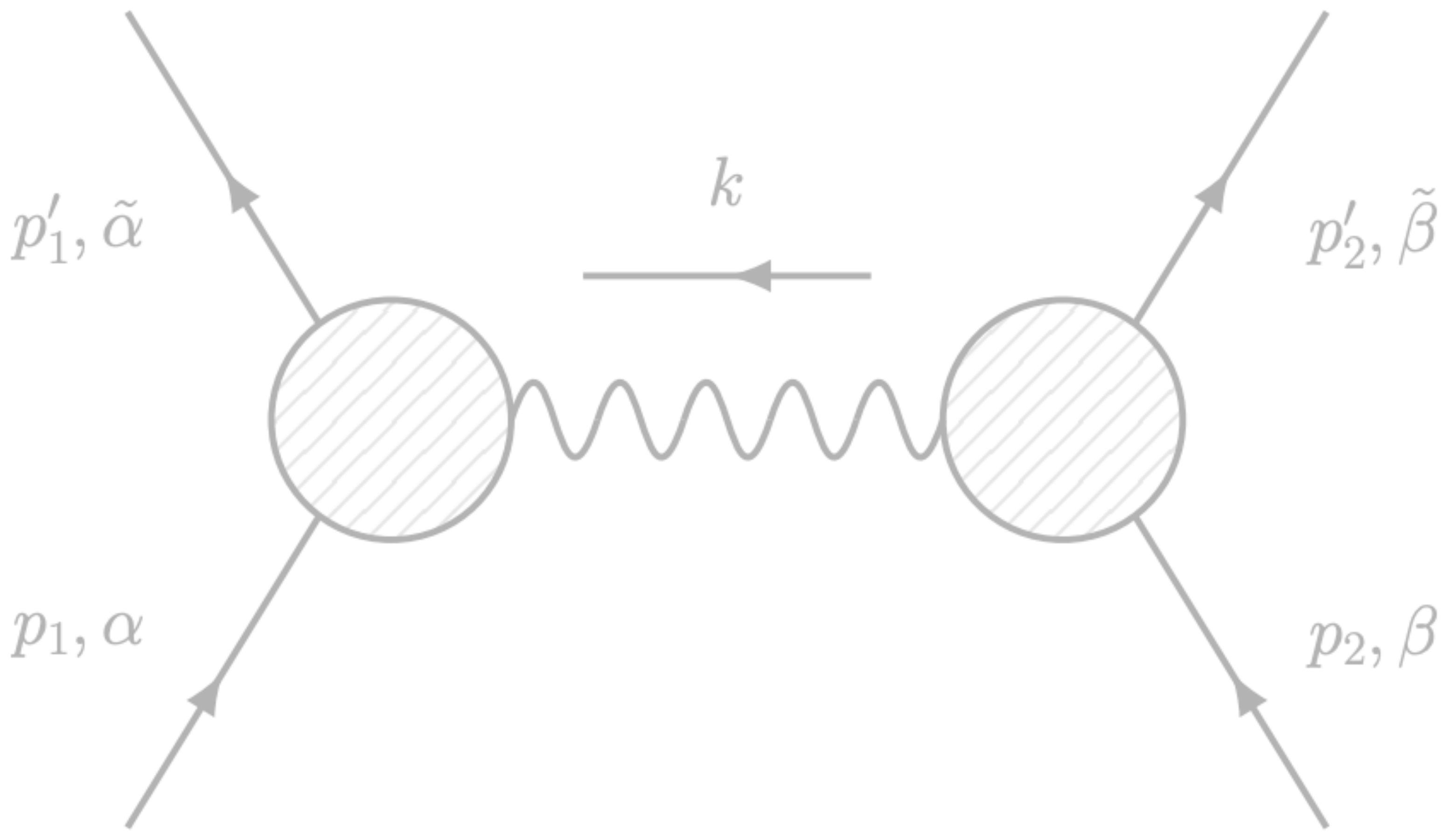
Factorizes into two three-points.

Pictorically…

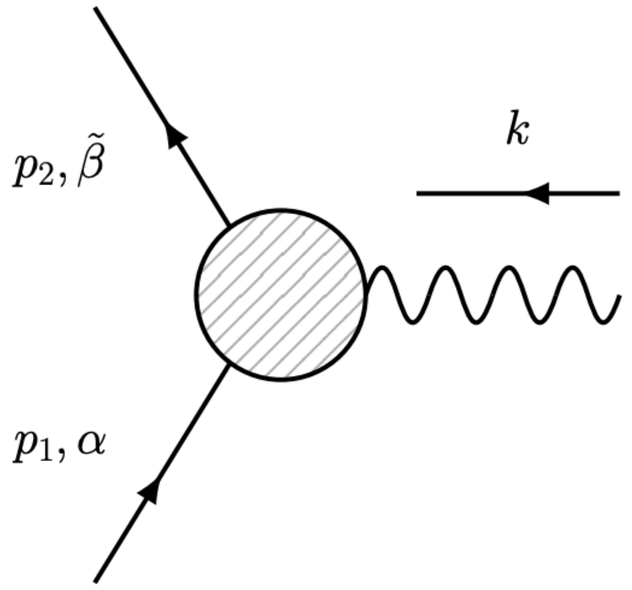


Classical limit: $\hbar \to 0$

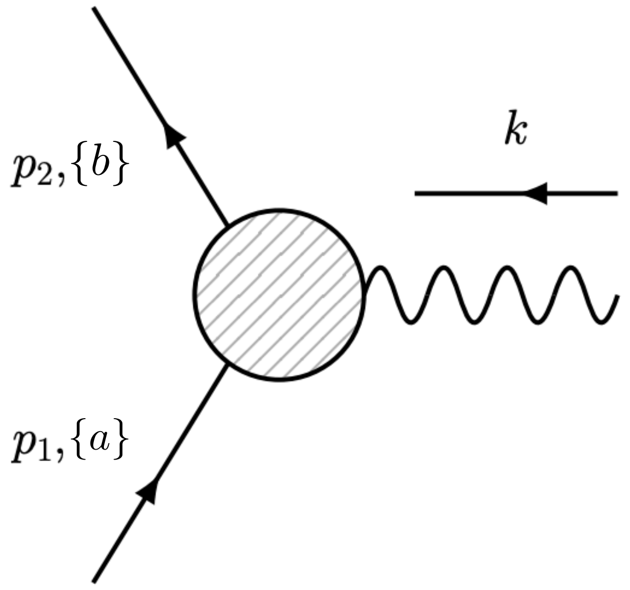
$$\Delta P_a^\mu = -\hbar \frac{\partial}{\partial b_\mu} \int_{p_a, p_b} |\psi_a(p_a)|^2 |\psi_b(p_b)|^2$$



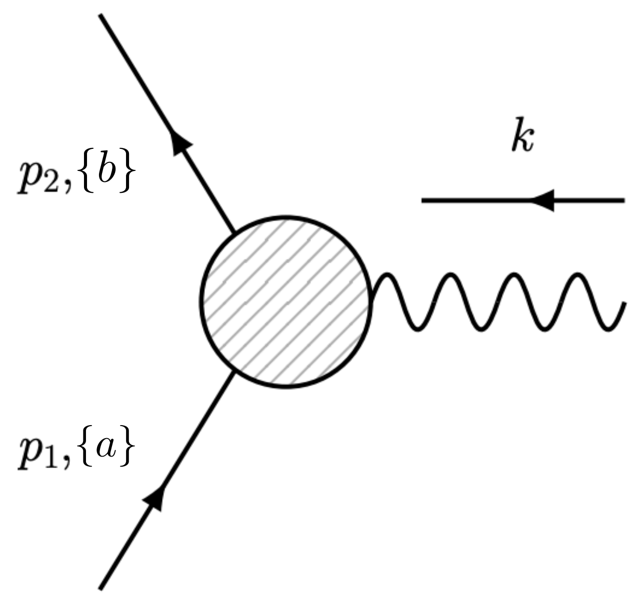
Coherent amplitude as a coherent sum of definite-spin amplitudes



$$= e^{-(\|\alpha\|^2 + \|\beta\|^2)/2} \sum_{s_1, s_2} \frac{(\tilde{\beta}_b)^{\odot 2s_2} (\alpha^a)^{\odot 2s_1}}{\sqrt{(2s_1)!(2s_2)!}} \cdot$$



Definite-spin amplitudes



Why do we use the spinor-helicity formalism?

- Off-shell Feynman Rules:

Four-momenta, polarization vectors/tensors and Dirac spinors

p_i^μ $\varepsilon^\mu(p_i)$ $\varepsilon^{\mu\nu}(p_i)$ $\bar{v}^b(p_i)$ $u^a(p_i)$

Gauge-dependent terms, uses the SO(1,3) Lorentz group

Difficult to go to higher-spins

- On-shell amplitudes:

spinor-helicity building blocks $\langle i^a j^b \rangle$ $[i^a j^b]$

Gauge-independent terms, uses particles' little-group

U(1) massless

SU(2) massive

Definite-spin amplitudes - Spinor-helicity formalism

[Arkani-Hamed, Huang, Huang 2017]

[Ochirov 2018]

Little-group: SU(2) labels $a, b = 1, 2$

Split the four-momenta into two Weyl spinors

$$p_{\alpha\dot{\beta}} = p_\mu \sigma^\mu_{\alpha\dot{\beta}} = \lambda^a_{p\alpha} \epsilon_{ab} \tilde{\lambda}^b_{p\dot{\beta}} \equiv |p^a\rangle_\alpha [p_a|_{\dot{\beta}}$$

Spin-1/2

$$u^{Aa}_p = \begin{pmatrix} |p^a\rangle \\ |p^a] \end{pmatrix}$$

Spin-1

$$\varepsilon^{ab}_{p\mu} = \frac{i\langle p^{(a}|\sigma_\mu|p^{b)}]}{\sqrt{2}m}$$

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Similar for massless. Little group U(1)

$$k_{\alpha\dot{\beta}} = k_\mu \sigma^\mu_{\alpha\dot{\beta}} = |k\rangle_\alpha [k|_{\dot{\beta}},$$

Spin-1

$$\varepsilon^\mu_{p+} = \frac{1}{\sqrt{2}} \frac{\langle q|\sigma^\mu|p]}{\langle qp\rangle} \qquad \varepsilon^\mu_{p-} = -\frac{1}{\sqrt{2}} \frac{[q|\bar{\sigma}^\mu|p\rangle}{[qp]}$$

Definite-spin amplitudes - Spinor-helicity formalism

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Example (spin-1/2): minimal coupling

$$\mathcal{A}(1^a_\psi, 2^b_\psi, 3^+_\gamma) = i\frac{g}{\sqrt{2}}\bar{v}^a_1\gamma^\mu u^b_2\,\varepsilon^+_\mu(q) \quad \to \quad ig\,x\,\langle 1^a 2^b\rangle$$

where

$$x = \frac{\langle q|p_1|3]}{m\langle 3q\rangle} = -\frac{\sqrt{2}}{m}(p_1\cdot\varepsilon^+_3)$$

Two massive vectors couplings

[Arkani-Hamed, Huang, Huang 2017]

Two massive spin-1 and a graviton: $$\mathcal{A}^{\{b\}}_{\min\{a\}} = -\frac{\kappa}{2}\langle 2^b 1_a\rangle^{\odot 2} x^2$$

$WW\gamma$ in the SM $$\mathcal{A}^{\{b\}}_{\min\{a\}} = -\frac{2\sqrt{2}s_\theta}{m_W\, v}\langle 2^b 1_a\rangle^{\odot 2} x$$

$\{a\} = \{a_1, a_2\}$

$\odot 2$ symmetrization

Definite-spin scattering amplitudes

[Arkani-Hamed, Huang, Huang 2017]

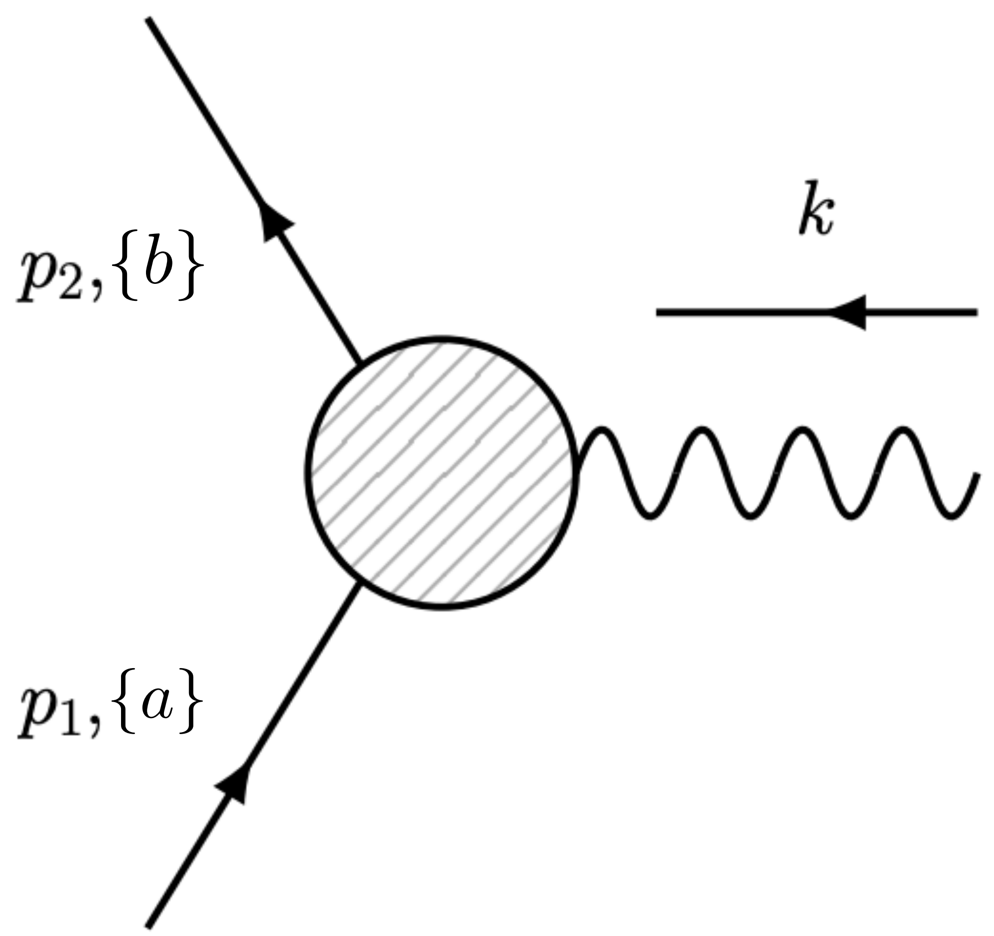
Spin-2s: 2s indices

$\{a\} = \{a_1, a_2, ..., a_{2s}\}$

$\odot 2s$ symmetrization

Using the particles' little-group: minimal coupling with a graviton

Best behavior in the high-energy limit



$$\mathcal{A}^{(0)\{b\}}_{\text{min}\;\{a\}}(p_2, s|p_1, s; k, +) = -\frac{\kappa}{2}\frac{\langle 2^b 1_a\rangle^{\odot 2s}}{m^{2s-2}}x^2,$$

