Statistics

or “How to find answers to your questions”

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en mathématique et physique

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Program for today

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Confidence Intervals in nontrivial cases
Test of hypotheses
Lesson 3
- CLs
- Significance

Truth and models

- **Lesson 1 - Fundaments**
  - Bayesian and frequentist probability, theory of measure, correlation and causality, distributions
- **Lesson 2 - Point and Interval estimation**
  - Maximum likelihood methods, confidence intervals, most probable values, credible intervals
- **Lesson 3 - Advanced interval estimation, test of hypotheses**
  - Interval estimation near the physical boundary of a parameter
  - Frequentist and Bayesian tests, CLs, significance, look-elsewhere effect, reproducibility crysis
- **Lesson 4 - Commonly-used methods in particle physics**
  - Unfolding, ABCD, ABC, MCMC, estimating efficiencies
- **Lesson 5 - Machine Learning**
  - Overview and mathematical foundations, generalities most used algorithms, automatic Differentiation and Deep Learning

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- Measure N times the same quantity: values x_i and uncertainties σ_i. MLE and variance are:

$$\hat{x}_{ML} = \frac{\sum_{i=1}^{N} \frac{x_i}{\sigma_i^2}}{\sum_{i=1}^{N} \frac{1}{\sigma_i^2}}$$

$$\frac{1}{\hat{\sigma}_x^2} = \sum_{i=1}^{N} \frac{1}{\sigma_i^2}$$

- The MLE is obtained when each measurement is weighted by its own variance
  - This is because the variance is essentially an estimate of how much information lies in each measurement
- This works if the p.d.f. is known
  - Compare this method with an alternative one that does not assume knowledge of the p.d.f.
  - The second method will be the only one applicable to cases in which the p.d.f. is unknown

- Take a set of measures sampled from an unknown p.d.f. $f(\vec{x}, \vec{\theta})$
- Compute the expected value and variance of a combination of such measurements described by a function $g(\vec{x})$.
- The expected value and variance of x_i are elementary:

$$\mu = E[x] V_{ij} = E[x_i x_j] - \mu_i \mu_j$$

- If we want to extract the p.d.f. of $g(\vec{x})$, we would normally use the jacobian of the transformation of f to g, but in this case we assumed $f(\vec{x})$ is unknown.

- We don't know f, but we can still write an expansion in series for it:

$$g(\vec{x}) \simeq g(\vec{\mu}) + \sum_{i=1}^{N} \left(\frac{\partial g}{\partial x_i}\right)\Big|_{x=\mu} (x_i - \mu_i)$$

- We can compute the expected value and variance of g by using the expansion:

$$E\left[g(\vec{x})\right] \simeq g(\mu), \qquad (E[x_i - \mu_i] = 0)$$

$$\sigma_g^2 = \sum_{ij=1}^{N} \left[\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j}\right]\Big|_{\vec{x}=\vec{\mu}} V_{ij}$$

- The variances are propagated to g by means of their jacobian!
- For a sum of measurements, $y = g(\vec{x}) = x_1 + x_2$, the variance of y is $\sigma_y^2 = \sigma_1^2 + \sigma_2^2 + 2V_{12}$, which is reduced to the sum of squares for independent measurements

- Let's compare the two ways of combining measurements, and check the role of the Fisher Information
- Let's estimate the time taken for a laser light pulse to go from the Earth to the Moon and back (in units of Earth-to-Moon-Time EMT)
  - On the Moon we have a receiver built by NASA. It's very good but placed in unfavourable conditions, yielding only a 2% precision on Earth-to-Moon
  - On Earth we have a receiver made out of scrap material. It is however placed in favourable conditions, yielding a 5% precisionon Moon-to-Earth

$$N_{EM} = 0.99 \pm 0.02\ EMT$$

$$N_{ME} = 1.05 \pm 0.05\ EMT$$

- Evidently, the time to moon and back is $N_{EME} = N_{EM} + N_{ME}$, and we can apply Eq. 6: Do it!

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- Resulting estimate:
  - $N_{EME} = 0.99 + 1.05 \pm \sqrt{0.02^2 + 0.05^2}\ EMT = 2.05 \pm 0.05\ EMT$, corresponding to a precision of $\frac{\sigma_{N_{EME}}}{N_{EME}} \sim 2.4\%$.