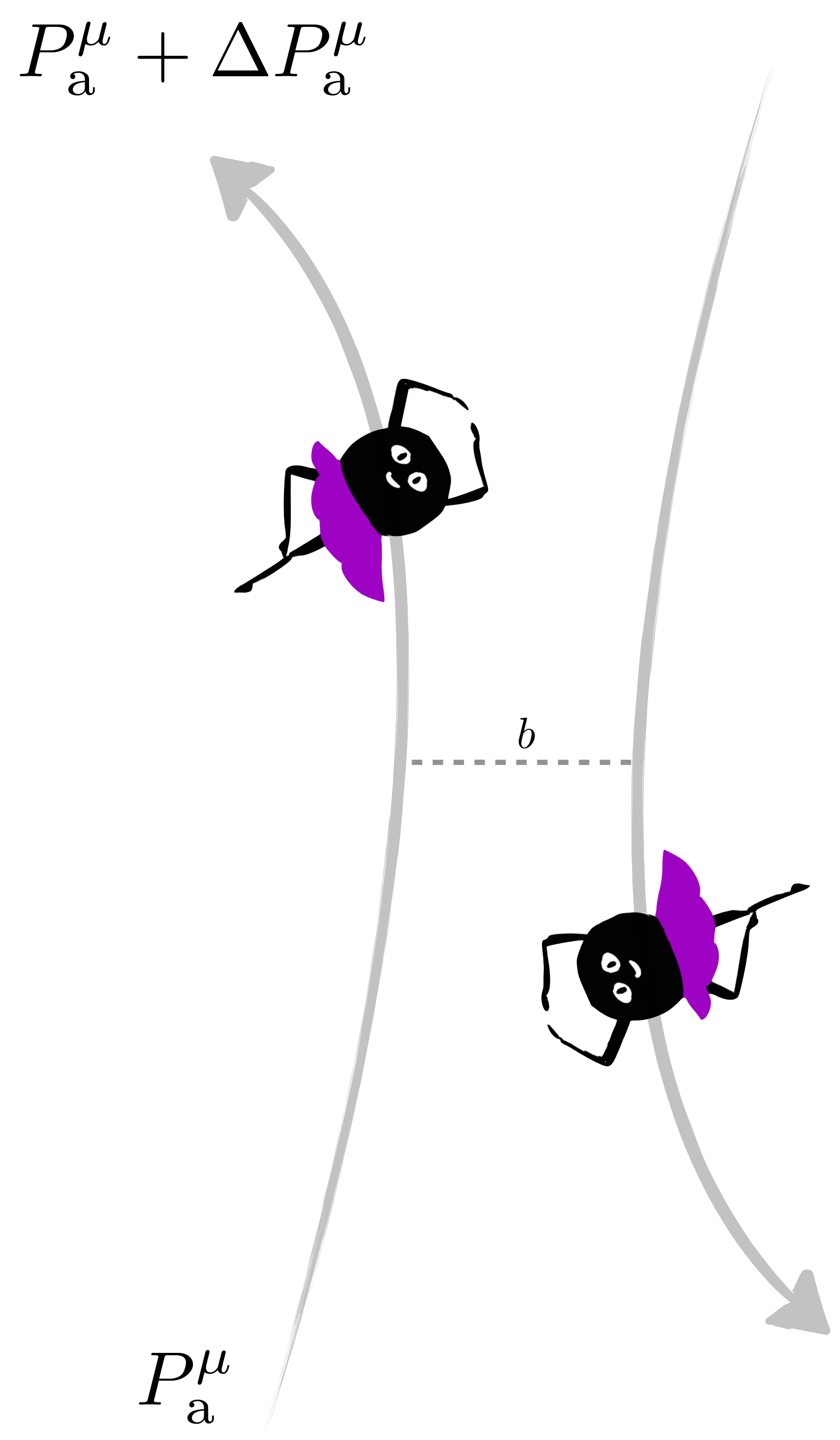
$$x = \frac{[k|p_1|r\rangle}{m\langle k\,r\rangle} = \frac{m[k\,r]}{\langle k|p_1|r]} = -\frac{\sqrt{2}}{m}(p_1\cdot\varepsilon^+) = \left[\frac{\sqrt{2}}{m}(p_1\cdot\varepsilon^-)\right]^{-1}.$$

*Non-minimal later!

*Similar for positive helicities

Use definite-spin amplitudes to contract with coherent sates

Pictorically…



Classical limit: $\hbar \to 0$

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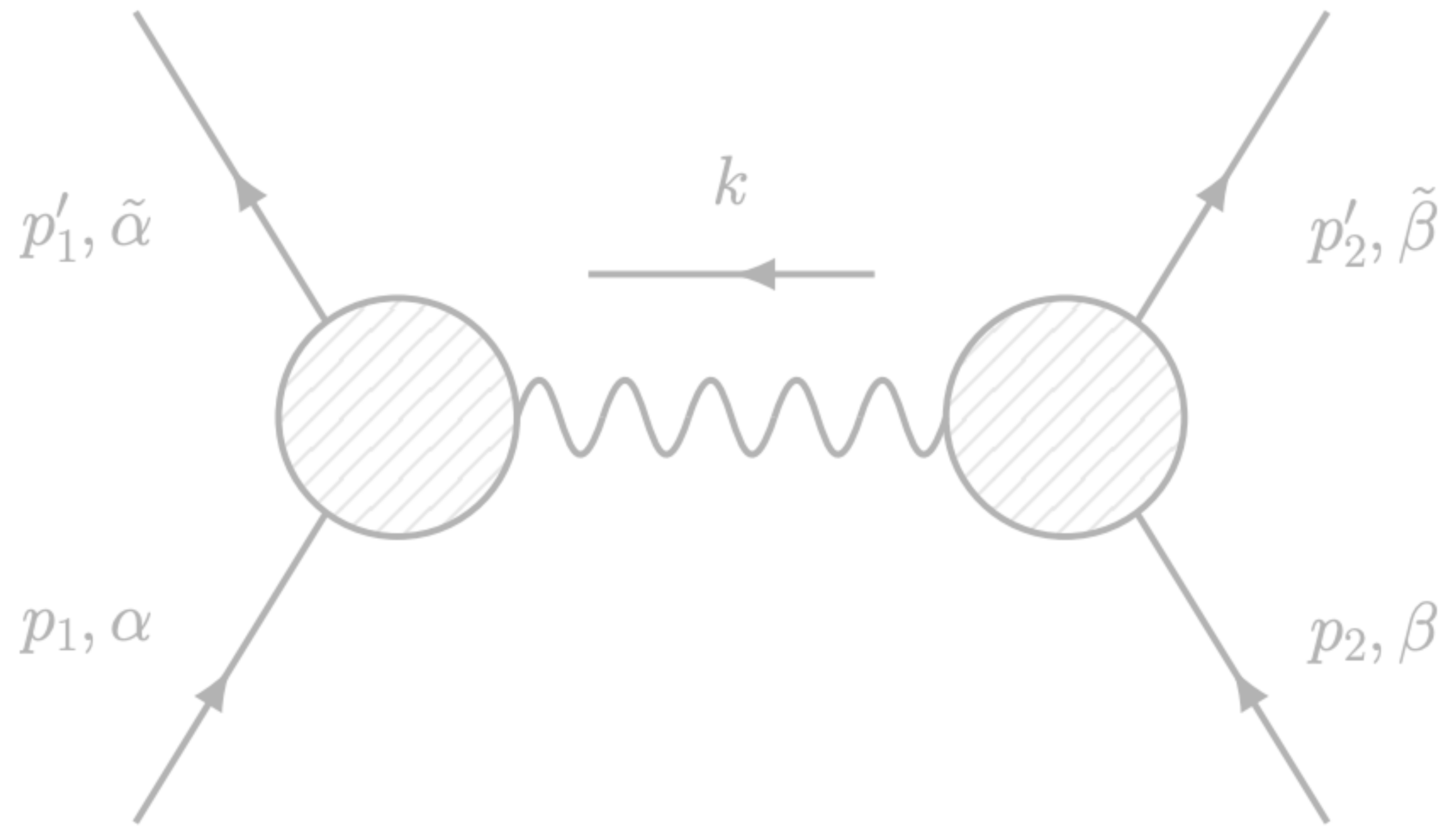
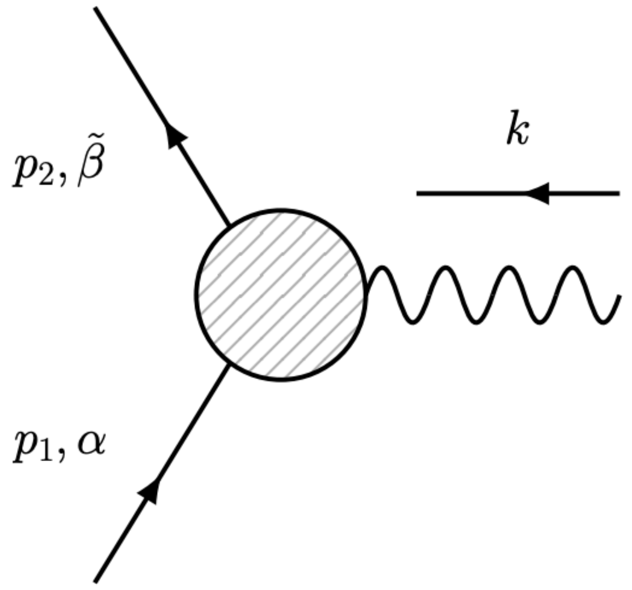
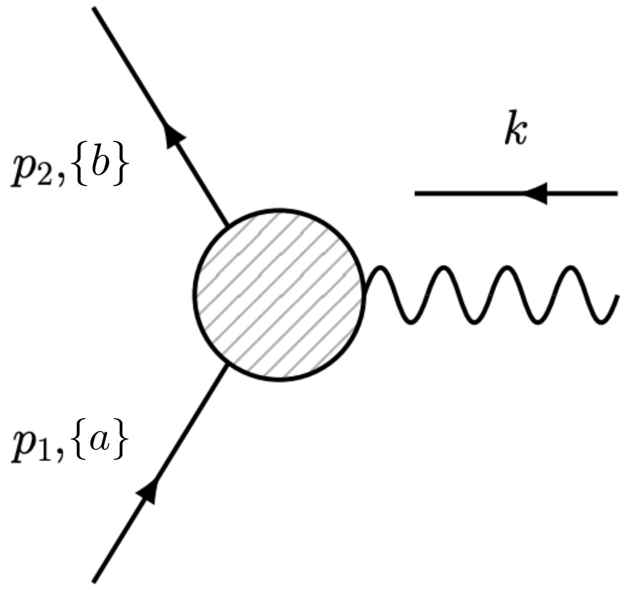


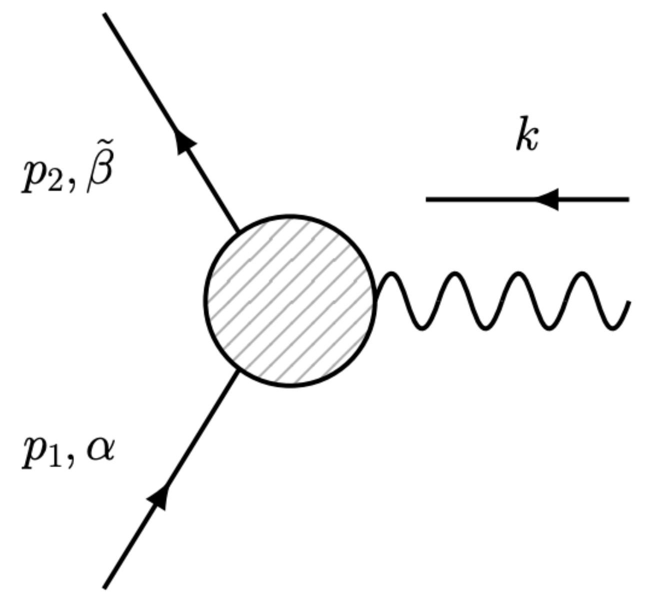
Coherent amplitude as a coherent sum of definite-spin amplitudes



$$= e^{-(\|\alpha\|^2+\|\beta\|^2)/2} \sum_{s_1,s_2} \frac{(\tilde{\beta}_b)^{\odot 2s_2} (\alpha^a)^{\odot 2s_1}}{\sqrt{(2s_1)!(2s_2)!}} \cdot$$



Coherent states and
Coherent scattering amplitudes



Why coherent-states?

Provide a rigorous framework for quantum-classical transitions

Schwinger's construction for spin-coherent sates
[Schwinger 1952]

SU(2) ⟷

Massive little-group of definite-momenta amplitudes
[Arkani-Hamed, Huang, Huang 2017]

Contract with the LG index of definite-spin amplitudes

We want to identify the classical spin from spin-coherent states

We employ the KMOC formalism with the aid of coherent-states

Classical coherent states

Quantum Harmonic Oscillator

$$H = \hbar\omega(a^\dagger a + 1/2) \longrightarrow E_n = \hbar\omega(n + 1/2)$$

$\hbar \to 0$, $n \to \infty$ classical limit

Uncertainties: $\Delta_n x = \sqrt{\frac{\hbar}{m\omega}(n+1/2)}$ $\quad \Delta_n p = \sqrt{m\omega\hbar(n+1/2)}$

Finite errors in the classical limit !!

Coherent states: $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \longrightarrow |\alpha\rangle = e^{-|\alpha|^2/2}\, e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}}|0\rangle$

Uncertainties: $\Delta_\alpha x = \sqrt{\frac{\hbar}{2m\omega}}$ $\quad \Delta_\alpha p = \sqrt{\frac{m\omega\hbar}{2}}$

Vanish in the classical limit

$$E_\alpha = \hbar\omega(\,||\alpha||^2 + 1/2)$$

For the energy to be finite $\quad ||\alpha||^2 \to \infty$

in the classical limit

Saturates the uncertainty principle

Expectation values evolve classically

Spin-states

[Schwinger, 1952]

Schwinger's construction: general spin from zero-spin with 2 creation ops.

along the z-axis

$$|s, s_z\rangle = \frac{\left(a_1^\dagger\right)^{s+s_z}\left(a_2^\dagger\right)^{s-s_z}}{\sqrt{(s+s_z)!(s-s_z)!}}|0\rangle, \qquad s_z = -s, -s+1, \ldots, s-1, s.$$

Covariantize it:

$$[a^a, a_b^\dagger] = \delta_b^a, \qquad \boldsymbol{S} = \frac{\hbar}{2} a_a^\dagger \boldsymbol{\sigma}^a{}_b a^b \qquad \Rightarrow \qquad [S^i, S^j] = i\hbar\epsilon^{ijk} S^k.$$

SU(2)-covariant s-spin states

$$|s, \{a\}\rangle \equiv |s, \{a_1 \ldots a_{2s}\}\rangle = \frac{1}{\sqrt{(2s)!}} a_{a_1}^\dagger a_{a_2}^\dagger \ldots a_{a_{2s}}^\dagger |0\rangle \equiv \frac{\left(a_a^\dagger\right)^{\odot 2s}}{\sqrt{(2s)!}}|0\rangle.$$

Coherent Spin-states

Coherent spin-states defined as

$$|\alpha\rangle = e^{-\tilde{\alpha}_a \alpha^a/2} e^{\alpha^a a_a^\dagger}|0\rangle \qquad \Rightarrow \qquad a^a|\alpha\rangle = \alpha^a|\alpha\rangle,$$

In terms of definite spin:

$$|\alpha\rangle = e^{-\tilde{\alpha}_a \alpha^a/2} \sum_{s=0,1/2}^{\infty} \sum_{a_1,\ldots,a_{2s}} \frac{\alpha^{a_1}\cdots\alpha^{a_{2s}}}{\sqrt{(2s)!}}|s,\{a_1\ldots a_{2s}\}\rangle \equiv e^{-(\tilde{\alpha}\alpha)/2} \sum_{2s=0}^{\infty} \frac{(\alpha^a)^{\odot 2s}}{\sqrt{(2s)!}} \cdot |s,\{a\}\rangle,$$

We want the coherent state in terms of definite spin…

because we know the general definite-spin amplitudes

Classical limit and crucial property

Lorentz-covariant SU(2) spin operator

$$\sigma_{p\mu,a}{}^{b} = -\frac{1}{2m}\Big(\langle p_a|\sigma_\mu|p^b] + [p_a|\bar{\sigma}_\mu|p^b\rangle\Big),$$

$$\langle S^i\rangle_\alpha = \frac{\hbar}{2}(\tilde{\alpha}\sigma^i\alpha)$$

Implies that classical spin is obtained when

$$\|\alpha\| \equiv \sqrt{\tilde{\alpha}_a\alpha^a}$$

$$= \sqrt{2|\boldsymbol{s}_{\rm cl}|/\hbar} = \mathcal{O}(\hbar^{-1/2}).$$

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$$\langle S^i \rangle_\alpha = \frac{\hbar}{2}(\tilde{\alpha}\sigma^i\alpha)$$

Implies that classical spin is obtained when

$$\langle S^i S^j \rangle_\alpha = \langle S^i \rangle_\alpha \langle S^j \rangle_\alpha + \frac{\hbar^2}{4}\big[\delta^{ij}(\tilde{\alpha}\alpha) + i\epsilon^{ijk}(\tilde{\alpha}\sigma^k\alpha)\big].$$

$$\|\alpha\| \equiv \sqrt{\tilde{\alpha}_a\alpha^a}$$

$$= \sqrt{2|\boldsymbol{s}_{\rm cl}|/\hbar} = \mathcal{O}(\hbar^{-1/2}).$$

$\Rightarrow$

In this limit,

$\langle S^i S^j \rangle_\alpha$ factorizes into $\langle S^i \rangle_\alpha \langle S^j \rangle_\alpha$

Taking the classical limit (KMOC + coherent)

$$\langle S_p^\mu \rangle_\alpha = \frac{\hbar}{2}(\tilde{\alpha}\sigma_p^\mu\alpha) \xrightarrow[\hbar\to 0]{} s_{\rm cl}^\mu, \qquad p\cdot s_{\rm cl} = 0,$$

$$\langle S_p^\mu S_p^\nu \rangle_\alpha = \langle S_p^\mu \rangle_\alpha \langle S_p^\nu \rangle_\alpha + \mathcal{O}(\hbar) \xrightarrow[\hbar\to 0]{} s_{\rm cl}^\mu s_{\rm cl}^\nu, \qquad \text{etc.}$$

Dressing the Minimal coupling

Minimal 3-point $\mathcal{A}_{\text{min}}^{(0)\{b\}}{}_{\{a\}}(p_2, s|p_1, s; k, +) = -\frac{\kappa}{2} \frac{\langle 2^b 1_a \rangle^{\odot 2s}}{m^{2s-2}} x^2,$

$$\mathcal{A}_3^h \equiv \mathcal{A}^{(0)}(p_2, \beta|p_1, \alpha; k, h) \;\; = e^{-(\|\alpha\|^2 + \|\beta\|^2)/2} \sum_{s_1, s_2} \frac{(\tilde{\beta}_b)^{\odot 2s_2} (\alpha^a)^{\odot 2s_1}}{\sqrt{(2s_1)!(2s_2)!}} \cdot \mathcal{A}^{(0)\{b\}}{}_{\{a\}}(p_2, s_2|p_1, s_1; k, h),$$

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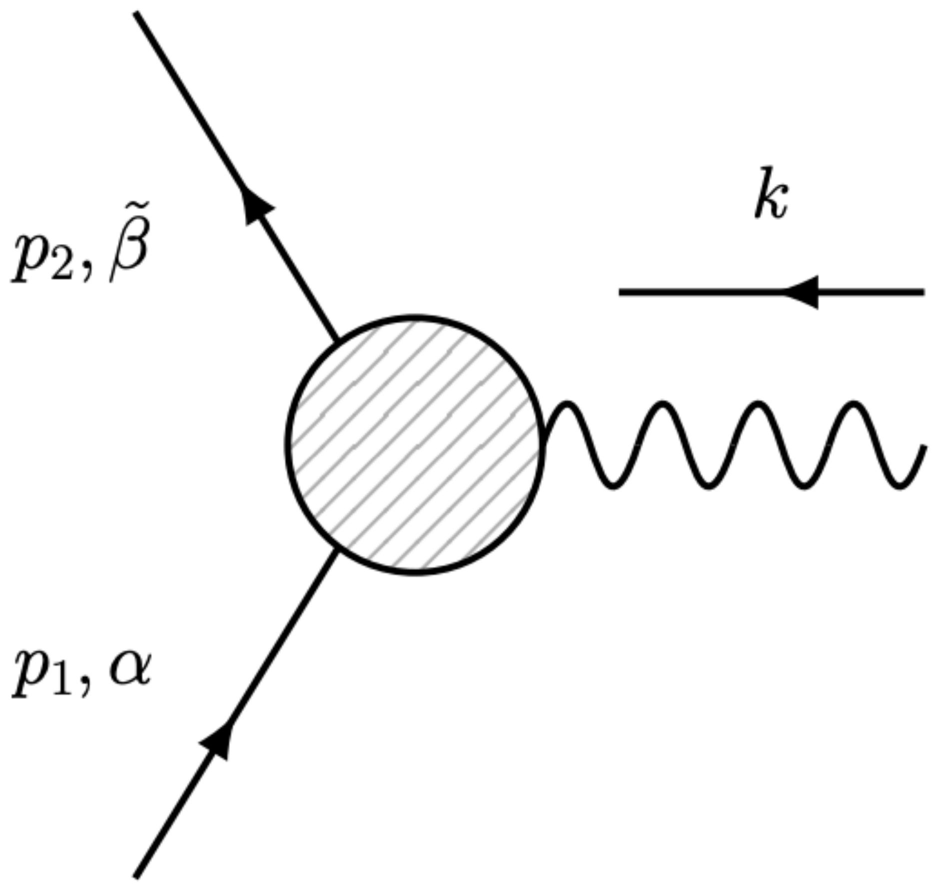


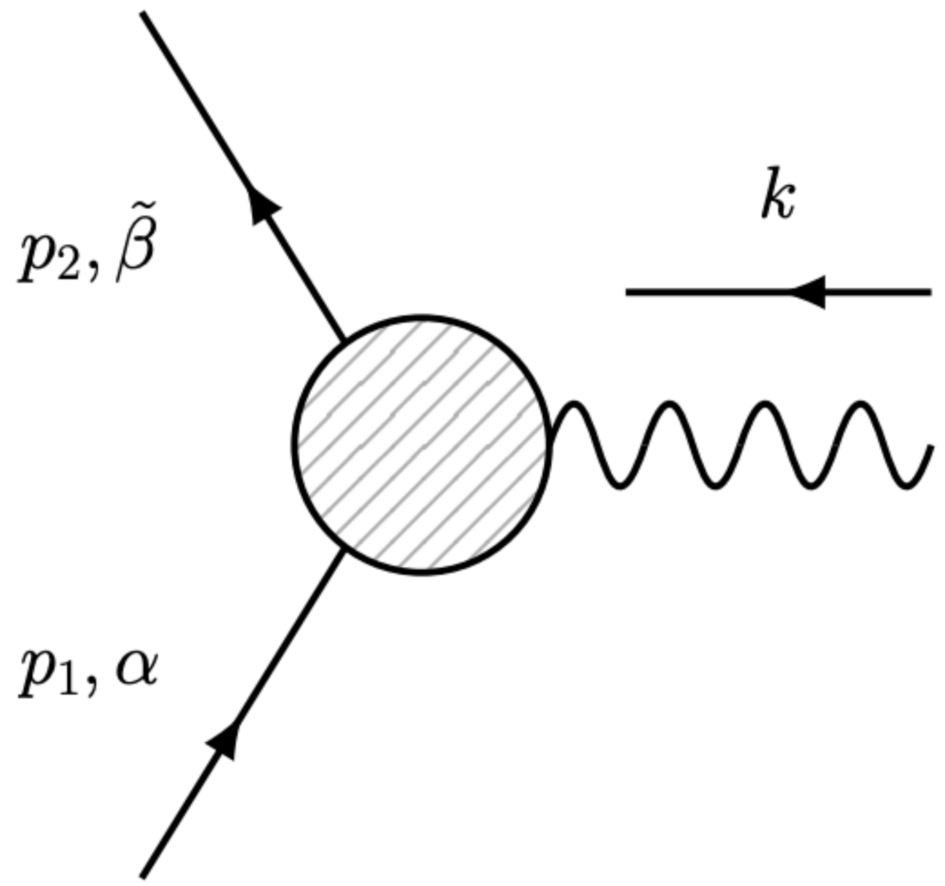
$$\mathcal{A}_{3,\text{min}}^{+} = -\frac{\kappa}{2}x^2 e^{-(\|\alpha\|^2+\|\beta\|^2)/2}\sum_{2s=0}^{\infty}\frac{1}{(2s)!}(\tilde{\beta}_b)^{\odot 2s}\cdot\frac{\langle 2^b 1_a\rangle^{\odot 2s}}{m^{2s-2}}\cdot(\alpha^a)^{\odot 2s}$$
$$= -\frac{\kappa}{2}m^2 x^2 e^{-(\|\alpha\|^2+\|\beta\|^2)/2}\exp\left\{\frac{1}{m}\tilde{\beta}_b\langle 2^b 1_a\rangle\alpha^a\right\}.$$

It exponentiates!