Combination of measurements: example 2/

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- We can therefore use Eq. 4 to compute $\frac{N_{EME}}{2}$ and multiply the result by 2, obtaining

$$N_{EME} = 2.00 \pm 0.03\ EMT$$

- This estimate corresponds to a precision of only 1.5%!!!
- The dramatic improvement in the precision of the measurement, from 2.4% to 1.5%, is a direct consequence of having used additional information under the form of a relationship (constraint) between the two available measurements.
- A good physicist exploits as many constraints as possible in order to improve the precision of a measurement
  - Sometimes the contraints are arbitrary or correspond to special cases
  - Is is very important to explicitly mention any constraint used to derive a measurement, when quoting the result.

What about asymmetric uncertainties?

- Now suppose my receivers operate by taking data and performing a maximum likelihood fit to estimate N_{EM} and N_{ME}
- Can I combine these two measurements with the two methods seen above?
  - $N_{EM} = 0.99 \pm 0.03$
  - $N_{ME} = 1.10^{+0.05}_{-0.01}$
- For example, $N_{EMT} = 2.09^{+0.06}_{-0.03}$

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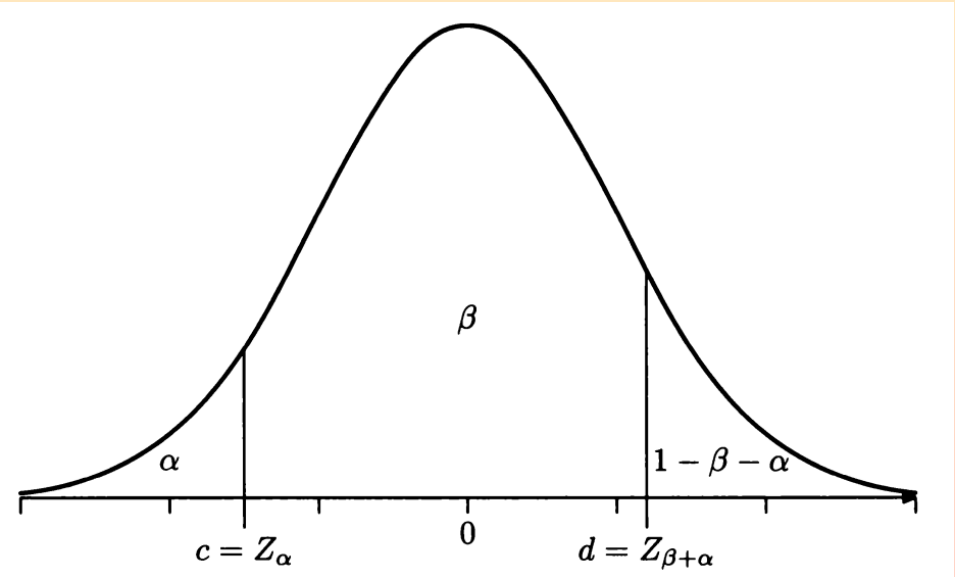
- Why?
- The naïve quadrature of the two uncertainties is wrong!
  - The naïve combination is an expression of the Central Limit Theorem
  - The resulting combination is expected to be more symmetric than the measurements it originates from
  - Symmetric uncertainties usually assume a Gaussian approximation of the likelihood
  - Asymmetric uncertainties? One would need a study of the non-linearity (large biases might be introduced if ignoring this)
- Intrinsic difference between averaging and most probable value
  - Averaging results in average value and variance that propagate linearly
  - Taking the mode (essentially what MLE does) does not add up linearly!
- With asymmetric uncertainties from MLE fits, always combine the likelihoods (better in an individual simultaneous fit)

Confidence Intervals in nontrivial cases

Confidence intervals!