Boost to the same momenta

On-shell kinematics

$$p_{\rm a} = (p_1 + p_2)/2 = p_1 + k/2 = p_2 - k/2$$



$$p_1^\rho = \exp\left[-\frac{i p_{\rm a}^\mu k^\nu}{2m^2}\Sigma_{\mu\nu}\right]^\rho{}_\sigma p_{\rm a}^\sigma,$$

$$|1_a\rangle = U_a{}^b(p_1, p_{\rm a})\left(|{\rm a}_b\rangle - \frac{1}{4m}|k|{\rm a}_b]\right),$$

(Similar for 2)

The exponent $$\tilde{\beta}_b(p_2)\langle 2^b 1_a\rangle\alpha^a(p_1) = \tilde{\beta}_b(p_{\rm a})\left(\langle {\rm a}^b {\rm a}_a\rangle - \frac{1}{4m}\left([{\rm a}^b|k|{\rm a}_a\rangle + \langle {\rm a}^b|k|{\rm a}_a]\right)\right)\alpha^a(p_{\rm a}).$$

spinless term

spin generator

Boost to the same momenta

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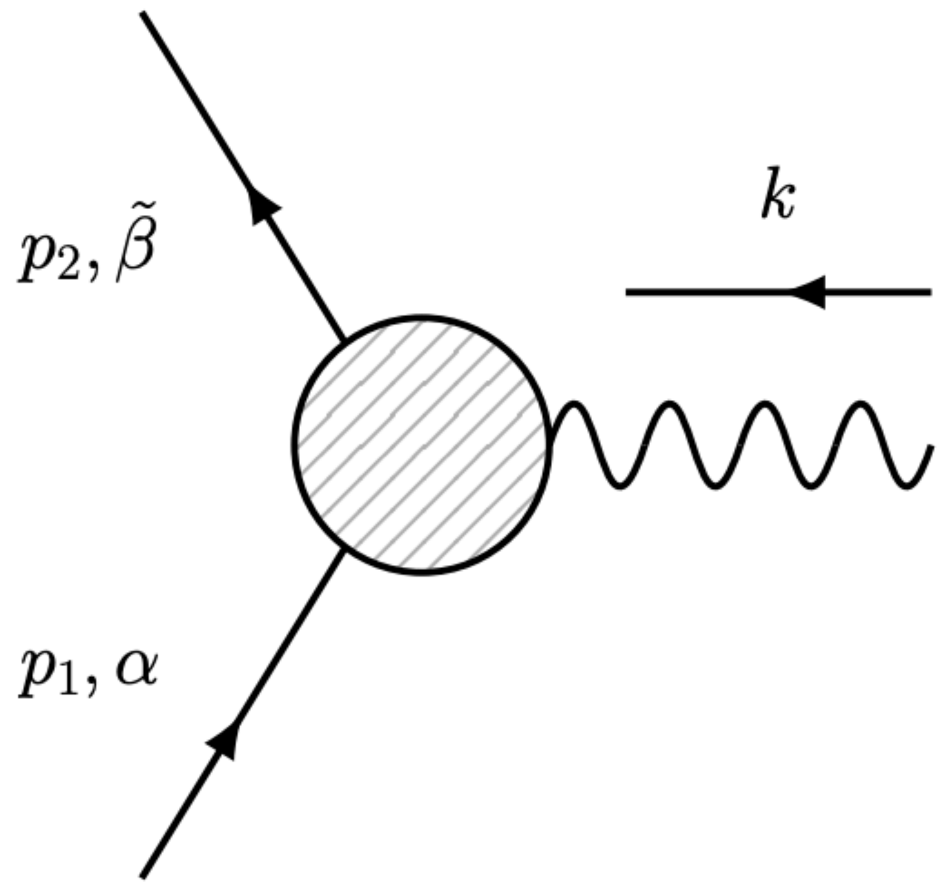

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spinless term — spin generator

$$\mathcal{A}_{3,\rm min}^{\pm} = -\frac{\kappa}{2} m^2 x^{\pm 2} e^{-(\|\alpha\|^2 + \|\beta\|^2)/2 + \tilde{\beta}\alpha} \exp\left\{\mp \frac{\hbar}{2m}\bar{k}_\mu(\tilde{\beta}\sigma^\mu_{p_{\rm a}}\alpha)\right\}$$

overlap between coherent states

Classical limit and classical three-points

Factored out the standard coherent-state overlap: $\langle\beta|\alpha\rangle = e^{-(\|\alpha\|^2+\|\beta\|^2)/2+\tilde{\beta}\alpha}$

In the classical limit, we take: $\tilde{\beta}_a = (\alpha^a)^*$ $\longrightarrow$ Exact cancellation between the spinless term and the normalization

and we can identify the spin expectation value

$$\mathcal{A}^{\pm}_{3,\mathrm{min}}\Big|_{\beta=\alpha} = -\frac{\kappa}{2}m^2x^{\pm 2}\exp\left\{\mp\frac{1}{m}\bar{k}_\mu\langle S^\mu_{p_\mathrm{a}}\rangle_\alpha\right\} = -\frac{\kappa}{2}m^2x^{\pm 2}e^{\mp\bar{k}\cdot a_\mathrm{a}}.$$

Matches the Kerr BH ‘amplitude’

Can use directly to built four-points.

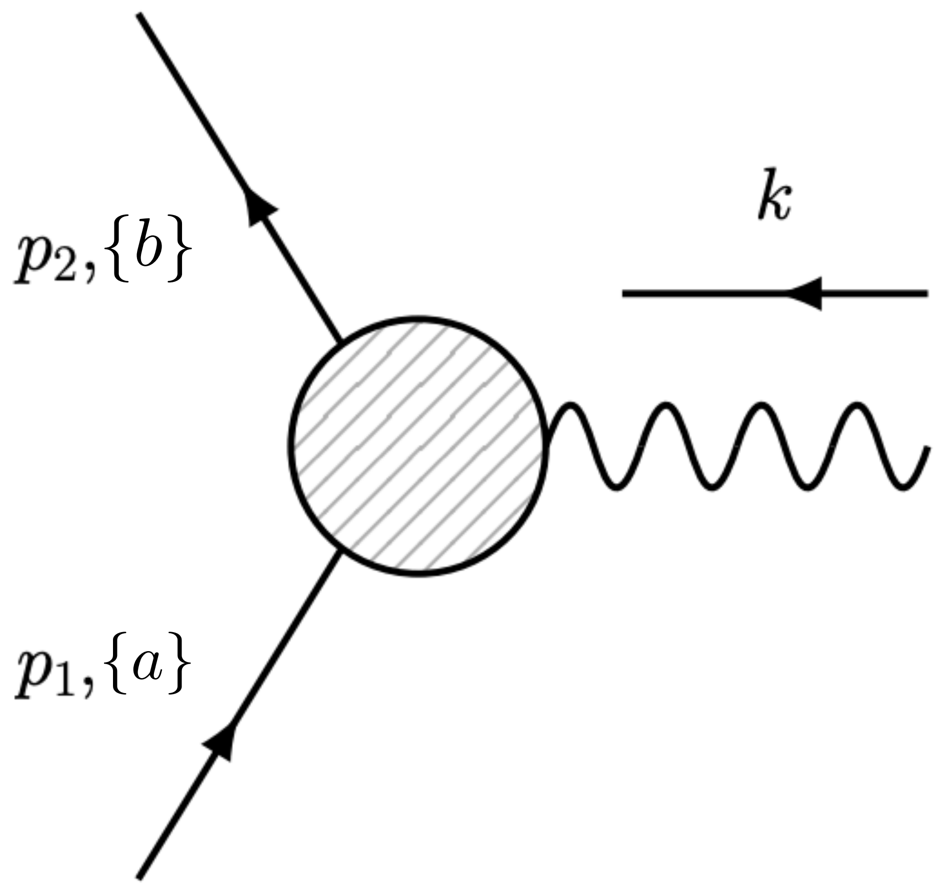
Matches 1PM results

notation: $k^\mu = \hbar\bar{k}^\mu$

$a^\mu_\mathrm{a} \equiv \frac{1}{m_\mathrm{a}}\langle S^\mu_{p_\mathrm{a}}\rangle_\alpha$

[see Guevara, Ochirov, Vines, 19’]

General
three-point amplitude
and
Kerr BHs



General Three-point amplitude: bootstrapping

$$x = \frac{\langle q|p_1|3]}{m\langle 3q\rangle} = -\frac{\sqrt{2}}{m}(p_1 \cdot \varepsilon_3^+)$$

Spin-1/2: minimal

$$\mathcal{A}(1_\psi^a, 2_\psi^b, 3_\gamma^+) = i\frac{g}{\sqrt{2}}\bar{v}_1^a \gamma^\mu u_2^b\, \varepsilon_\mu^+(q) \quad \to \quad ig\, x\, \langle 1^a 2^b\rangle$$

Spin-1/2: dipole (higher-dim operator)

$$\mathcal{A}_{\text{dipole}}(1_\psi^a, 2_\psi^b, 3_\gamma^+) = i\frac{g}{\sqrt{2}}\bar{v}_1^a \sigma_{\mu\nu} u_2^b\, q^\mu \varepsilon_\nu^+(q) \quad \to \quad igx^2\langle 1^a q\rangle\langle q 2^b\rangle$$

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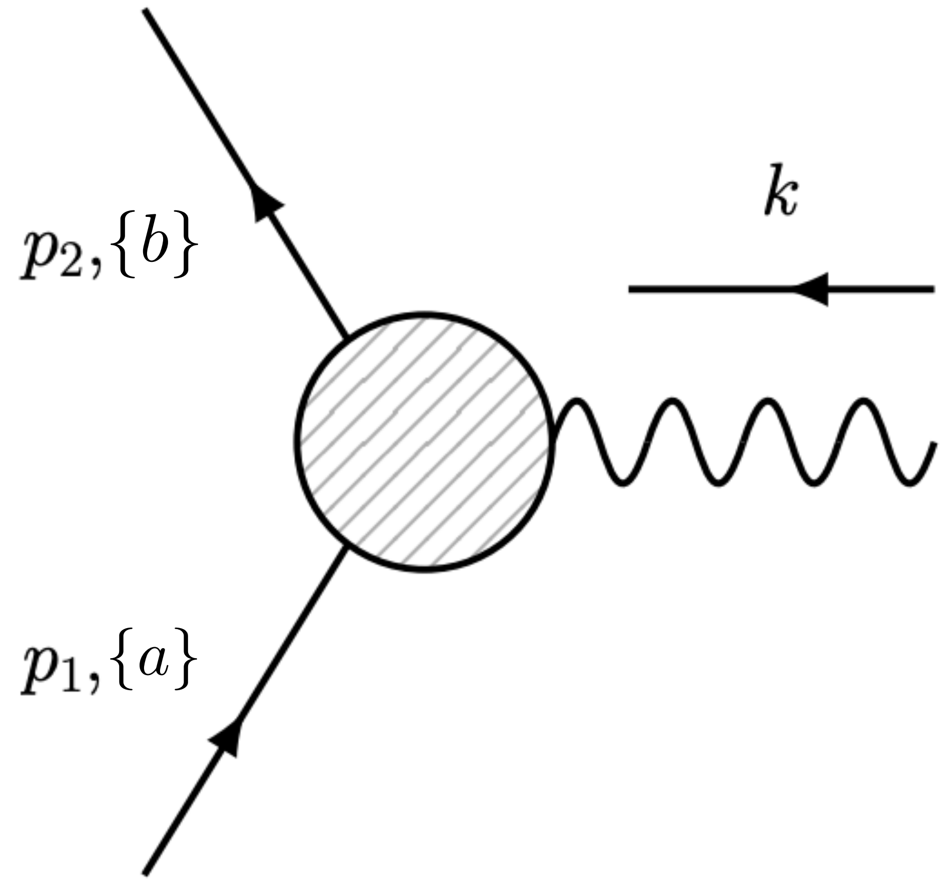
For general spin, we have 2s+1 terms

[Arkani-Hamed, Huang, Huang 2017]



$$\mathcal{A}_{\text{gen}}^{(0)\{b\}}{}_{\{a\}}(p_2, s|p_1, s; k, +) = -\frac{\kappa}{2}\sum_{n=0}^{2s} g_n^+ \frac{x^{n+2}\langle 2^b 1_a\rangle^{\odot(2s-n)}}{m^{2s+n-2}} \odot \left(\langle 2^b k\rangle\langle k 1_a\rangle\right)^{\odot n}$$

How to connect it with Kerr BHs?

Worldline effective action vs. three-point

[Porto, Rothstein, 06']
[Porto, Rothstein, 08']
[Levi, Steinhoff, 15']

Expanding the curvature tensor $R_{\lambda\mu\nu\rho}$ in terms of linear grav. pertubation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ in the effective action

$$S_{\text{Int}} = -\frac{m}{2}\int d\tau \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} C_{\text{ES}^{2n}} (a\cdot\partial)^{2n} u^\mu u^\nu h_{\mu\nu} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} C_{\text{BS}^{2n+1}} (a\cdot\partial)^{2n} u^\mu \epsilon^{\nu\rho\sigma\tau} u_\rho a_\sigma \partial_\tau h_{\mu\nu}\right]_{x=r(\tau)} + \mathcal{O}(h^2).$$

Interpreted as the interaction

$$S_{\text{Int}} = -\frac{1}{2}\int d^4x\, h_{\mu\nu}(x) T^{\mu\nu}_{\text{gen}}(x) = -\frac{1}{2}\int \frac{d^4\bar{k}}{(2\pi)^4} h_{\mu\nu}(\bar{k}) T^{\mu\nu}_{\text{gen}}(-\bar{k}),$$

General stress-tensor: $$T^{\mu\nu}_{\text{gen}}(\bar{k}) = m\int d\tau\, e^{i\bar{k}\cdot r(\tau)} \sum_{n=0}^{\infty} (\bar{k}\cdot a)^{2n} \left[\frac{C_{\text{ES}^{2n}}}{(2n)!} u^\mu u^\nu + \frac{C_{\text{BS}^{2n+1}}}{(2n+1)!} i u^{(\mu} \epsilon^{\nu)\rho\sigma\tau} u_\rho a_\sigma \bar{k}_\tau\right].$$

Kerr BH corresponds: $C_{\text{ES}^{2n}} = -C_{\text{BS}^{2n+1}} = 1$

Worldline effective action vs. three-point

To obtain the amplitude from the action:

Straight particle trajectory coupled to an on-shell graviton

$$h^{\mu\nu}(\bar{k}) \;\to\; \kappa\, 2\pi\delta(\bar{k}^2)\varepsilon_k^\mu\varepsilon_k^\nu, \qquad r^\mu(\tau) = \frac{p^\mu}{m}\tau \qquad \Rightarrow \qquad u^\mu(\tau) = \frac{p^\mu}{m}.$$

The interaction:

$$S_{\rm Int} = \int \frac{d^4\bar{k}}{(2\pi)^2}\delta(\bar{k}^2)\delta(2p\cdot\bar{k})\mathcal{A}_{\rm gen}(p,k),$$

Amplitude

$$\mathcal{A}^{\pm}_{\rm gen}(p,k) = -\kappa(p\cdot\varepsilon_k^{\pm})^2\left[\sum_{n=0}^{\infty}\frac{C_{{\rm ES}^{2n}}}{(2n)!}(\bar{k}\cdot a)^{2n} \pm \sum_{n=0}^{\infty}\frac{C_{{\rm BS}^{2n+1}}}{(2n+1)!}(\bar{k}\cdot a)^{2n+1}\right],$$

For Kerr:

$$C_{{\rm ES}^{2n}} = -C_{{\rm BS}^{2n+1}} = 1$$

$$\mathcal{A}^{\pm}_{\rm min}(p,k) = -\kappa(p\cdot\varepsilon_k^{\pm})^2\left[\cosh(\bar{k}\cdot a) \mp \sinh(\bar{k}\cdot a)\right] = -\frac{\kappa}{2}m^2x^{\pm 2}e^{\mp\bar{k}\cdot a}.$$

Same as before!

Kerr preferred solution: Non-minimal

$$\mathcal{A}^{(0)\{b\}}_{\text{gen}}{}_{\{a\}}(p_2, s|p_1, s; k, +) = -\frac{\kappa}{2}\sum_{n=0}^{2s} g_n^+ \frac{x^{n+2}\langle 2^b 1_a\rangle^{\odot(2s-n)}}{m^{2s+n-2}} \odot \left(\langle 2^b k\rangle\langle k 1_a\rangle\right)^{\odot n},$$

Matching with the previous amplitude

$$\mathcal{A}^{\pm}_{\text{gen}}(p, k) = -\kappa(p\cdot\varepsilon_k^{\pm})^2\left[\sum_{n=0}^{\infty}\frac{C_{\text{ES}^{2n}}}{(2n)!}(\bar{k}\cdot a)^{2n} \pm \sum_{n=0}^{\infty}\frac{C_{\text{BS}^{2n+1}}}{(2n+1)!}(\bar{k}\cdot a)^{2n+1}\right],$$

The wilson coefficients

$$C_{\text{ES}^{2n}} = \sum_{r=0}^{2n}\frac{(2n)!(-2)^r g_r^{\pm}}{(2n-r)!\|\alpha\|^{2r}}$$

(Same for the magnetic)

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$$\mathcal{A}^{(0)\{b\}}_{\text{gen}\ \{a\}}(p_2, s|p_1, s; k, +) = -\frac{\kappa}{2}\sum_{n=0}^{2s} g_n^+ \frac{x^{n+2}\langle 2^b 1_a\rangle^{\odot(2s-n)}}{m^{2s+n-2}} \odot \left(\langle 2^b k\rangle\langle k 1_a\rangle\right)^{\odot n},$$

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The wilson coefficients

$$C_{\text{ES}^{2n}} = \sum_{r=0}^{2n}\frac{(2n)!(-2)^r g_r^{\pm}}{(2n-r)!\|\alpha\|^{2r}}$$

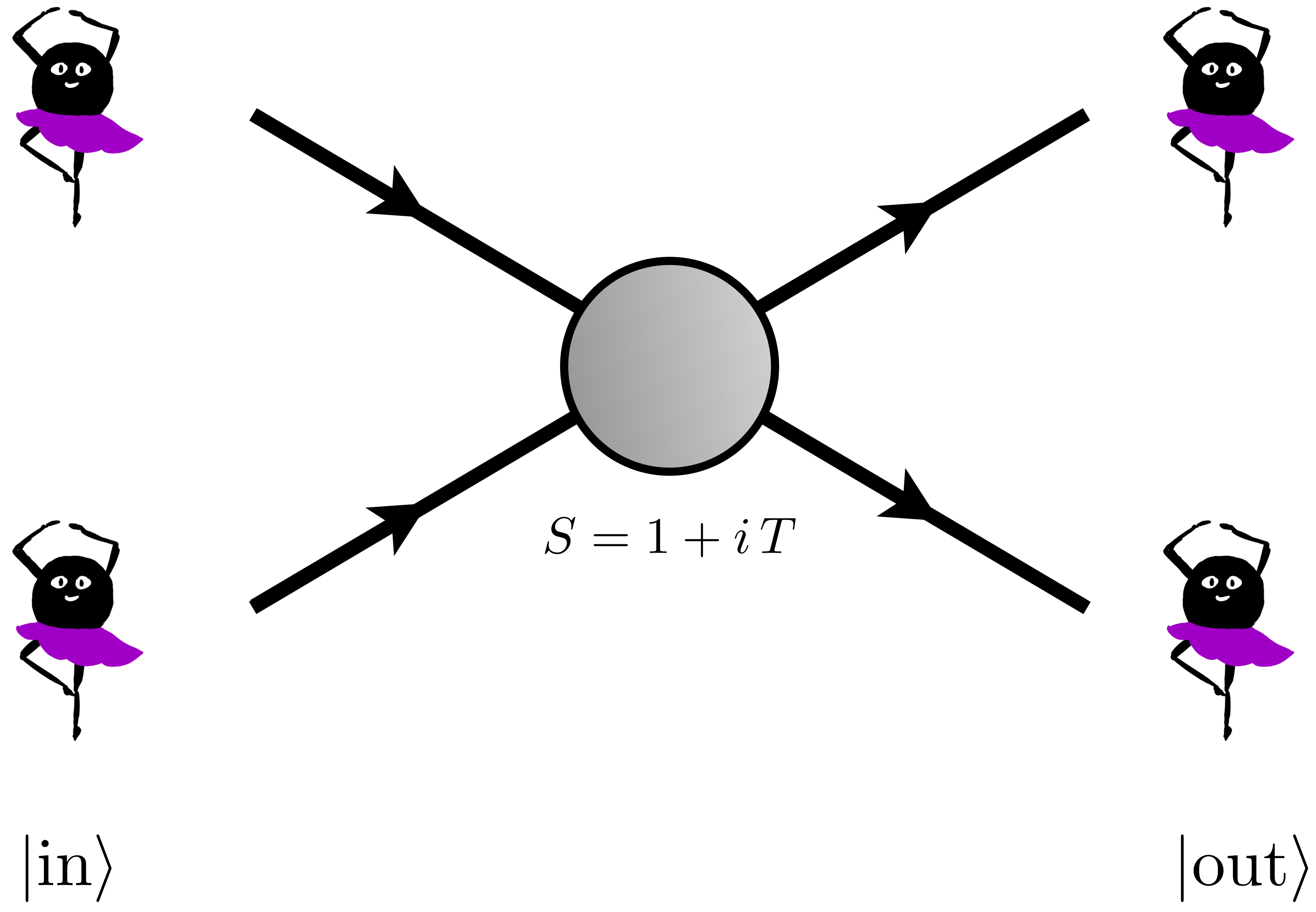
(Same for the magnetic)

Classically suppressed unless $g^{\pm}_{n>0}$ scales with $\mathcal{O}(\hbar^{-n})$

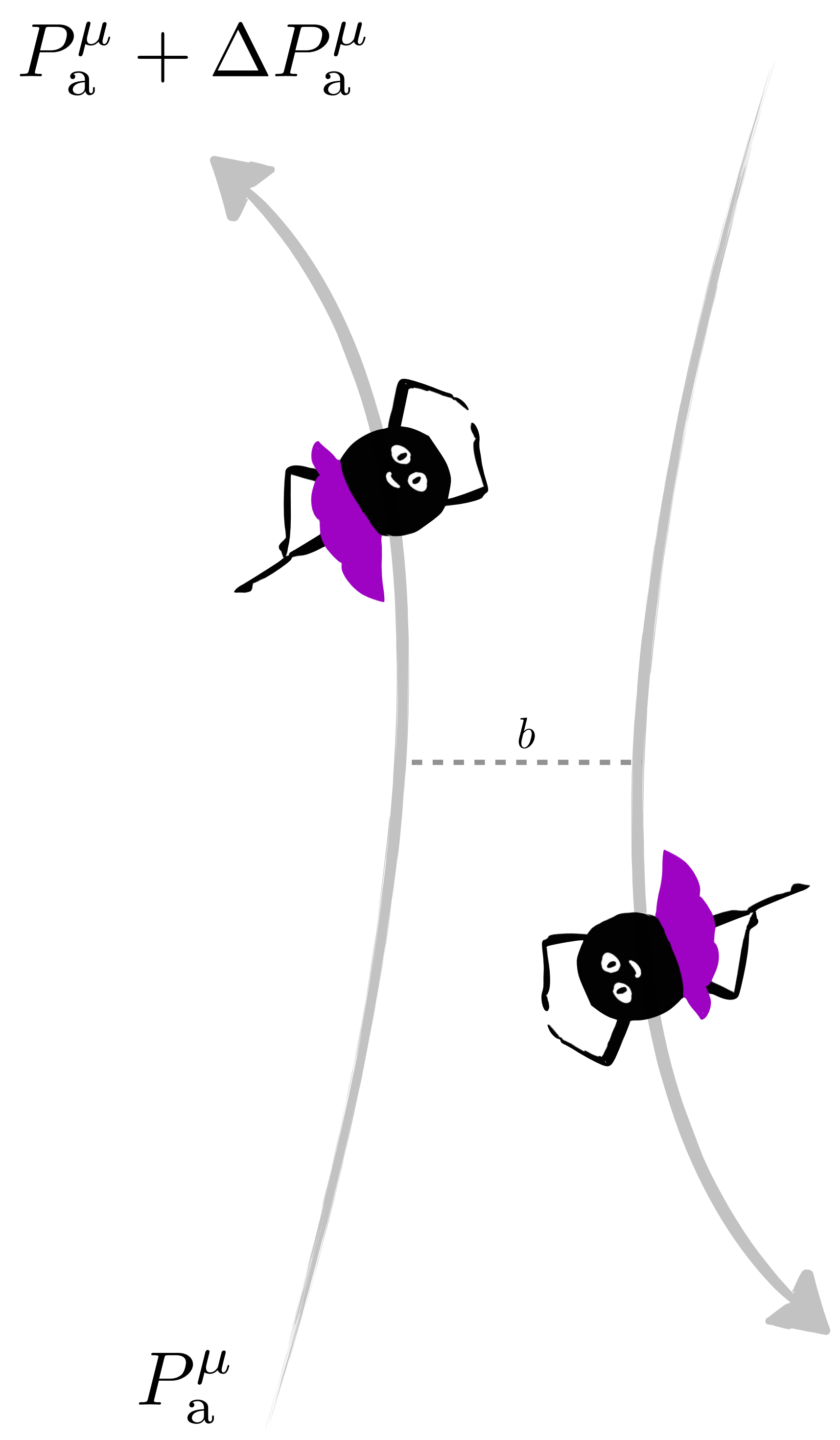
In order to model general spinning body, non-minimal couplings depends on the spin via $\|\alpha\|^2 = \frac{2m}{\hbar}\sqrt{-a^2}.$