- Confidence interval for θ with probability content β
  - The range $\theta_a < \theta < \theta_b$ containing the true value θ_0 with probability β
  - The physicists sometimes improperly say the uncertainty on the parameter θ
- Given a p.d.f., the probability content is $\beta = P(a \leq X \leq b) = \int_a^b f(X|\theta)dX$
- If θ is unknown (as is usually the case), use auxiliary variable $Z = Z(X, \theta)$ with p.d.f. $g(Z)$ independent of θ
- If Z can be found, then the problem is to estimate interval $P(\theta_a \leq \theta_0 \leq \theta_b) = \beta$
  - Confidence interval
  - A method yielding an interval satisfying this property has coverage

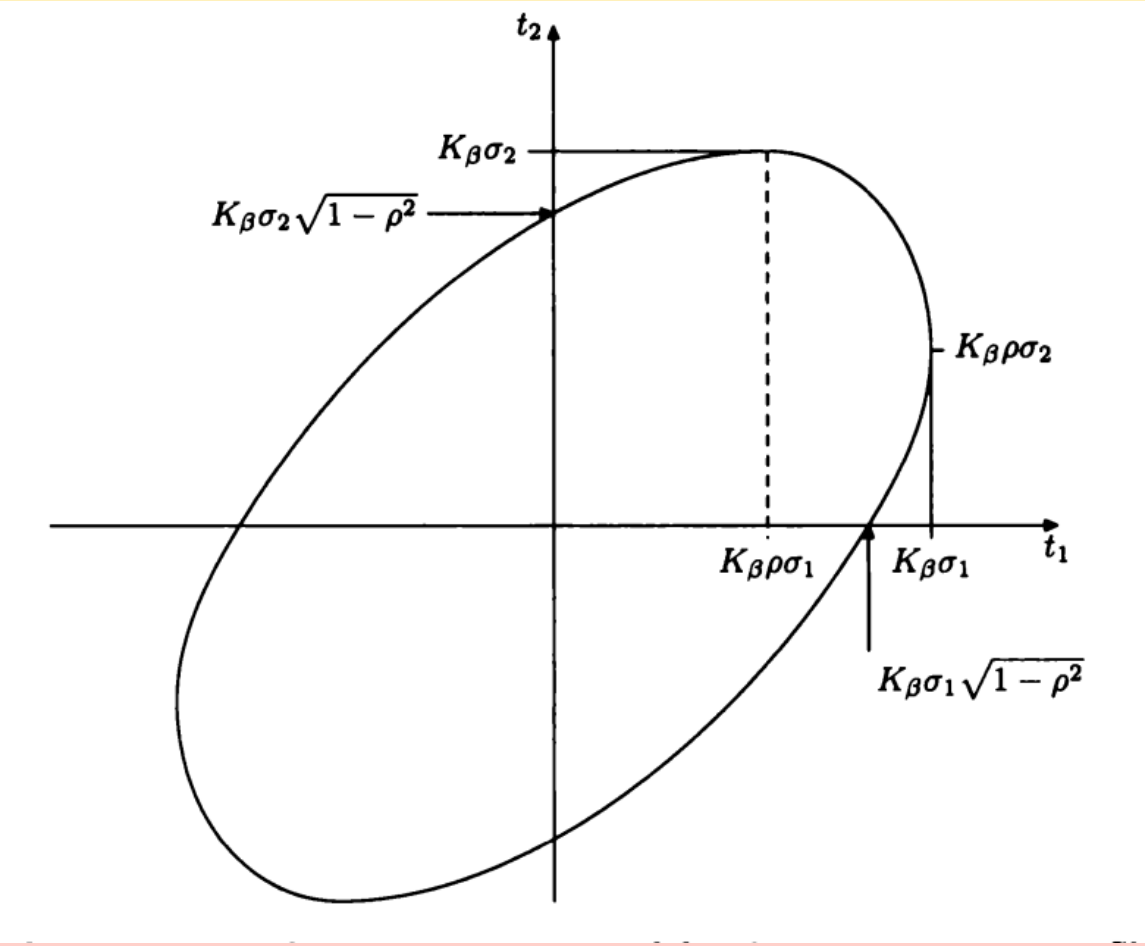
- Example: if $f(X|\theta) = N(\mu, \sigma^2)$ with unknown μ, σ, choose $Z = \frac{X-\mu}{\sigma}$
- Find $[c, d]$ in $\beta = P(c \leq Z \leq d) = \Phi(d) - \Phi(c)$ by finding $[Z_\alpha, Z_{\alpha+\beta}]$
- Infinite interval choices: here central interval $\alpha = \frac{1-\beta}{2}$



Plot from James, 2nd ed.