(Expect for a Kerr BH) $g_0^{\pm} = 1 \quad g^{\pm}_{n>0} = 0$

**Kosower
Maybee
O’Connell
(KMOC) formalism**

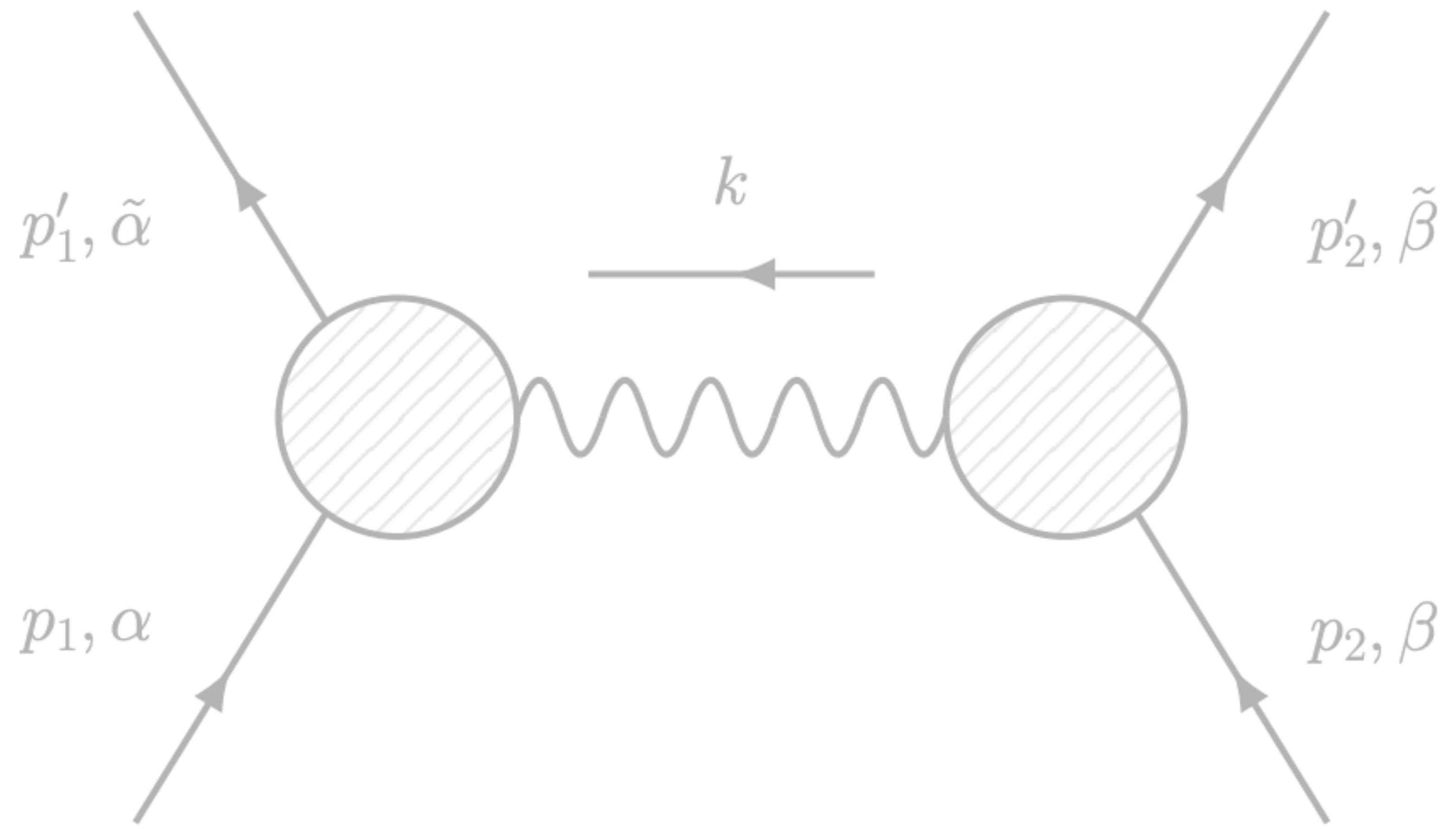


Pictorically…



Classical limit: $\hbar \to 0 \quad ||\alpha||^2 \to \infty$

$$\Delta P^\mu_a = -\hbar \frac{\partial}{\partial b_\mu} \int_{p_a, p_b} |\psi_a(p_a)|^2 |\psi_b(p_b)|^2$$



The KMOC formalism:
- quantum expectation values
- chosen initial quantum states
- classical observables when $\hbar \to 0 \quad ||\alpha||^2 \to \infty$

Kosower Maybee O'Connell (KMOC) formalism

[Kosower, Maybee,O'Connell 18]
[Maybee,O'Connell, Vines 19]
[de la Cruz, Maybee,O'Connell 20]

Changing in an operator due to scattering

$$\Delta O = \langle \text{out}|O|\text{out}\rangle - \langle \text{in}|O|\text{in}\rangle = \langle \text{in}|S^\dagger O S|\text{in}\rangle - \langle \text{in}|O|\text{in}\rangle$$

Using $S = 1 + iT$ and optical theorem $T^\dagger = T - iT^\dagger T$

$$\Delta O = i\langle \text{in}|\underline{[O,T]}|\text{in}\rangle + \langle \text{in}|\underline{T^\dagger [O,T]}|\text{in}\rangle$$

leading order next-to-leading order

Alternatively, we can write (indifferent at LO)

$$\Delta O = \langle \text{in}|S^\dagger O S|\text{in}\rangle - \langle \text{in}|O|\text{in}\rangle = \underbrace{i\langle \text{in}|[OT - T^\dagger O]|\text{in}\rangle}_{\Delta_1 O} + \underbrace{\langle \text{in}|T^\dagger O T|\text{in}\rangle}_{\Delta_2 O},$$

We need to prepare well-defined the initial state states.

Kosower Maybee O'Connell (KMOC) formalism

Incoming (spineless) state: $|\mathrm{in}\rangle = \int_{p_1}\int_{p_2} \psi_\mathrm{a}(p_1)\psi_\mathrm{b}(p_2)e^{ib\cdot p_1/\hbar}|p_1;p_2\rangle$

definite momenta state

$$\int_p \equiv \int \frac{d^4p}{(2\pi)^3}\Theta(p^0)\delta(p^2-m^2)$$

Wave functions

Particles well separated by impact-parameter

$$\psi_\xi(p) = \frac{1}{m}\left[\frac{8\pi^2}{\xi K_1(2/\xi)}\right]^{1/2}\exp\left(-\frac{p\cdot u}{\xi m}\right).$$

Well-behaved classical exp. values

Kosower Maybee O'Connell (KMOC) formalism

Incoming (spineless) state: $$|\text{in}\rangle = \int_{p_1}\int_{p_2} \psi_{\text{a}}(p_1)\psi_{\text{b}}(p_2)e^{ib\cdot p_1/\hbar}\underline{|p_1;p_2\rangle}$$

[Kosower, Maybee,O'Connell 18]

definite momenta state

Incoming (spinning) state: $$|\text{in}\rangle = \sum_{a_1,a_2}\int_{p_1}\int_{p_2} \psi_{\text{a}}(p_1)\psi_{\text{b}}(p_2)\xi_{a_1}\xi_{a_2}e^{ib\cdot p_1/\hbar}|p_1,p_2;\underline{a_1,a_2}\rangle$$

[Maybee,O'Connell, Vines 19]

Quantum
spin-indices

Kosower Maybee O'Connell (KMOC) formalism

Incoming (spineless) state: $$|\text{in}\rangle = \int_{p_1}\int_{p_2} \psi_{\text{a}}(p_1)\psi_{\text{b}}(p_2)e^{ib\cdot p_1/\hbar}\underline{|p_1;p_2\rangle}$$

[Kosower, Maybee,O'Connell 18]

definite momenta state

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[Maybee,O'Connell, Vines 19]

Quantum spin-indices

Incoming (coherent) state:

[RA, Ochirov 21]

$$|\text{in}\rangle = \int_{p_1}\int_{p_2} \psi_{\text{a}}(p_1)\psi_{\text{b}}(p_2)e^{ib\cdot p_1/\hbar}|p_1,\alpha;p_2,\beta\rangle$$

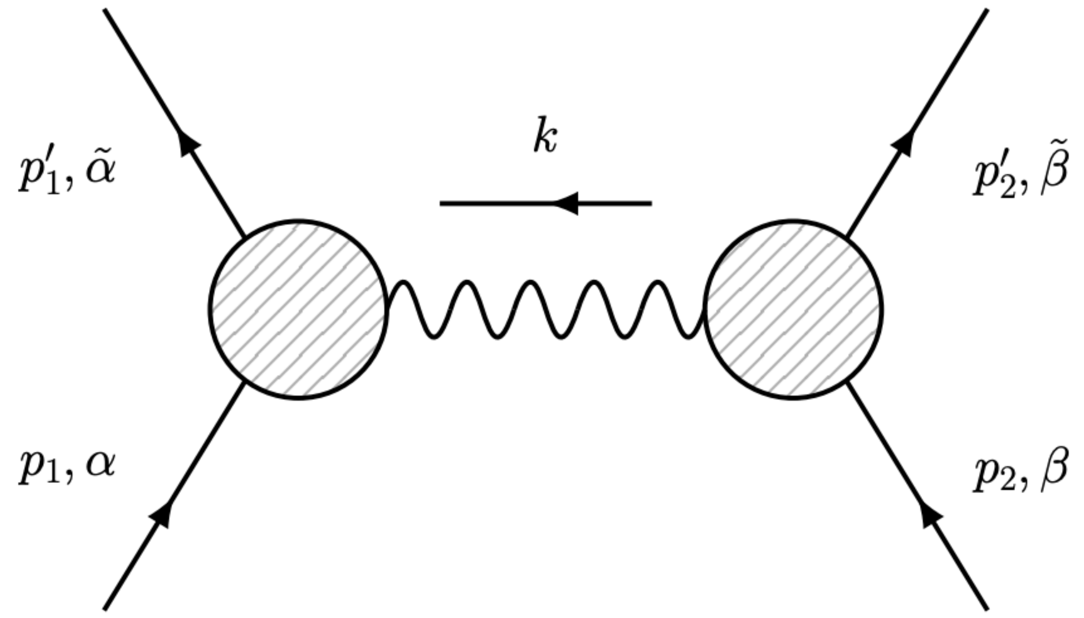
$$= \underline{e^{-(\|\alpha\|^2+\|\beta\|^2)/2}}\sum_{s_1,s_2}\int_{p_1,p_2} e^{ib\cdot p_1/\hbar}\psi_{\text{a}}(p_1)\psi_{\text{b}}(p_2)\frac{(\alpha^a)^{\odot 2s_1}(\beta^b)^{\odot 2s_2}}{\sqrt{(2s_1)!(2s_2)!}}\cdot \underline{|p_1,s_1,\{a\};p_2,s_2,\{b\}\rangle}.$$

normalization

definite spin-state

Kosower Maybee O'Connell (KMOC) formalism

$$\Delta O = \langle \text{in}|S^\dagger O S|\text{in}\rangle - \langle \text{in}|O|\text{in}\rangle = \underbrace{i\langle \text{in}|[OT - T^\dagger O]|\text{in}\rangle}_{\Delta_1 O} + \underbrace{\langle \text{in}|T^\dagger O T|\text{in}\rangle}_{\Delta_2 O},$$



Focusing on the first term

$$\Delta_1 O = \int_{p_1', p_2', p_1, p_2} e^{-ik\cdot b/\hbar} \psi_\text{a}^*(p_1')\psi_\text{b}^*(p_2')\psi_\text{a}(p_1)\psi_\text{b}(p_2) i\langle p_1', \alpha; p_2', \beta|[OT - T^\dagger O]|p_1, \alpha; p_2, \beta\rangle,$$

For the momentum operator:

$$\Delta P_\text{a}^\mu = \int_{p_1, p_2}\int_k e^{-ik\cdot b/\hbar}\psi_\text{a}^*(p_1+k)\psi_\text{b}^*(p_2-k)\psi_\text{a}(p_1)\psi_\text{b}(p_2) \times \Big\{(p_1+k)^\mu i\mathcal{A}(p_1+k, \alpha; p_2-k, \beta|p_1, \alpha; p_2, \beta) - p_1^\mu i\mathcal{A}^*(p_1, \alpha; p_2, \beta|p_1+k, \alpha; p_2-k, \beta)\Big\}$$

Result in full QFT, need to take the classical limit

Kosower Maybee O'Connell (KMOC) formalism

Classical limit:

From the wave-functions
$$\psi_\xi(p) = \frac{1}{m}\left[\frac{8\pi^2}{\xi K_1(2/\xi)}\right]^{1/2} \exp\left(-\frac{p\cdot u}{\xi m}\right).$$
$$\xi \approx \frac{2\sigma_p^2}{3m^2} \;\to\; 0.$$

From the coherent-states
$$\|\alpha\|^2 = \frac{2}{\hbar}\sqrt{-s_{\mathrm{cl}}^2} \;\to\; \infty.$$

Avoid head-on or deep-inelastic collisions
$$|b| \equiv \sqrt{-b^2} \;\gg\; (\sigma_x)_{\mathrm{a,b}} \geq \frac{\hbar}{2(\sigma_p)_{\mathrm{a,b}}} \propto \frac{\hbar}{\sqrt{\xi} m_{\mathrm{a,b}}}.$$

i.e.
$$|b| \gg \sigma_x \gg (\lambda_{\mathrm{Compton}})_{\mathrm{a,b}}$$

Heuristically:

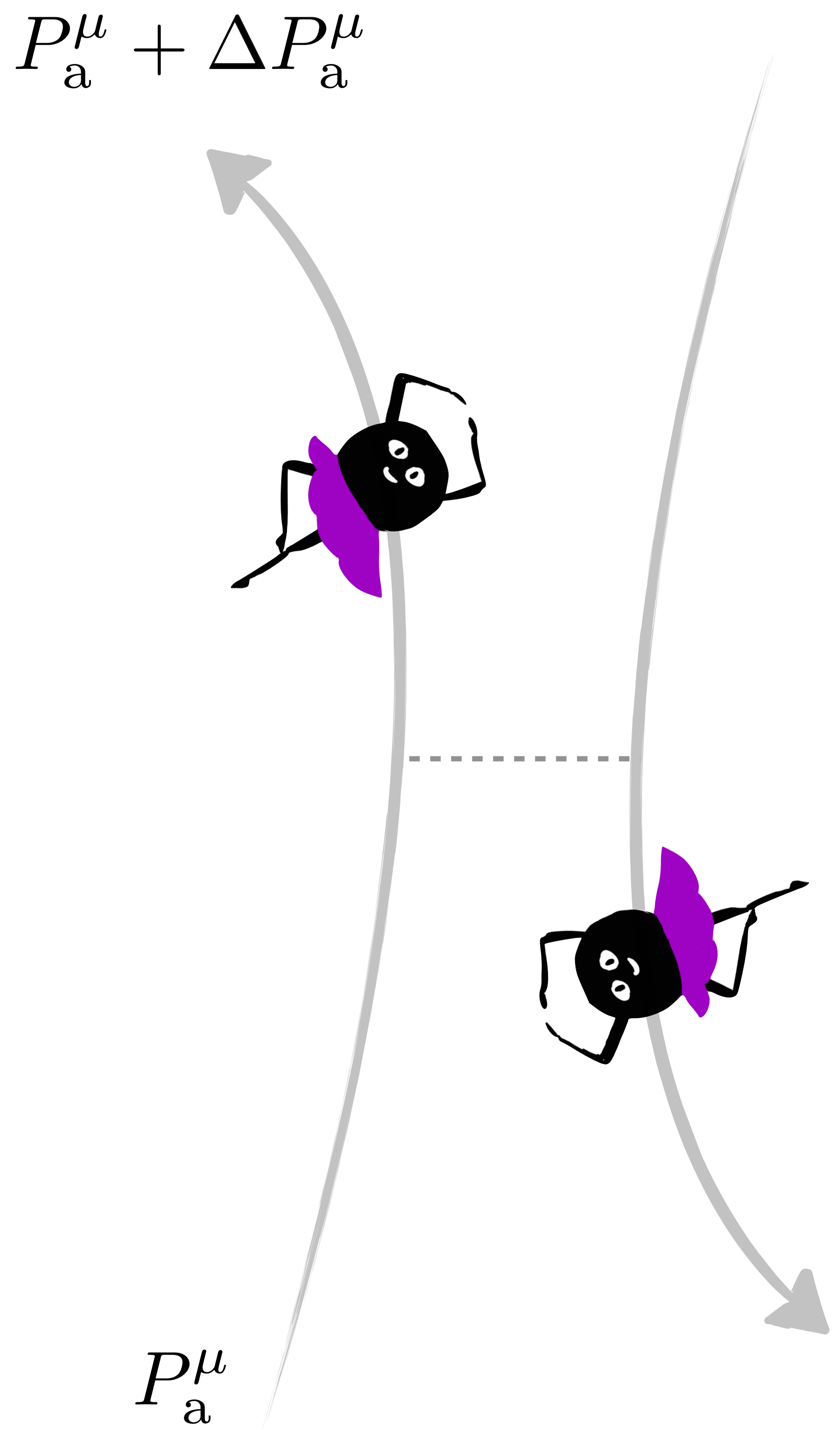
$$\sigma_x, \sigma_p \propto \hbar^{1/2}, \qquad \xi \propto \hbar, \qquad \|\alpha\| \propto \hbar^{-1/2}, \qquad |k| \propto \hbar, \qquad k\cdot u_{\mathrm{a,b}} \propto \hbar^{3/2}.$$

KMOC using coherent states

After some manipulation
Leading classical impulse:

$$\Delta P_{\mathrm{a}}^{\mu} = -\hbar \frac{\partial}{\partial b_{\mu}} \int_{p_{\mathrm{a}}, p_{\mathrm{b}}} |\psi_{\mathrm{a}}(p_{\mathrm{a}})|^2 |\psi_{\mathrm{b}}(p_{\mathrm{b}})|^2 \int_k e^{-i\bar{k}\cdot b} \mathcal{A}^{(0)}(k)$$

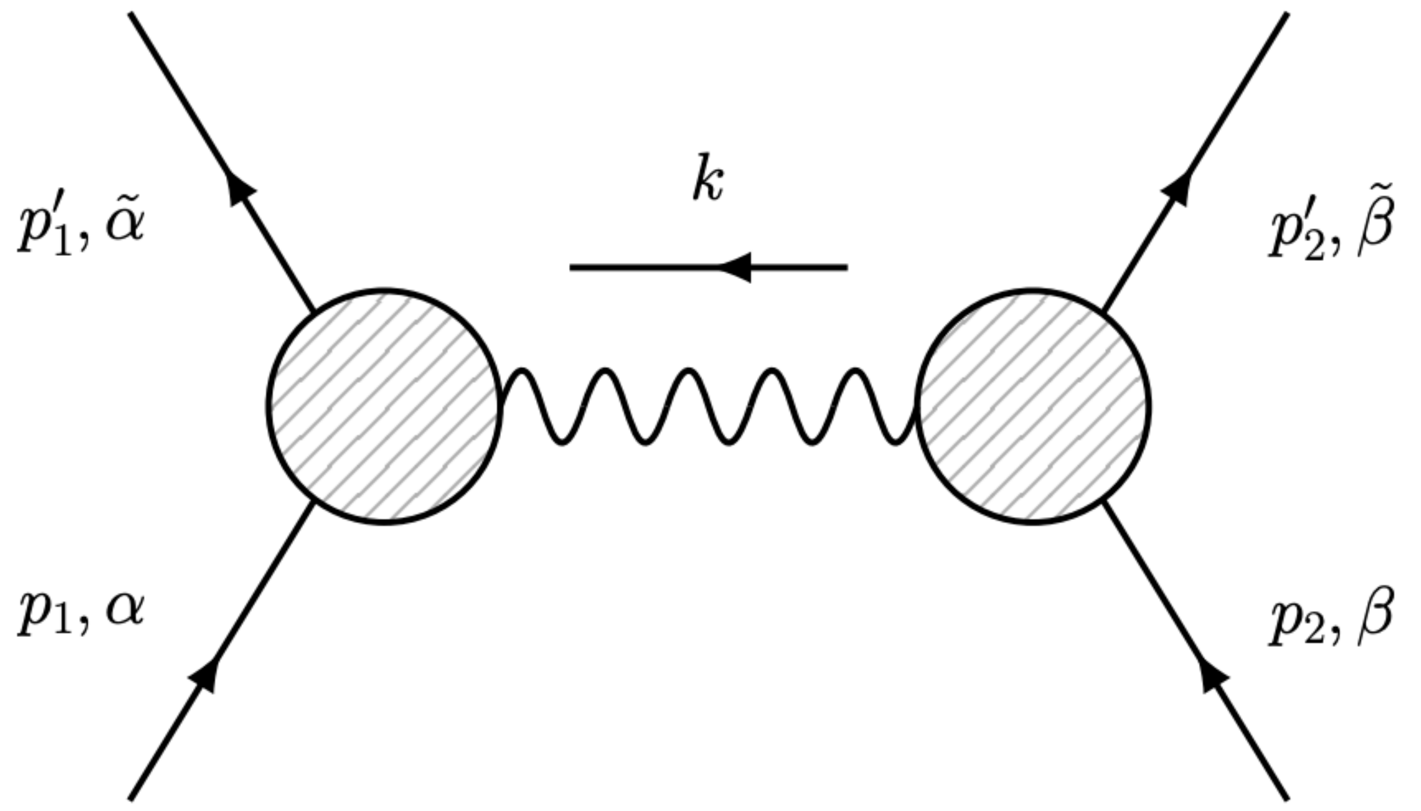
$$\Delta S_{\mathrm{a}}^{\mu} = \frac{\hbar}{m_{\mathrm{a}}} \int_{p_{\mathrm{a}}, p_{\mathrm{b}}} |\psi_{\mathrm{a}}(p_{\mathrm{a}})|^2 |\psi_{\mathrm{b}}(p_{\mathrm{b}})|^2 \left[p_{\mathrm{a}}^{\mu} a_{\mathrm{a}}^{\nu} \frac{\partial}{\partial b^{\nu}} - \epsilon^{\mu\nu\rho\sigma} p_{\mathrm{a}\nu} a_{\mathrm{a}\rho} \frac{\partial}{\partial a_{\mathrm{a}}^{\sigma}} \right] \int_k e^{-i\bar{k}\cdot b} \mathcal{A}^{(0)}(k)$$



in both, we need the eikonal coherent-spin amplitudes

Coherent scattering amplitudes

Classical limit dominated by $t = k^2 = \hbar^2 \bar{k}^2$



Four-point factorizes…

$$\mathcal{A}^{(0)}(p_1', \alpha; p_2', \beta | p_1, \alpha; p_2, \beta) = -\frac{1}{\hbar^2 \bar{k}^2} \sum_{\pm} \mathcal{A}^{(0)}(p_1', \alpha | p_1, \alpha; k, \pm) \times \mathcal{A}^{(0)}(p_2', \beta; k, \mp | p_2, \beta) + \mathcal{O}(1/\hbar),$$

into three-points

Holomorphic Classical Limit

and kinematics

[Cachazo, Guevara, 17']

[Guevara, 17']

[Guevara, Ochirov, Vines 18']

[Guevara, Ochirov, Vines 19']

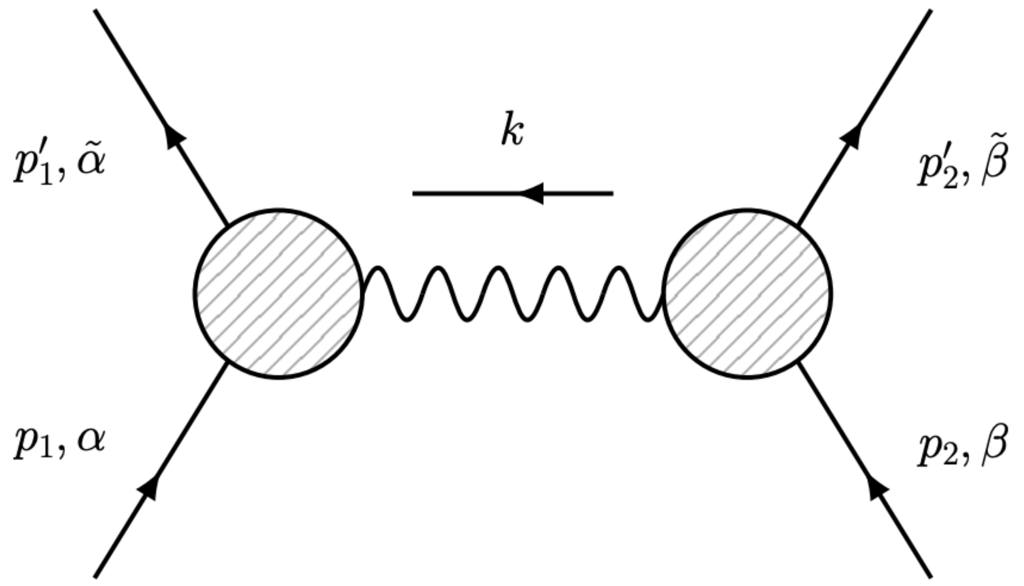
$$x_a / x_b = \gamma(1 - v)$$

$$x_b / x_a = \gamma(1 + v). \qquad \gamma = \frac{1}{\sqrt{1 - v^2}} = \frac{p_a \cdot p_b}{m_a m_b}.$$

Four-point coherent amplitudes (general case)

notation

$$a^\mu_{\rm a} \equiv \frac{1}{m_{\rm a}} \langle S^\mu_{p_{\rm a}} \rangle_\alpha,$$
$$C_{2n} \equiv C_{{\rm ES}^{2n}}$$
$$C_{2n+1} \equiv C_{{\rm BS}^{2n+1}}$$



Using the three-points

$$\mathcal{A}^{(0)}(p_1', \alpha | p_1, \alpha; k, \pm) = -\frac{\kappa}{2} m_{\rm a}^2 x_{\rm a}^{\pm 2} \sum_{n=0}^{\infty} \frac{C_{{\rm a}n}}{n!} \left(\pm \bar{k} \cdot a_{\rm a}\right)^n + \mathcal{O}(\hbar^0),$$

Beyond t-pole and using some notation [Vines 17']

$$\mathcal{A}^{(0)}(k) = -\frac{8\pi G m_{\rm a}^2 m_{\rm b}^2 \gamma^2}{\hbar^3 \bar{k}^2} \times \sum_{\pm} (1 \mp v)^2 \sum_{n_1, n_2 = 0}^{\infty} \frac{C_{{\rm a}n_1} C_{{\rm b}n_2}}{n_1! n_2!} \left(\pm i\bar{k} \cdot [w * a_{\rm a}]\right)^{n_1} \left(\pm i\bar{k} \cdot [w * a_{\rm b}]\right)^{n_2} + \mathcal{O}(\hbar^{-5/2})$$

Multipole expansion of particle 1 and particle 2

modeled by Wilson coefficients!

$$C_{{\rm a}n_1} C_{{\rm b}n_2}$$

$$w^{\mu\nu} = \frac{2 p_{\rm a}^{[\mu} p_{\rm b}^{\nu]}}{m_{\rm a} m_{\rm b} \gamma v},$$
$$[w * a]_\lambda = (*w)_{\lambda\mu} a^\mu = \frac{\epsilon_{\lambda\mu\nu\rho} p_{\rm a}^\mu p_{\rm b}^\nu a^\rho}{m_{\rm a} m_{\rm b} \gamma v},$$

Impulse Observables from elastic scattering

After the Fourier transform to the impact parameter space…

Linear impulse

$$\Delta P_{\mathrm{a}}^{\mu} = -\hbar \frac{\partial}{\partial b_{\mu}} \int_{p_{\mathrm{a}},p_{\mathrm{b}}} |\psi_{\mathrm{a}}(p_{\mathrm{a}})|^2 |\psi_{\mathrm{b}}(p_{\mathrm{b}})|^2 \mathcal{A}_4^{(0)}(b)$$

$$\Delta P_{\mathrm{a}}^{\mu} = G m_{\mathrm{a}} m_{\mathrm{b}} \frac{\gamma}{v} \sum_{\pm} (1 \mp v)^2 \frac{[b \pm w * (a_{\mathrm{a}} + a_{\mathrm{b}})]^{\mu}}{[b \pm w * (a_{\mathrm{a}} + a_{\mathrm{b}})]^2} \bigg|_{\mathrm{cl}},$$

Angular impulse

$$\Delta S_{\mathrm{a}}^{\mu} = \frac{\hbar}{m_{\mathrm{a}}} \int_{p_{\mathrm{a}},p_{\mathrm{b}}} |\psi_{\mathrm{a}}(p_{\mathrm{a}})|^2 |\psi_{\mathrm{b}}(p_{\mathrm{b}})|^2 \left[p_{\mathrm{a}}^{\mu} a_{\mathrm{a}}^{\nu} \frac{\partial}{\partial b^{\nu}} - \epsilon^{\mu\nu\rho\sigma} p_{\mathrm{a}\nu} a_{\mathrm{a}\rho} \frac{\partial}{\partial a_{\mathrm{a}}^{\sigma}} \right] \mathcal{A}_4^{(0)}(b)$$

$$\begin{aligned}\Delta S_{\mathrm{a}}^{\mu} = -G m_{\mathrm{a}} m_{\mathrm{b}} \frac{\gamma}{v} \sum_{\pm} \frac{(1 \mp v)^2}{[b \pm w * (a_{\mathrm{a}} + a_{\mathrm{b}})]^2} \Bigg[& (a_{\mathrm{a}} \cdot [b \pm w * a_{\mathrm{b}}])\, u_{\mathrm{a}}^{\mu} \\ & \pm \frac{1}{\gamma v} \Big((u_{\mathrm{b}} \cdot a_{\mathrm{a}})\, [b \pm w * (a_{\mathrm{a}} + a_{\mathrm{b}})]^{\mu} - (a_{\mathrm{a}} \cdot [b \pm w * a_{\mathrm{b}}])\, [u_{\mathrm{b}} - \gamma u_{\mathrm{a}}]^{\mu} \Big) \Bigg] \Bigg|_{\mathrm{cl}}.\end{aligned}$$