Confidence intervals in many dimensions

- Generalization to multidimensional $\boldsymbol{\theta}$ is immediate
- Probability statement concerns the whole $\boldsymbol{\theta}$, not the individual θ_i
- Shape of the ellipsoid governed by the correlation coefficient (or the mutual information) between the parameters
- Arbitrariety in the choice of the interval is still present



Plot from James, 2nd ed.

- Coverage probability of a method for calculating a confidence interval $[\theta_1, \theta_2]$: $P(\theta_1 \leq \theta_{true} \leq \theta_2)$
  - Fraction of times, over a set of (usually hypothetical) measurements, that the resulting interval covers the true value of the parameter
  - Can sample with toys to study coverage
- Coverage is not a property of a specific confidence interval!
- **Coverage is a property of the method you use to compute your confidence interval**
  - It is calculated from the sampling distribution of your confidence intervals
- The nominal coverage is the value of confidence level you have built your method around (often 0.95)
- When actually derive a set of intervals, the fraction of them that contain θ_{true} ideally would be equal to the nominal coverage
  - You can build toy experiments in each of whose you sample N times for a known value of θ_{true}
  - You calculate the interval for each toy experiment
  - You count how many times the interval contains the true value
- Nominal coverage (CL) and the actual coverage (Co) observed with toys should agree
  - If all the assumptions you used in computing the intervals are valid
  - If they don't agree, it might be that $Co < CL$ (undercoverage) or $Co > CL$ (overcoverage)
  - It's OK to strive to be conservative, but one might be unnecessarily lowering the precision of the measurement
  - When $Co! = CL$ you usually want at least a convergence to equality in some limit