Matches Vines 17'

*cl. means initial momenta $p_{\mathrm{a,b}}^{\mu}$ localized on their classical values $m_{\mathrm{a,b}} u_{\mathrm{a,b}}^{\mu}$

Hamiltonian

$$H(\boldsymbol{r}, \boldsymbol{p}, \boldsymbol{S}_{\mathrm{a}}, \boldsymbol{S}_{\mathrm{b}}) = \sqrt{\boldsymbol{p}^2 + m_{\mathrm{a}}^2} + \sqrt{\boldsymbol{p}^2 + m_{\mathrm{b}}^2} + V(\boldsymbol{r}, \boldsymbol{p}, \boldsymbol{S}_{\mathrm{a}}, \boldsymbol{S}_{\mathrm{b}}).$$

LO potential from tree-level amplitude

$$V^{(1)}(\boldsymbol{r}, \boldsymbol{p}, \boldsymbol{S}_{\mathrm{a}}, \boldsymbol{S}_{\mathrm{b}}) = -\frac{\hbar^3}{4E_{\mathrm{a}}E_{\mathrm{b}}} \int \frac{d^3\bar{\boldsymbol{k}}}{(2\pi)^3} e^{i\bar{\boldsymbol{k}}\cdot\boldsymbol{r}} \mathcal{A}^{(0)}(\bar{\boldsymbol{k}}, \boldsymbol{p}, \boldsymbol{S}_{\mathrm{a}}, \boldsymbol{S}_{\mathrm{b}}),$$

COM kinematics

General spinning bodies

$$V^{(1)}(\boldsymbol{r}, \boldsymbol{p}, \boldsymbol{S}_{\mathrm{a}}, \boldsymbol{S}_{\mathrm{b}}) = -\frac{G m_{\mathrm{a}}^2 m_{\mathrm{b}}^2 \gamma^2}{2E_{\mathrm{a}}E_{\mathrm{b}}} \sum_{\pm} (1 \mp v)^2 \sum_{n_1, n_2 = 0}^{\infty} \frac{C_{\mathrm{a}n_1} C_{\mathrm{b}n_2}}{n_1! n_2!} \left(\pm \frac{1}{m_{\mathrm{a}}} [\hat{\boldsymbol{p}} \times \boldsymbol{S}_{\mathrm{a}}] \cdot \nabla_{\boldsymbol{r}} \right)^{n_1} \left(\pm \frac{1}{m_{\mathrm{b}}} [\hat{\boldsymbol{p}} \times \boldsymbol{S}_{\mathrm{b}}] \cdot \nabla_{\boldsymbol{r}} \right)^{n_2} \frac{1}{|\boldsymbol{r}|}.$$

Hamiltonian

$$H(\boldsymbol{r},\boldsymbol{p},\boldsymbol{S}_{\rm a},\boldsymbol{S}_{\rm b})=\sqrt{\boldsymbol{p}^2+m_{\rm a}^2}+\sqrt{\boldsymbol{p}^2+m_{\rm b}^2}+V(\boldsymbol{r},\boldsymbol{p},\boldsymbol{S}_{\rm a},\boldsymbol{S}_{\rm b}).$$

LO potential from tree-level amplitude

$$V^{(1)}(\boldsymbol{r},\boldsymbol{p},\boldsymbol{S}_{\rm a},\boldsymbol{S}_{\rm b})=-\frac{\hbar^3}{4E_{\rm a}E_{\rm b}}\int\frac{d^3\bar{\boldsymbol{k}}}{(2\pi)^3}e^{i\bar{\boldsymbol{k}}\cdot\boldsymbol{r}}\mathcal{A}^{(0)}(\bar{\boldsymbol{k}},\boldsymbol{p},\boldsymbol{S}_{\rm a},\boldsymbol{S}_{\rm b}),$$

COM kinematics

General spinning bodies

$$V^{(1)}(\boldsymbol{r},\boldsymbol{p},\boldsymbol{S}_{\rm a},\boldsymbol{S}_{\rm b})=-\frac{Gm_{\rm a}^2m_{\rm b}^2\gamma^2}{2E_{\rm a}E_{\rm b}}\sum_{\pm}(1\mp v)^2\sum_{n_1,n_2=0}^{\infty}\frac{C_{{\rm a}n_1}C_{{\rm b}n_2}}{n_1!n_2!}\left(\pm\frac{1}{m_{\rm a}}[\hat{\boldsymbol{p}}\times\boldsymbol{S}_{\rm a}]\cdot\nabla_{\boldsymbol{r}}\right)^{n_1}\left(\pm\frac{1}{m_{\rm b}}[\hat{\boldsymbol{p}}\times\boldsymbol{S}_{\rm b}]\cdot\nabla_{\boldsymbol{r}}\right)^{n_2}\frac{1}{|\boldsymbol{r}|}.$$

Kerr BH

$$V^{(1)}(\boldsymbol{r},\boldsymbol{p},\boldsymbol{S}_{\rm a},\boldsymbol{S}_{\rm b})=-\frac{Gm_{\rm a}^2m_{\rm b}^2\gamma^2}{2E_{\rm a}E_{\rm b}}\sum_{\pm}\frac{(1\pm v)^2}{|\boldsymbol{r}\pm\hat{\boldsymbol{p}}\times(\boldsymbol{a}_{\rm a}+\boldsymbol{a}_{\rm b})|}$$

Validated by integrating the EOM to obtain linear and angular impulses

Conclusion and outlook

- Extended the KMOC formalism to general spinning bodies (described by wilson coeffs.)
- Coherent states provide rigorous framework to extract classical observables from quantum scattering amplitudes
- Quantum amplitudes compatible with the Kerr Black hole is favored in the classical limit (also favored in spin-entanglement)
- Hamiltonian also obtained which can be used beyond the scattering

Thank you for your attention

Covariant Spin

[Guevara, Ochirov, Vines 2019]

General spin wave-functions

$$u_p^{Aa} = \begin{pmatrix} |p^a\rangle \\ |p^a] \end{pmatrix} \qquad \varepsilon_{p\mu}^{ab} = \frac{i\langle p^{(a}|\sigma_\mu|p^{b)}]}{\sqrt{2}m}$$

$$\text{integer } s: \quad \varepsilon_{p\mu_1\ldots\mu_s}^{\{a\}} = \varepsilon_{p\mu_1}^{(a_1a_2} \cdots \varepsilon_{p\mu_s}^{a_{2s-1}a_{2s})},$$

$$\text{half-integer } s: \quad u_{p\mu_1\ldots\mu_{\lfloor s\rfloor}}^{\{a\}} = u_p^{(a_1}\varepsilon_{p\mu_1}^{a_2a_3} \cdots \varepsilon_{p\mu_{\lfloor s\rfloor}}^{a_{2s-1}a_{2s})}.$$

Combined with the Pauli-Lubanski spin operator $\Sigma_\lambda = \frac{1}{2m}\epsilon_{\lambda\mu\nu\rho}\Sigma^{\mu\nu}p^\rho.$

One-particle matrix element:

$$\text{integer } s: \quad \frac{1}{(-1)^s}\,\varepsilon_{p\{a\}}\cdot\Sigma^\mu\cdot\varepsilon_p^{\{b\}} = s\,\sigma_{p\mu,(a_1}{}^{(b_1}\delta_{a_2}^{b_2}\cdots\delta_{a_{2s})}^{b_{2s})},$$

$$\text{half-integer } s: \quad \frac{1}{(-1)^{\lfloor s\rfloor}2m}\,\bar{u}_{p\{a\}}\cdot\Sigma^\mu\cdot u_p^{\{b\}} = s\,\sigma_{p\mu,(a_1}{}^{(b_1}\delta_{a_2}^{b_2}\cdots\delta_{a_{2s})}^{b_{2s})}.$$

Lorentz-covariant SU(2) spin operator: $\sigma_{p\mu,a}{}^{b} = -\frac{1}{2m}\Big(\langle p_a|\sigma_\mu|p^b] + [p_a|\bar{\sigma}_\mu|p^b\rangle\Big),$

The one-particle ang.mom representation (with hbar)

$$(S_p^\mu)_{s,\{a\}}{}^{s',\{b\}} = \hbar s\,\delta_s^{s'}\,\sigma^{\mu,}_{p\ (a_1}{}^{(b_1}\delta_{a_2}^{b_2}\cdots\delta_{a_{2s})}^{b_{2s})} = \hbar s\,\delta_s^{s'}\,\sigma^{\mu,}_{p\ a}{}^{b}\odot\left(\delta_a^b\right)^{\odot(2s-1)}$$

Pauli-Lubanski

Combined with the Pauli-Lubanski spin operator $\Sigma_\lambda = \frac{1}{2m}\epsilon_{\lambda\mu\nu\rho}\Sigma^{\mu\nu}p^\rho.$

Inner products: $\varepsilon_{p\{a\}} \cdot \varepsilon_p^{\{b\}} = (-1)^s\big(\delta_a^b\big)^{\odot 2s},$ $\qquad \bar{u}_{p\{a\}} \cdot u_p^{\{b\}} = (-1)^{\lfloor s \rfloor}\, 2m\big(\delta_a^b\big)^{\odot 2s}$

Generalization of Lorentz generator:

$$\Sigma^{\mu\nu,\sigma}{}_\tau = i[\eta^{\mu\sigma}\delta^\nu_\tau - \eta^{\nu\sigma}\delta^\mu_\tau]$$

$$\text{integer } s: \quad (\Sigma_s^{\mu\nu})^{\sigma_1\ldots\sigma_s}{}_{\tau_1\ldots\tau_s} = \Sigma^{\mu\nu,\sigma_1}{}_{\tau_1}\delta^{\sigma_2}_{\tau_2}\cdots\delta^{\sigma_s}_{\tau_s} + \ldots + \delta^{\sigma_1}_{\tau_1}\cdots\delta^{\sigma_{s-1}}_{\tau_{s-1}}\Sigma^{\mu\nu,\sigma_s}{}_{\tau_s},$$

$$\text{half-integer } s: \quad \Sigma_s^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu] + \Sigma^{\mu\nu}_{\lfloor s \rfloor},$$

One-particle matrix element:

$$\text{integer } s: \quad \frac{1}{(-1)^s}\,\varepsilon_{p\{a\}}\cdot\Sigma^\mu\cdot\varepsilon_p^{\{b\}} = s\,\sigma_{p\mu,(a_1}{}^{(b_1}\delta^{b_2}_{a_2}\cdots\delta^{b_{2s})}_{a_{2s})},$$

$$\text{half-integer } s: \quad \frac{1}{(-1)^{\lfloor s \rfloor}2m}\,\bar{u}_{p\{a\}}\cdot\Sigma^\mu\cdot u_p^{\{b\}} = s\,\sigma_{p\mu,(a_1}{}^{(b_1}\delta^{b_2}_{a_2}\cdots\delta^{b_{2s})}_{a_{2s})}.$$

PM vs. PN expansion

Viral theorem

$$v^2 \sim \frac{GM}{r} \ll 1$$

PN double expansion

1PN 2PN 3PN 4PN 5PN 5PN 6PN

1PM $$(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + v^{14} + \ldots)\, G$$

2PM $$(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \ldots)\, G^2$$

3PM $$(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + \ldots)\, G^3$$

4PM $$(1 + v^2 + v^4 + v^6 + v^8 + \ldots)\, G^4$$

5PM $$(1 + v^2 + v^4 + v^6 + \ldots)\, G^5$$

Eikonal phase

Fourier transform to the impact parameter

$$\mathcal{A}_4^{(0)}(b) = \int_k e^{-i\bar{k}\cdot b}\mathcal{A}^{(0)}(p_{\mathrm{a}}+k/2,\alpha;p_{\mathrm{b}}-k/2,\beta|p_{\mathrm{a}}-k/2,\alpha;p_{\mathrm{b}}+k/2,\beta)$$

Transfer momenta becomes derivatives in impact parameter space

$$\mathcal{A}_4^{(0)}(b) = -\frac{Gm_{\mathrm{a}}m_{\mathrm{b}}\gamma}{\hbar v}\sum_{\pm}(1\pm v)^2\sum_{n_1,n_2=0}^{\infty}\frac{(\pm 1)^{n_1+n_2}}{n_1!n_2!}C_{\mathrm{a}n_1}C_{\mathrm{b}n_2}\times\big([w*a_{\mathrm{a}}]\cdot\partial_{b_\perp}\big)^{n_1}\big([w*a_{\mathrm{b}}]\cdot\partial_{b_\perp}\big)^{n_2}\log\sqrt{-b_\perp^2}+\mathcal{O}(\hbar^{-1/2}).$$

For Kerr $C_{\mathrm{a}n} = C_{\mathrm{b}n} = (-1)^n$

$$\mathcal{A}_4^{(0)}(b) = -\frac{Gm_{\mathrm{a}}m_{\mathrm{b}}\gamma}{\hbar v}\sum_{\pm}(1\pm v)^2\log\sqrt{-\big(b_\perp\mp w*(a_{\mathrm{a}}+a_{\mathrm{b}})\big)^2}+\mathcal{O}(\hbar^{-1/2}),$$