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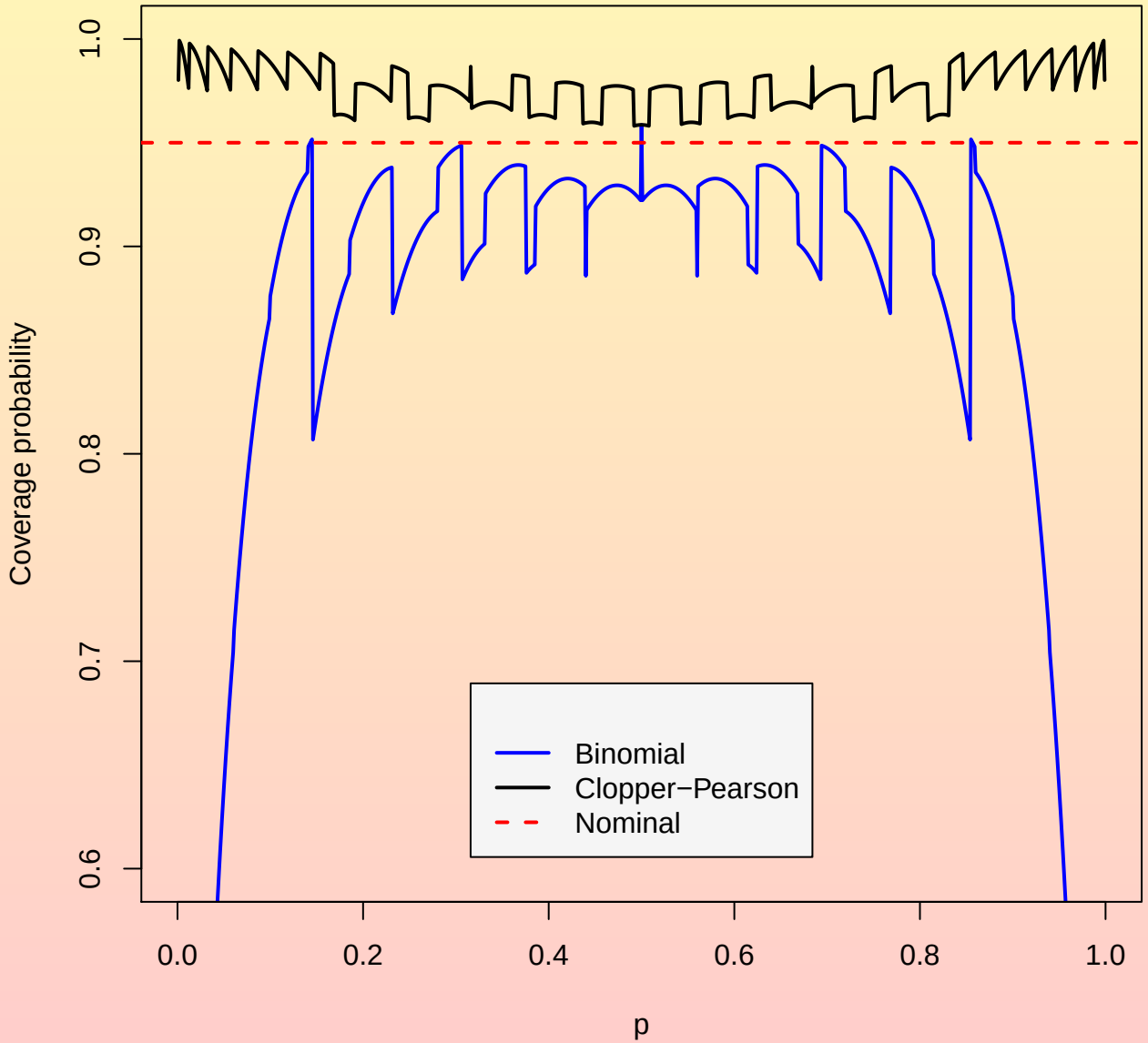
- For discrete distributions, the discreteness induces steps in the probability content of the interval
  - Continuous case: $P(a \leq X \leq b) = \int_a^b f(X|\theta)dX = \beta$
  - Discrete case: $P(a \leq X \leq b) = \sum_a^b f(X|\theta)dX \leq \beta$
- Binomial: find interval (r_{low}, r_{high}) such that $\sum_{r=r_{low}}^{r=r_{high}} \binom{r}{N} p^r (1-p)^{N-r} \leq 1-\alpha$
  - Also, $\binom{r}{N}$ computationally taxing for large r and N
  - Approximations are found in order to deal with the problem
- Gaussian approximation: $p \pm Z_{1-\alpha/2}\sqrt{\frac{p(1-p)}{N}}$
- Clopper Pearson: invert two single-tailed binomial tests
  $\sum_{r=0}^{N} \binom{r}{N} p^n (1-p_{low})^{N-n} \leq \alpha/2$
  $\sum_{r=0}^{N} \binom{r}{N} p^r (1-p_{high})^{N-r} \leq \alpha/2$
  - Single-tailed $\rightarrow$ use $\alpha/2$ instead of α

Coverage: the binomial case

- Gaussian approximation: $p \pm Z_{1-\alpha/2}\sqrt{\frac{p(1-p)}{N}}$
- Clopper Pearson: invert two single-tailed binomial tests, designed to overcover
  $\sum_{r=0}^{N} \binom{r}{N} p^n (1-p_{low})^{N-n} \leq \alpha/2$
  $\sum_{r=0}^{N} \binom{r}{N} p^r (1-p_{high})^{N-r} \leq \alpha/2$
  - Single-tailed $\rightarrow$ use $\alpha/2$ instead of α
- This afternoon we will study the coverage of intervals from a gaussian approximation and from the Clopper-Pearson method
- We will also study the coverage of intervals obtained from crossings with ΔlnL
- Question time: Coverage

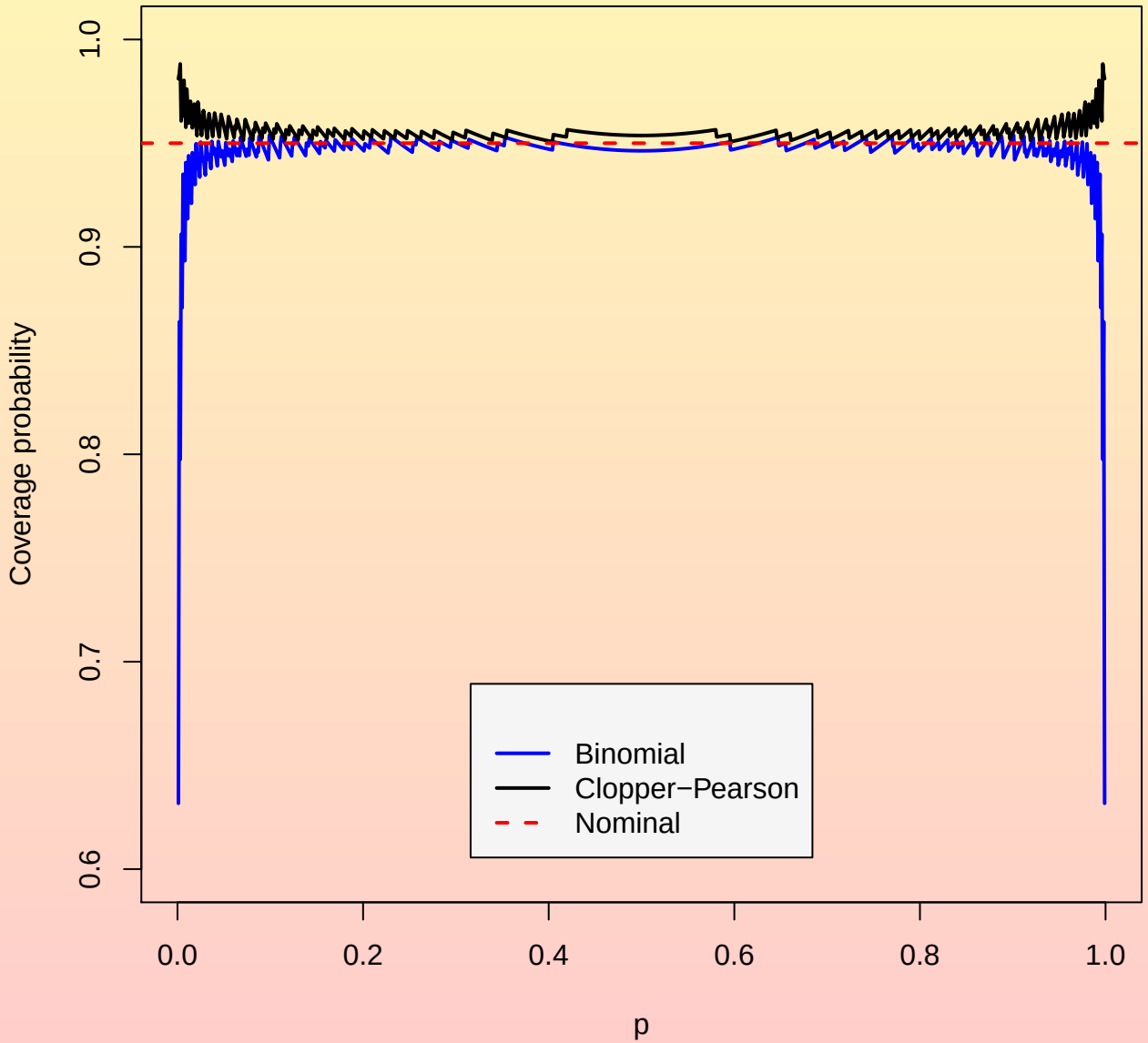
Coverage, $N = 20$

- Gaussian approximation bad for small sample sizes



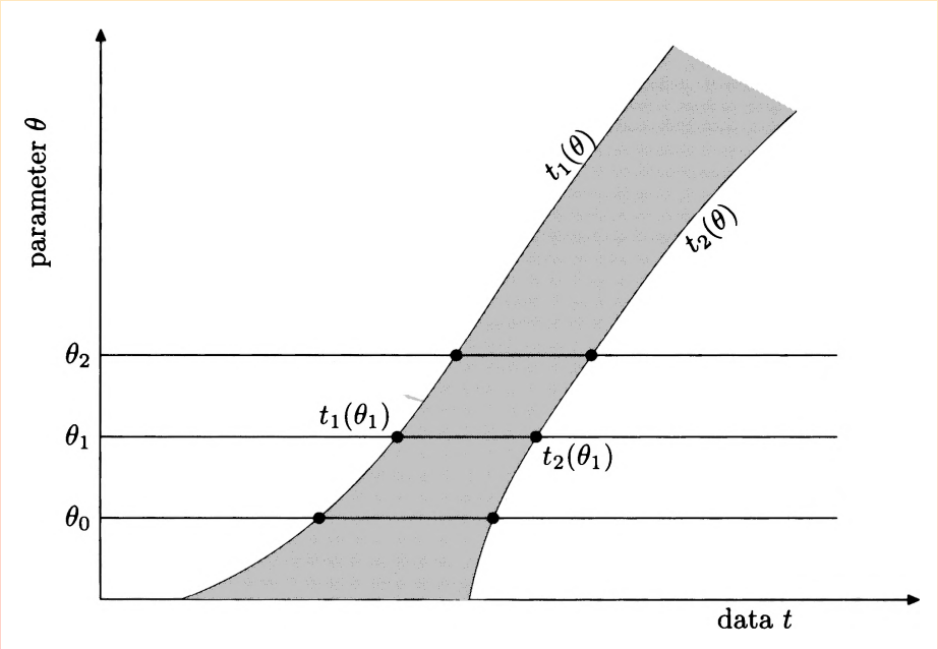
Coverage, $N = 1000$

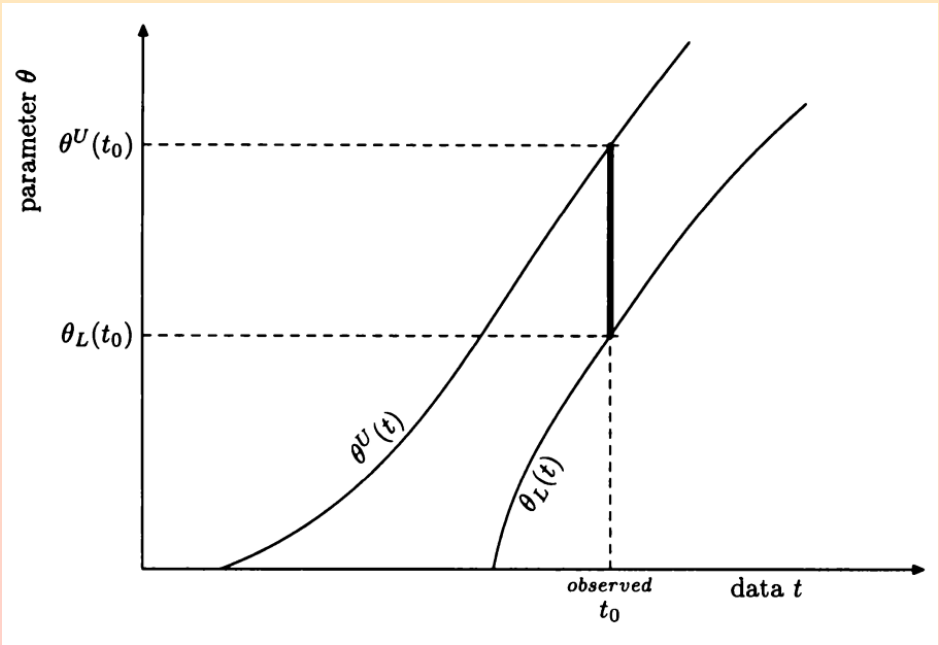
- Gaussian approximation bad near $p = 0$ and $p = 1$ even for large sample sizes



Confidence belts: the Neyman construction

- Unique solutions to finding confidence intervals are infinite
  - Central intervals, lower limits, upper limits, etc
- Let's suppose we have chosen a way
- Build horizontally: for each (hypothetical) value of θ, determine $t_1(\theta)$, $t_2(\theta)$ such that $\int_{t_1}^{t_2} P(t|\theta)dt = \beta$
- Read vertically: from the observed value t_0, determine $[\theta_L, \theta^U]$ by intersection
  - The resulting interval might be disconnected in severely non-linear cases
- Probability content statements to be seen in a frequentist way
  - Repeating many times the experiment, the fraction of $[\theta_L, \theta^U]$ containing θ_0 is β

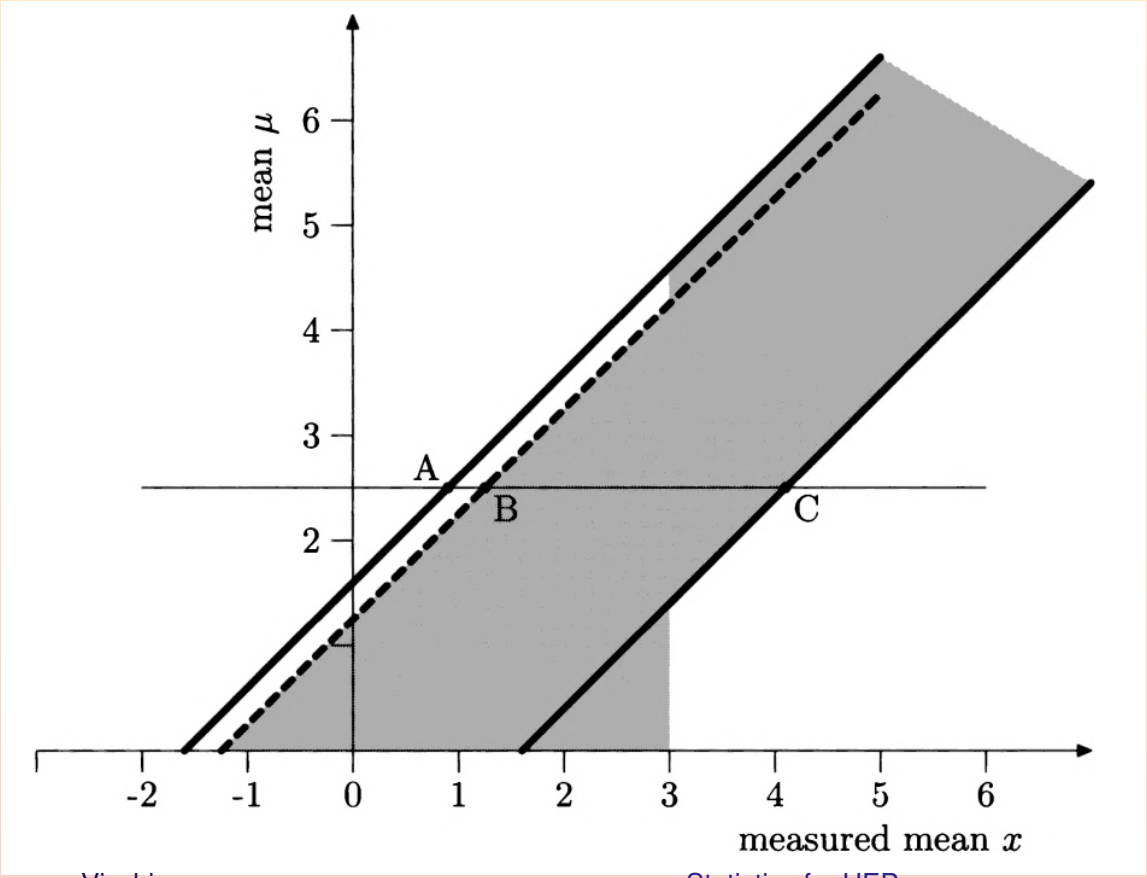




Plot from James, 2nd ed.

Upper limits for non-negative parameters

- Gaussian measurement (variance 1) of a non-negative parameter $\mu \sim 0$ (physical bound)
- Individual prescriptions are self-consistent
  - 90% central limit (solid lines)
  - 90% upper limit (single dashed line)
- Other choices are problematic (flip-flopping): never choose after seeing the data!
  - "quote upper limit if x_{obs} is less than 3σ from zero, and central limit above" (shaded)
  - Coverage not guaranteed anymore (see e.g. $\mu = 2.5$)
- Unphysical values and empty intervals: choose 90% central interval, measure $x_{obs} = -2.0$
  - Don't extrapolate to an unphysical interval for the true value of μ!
  - The interval is simply empty, i.e. does not contain any allowed value of μ
  - The method still has coverage (90% of other hypothetical intervals would cover the true value)

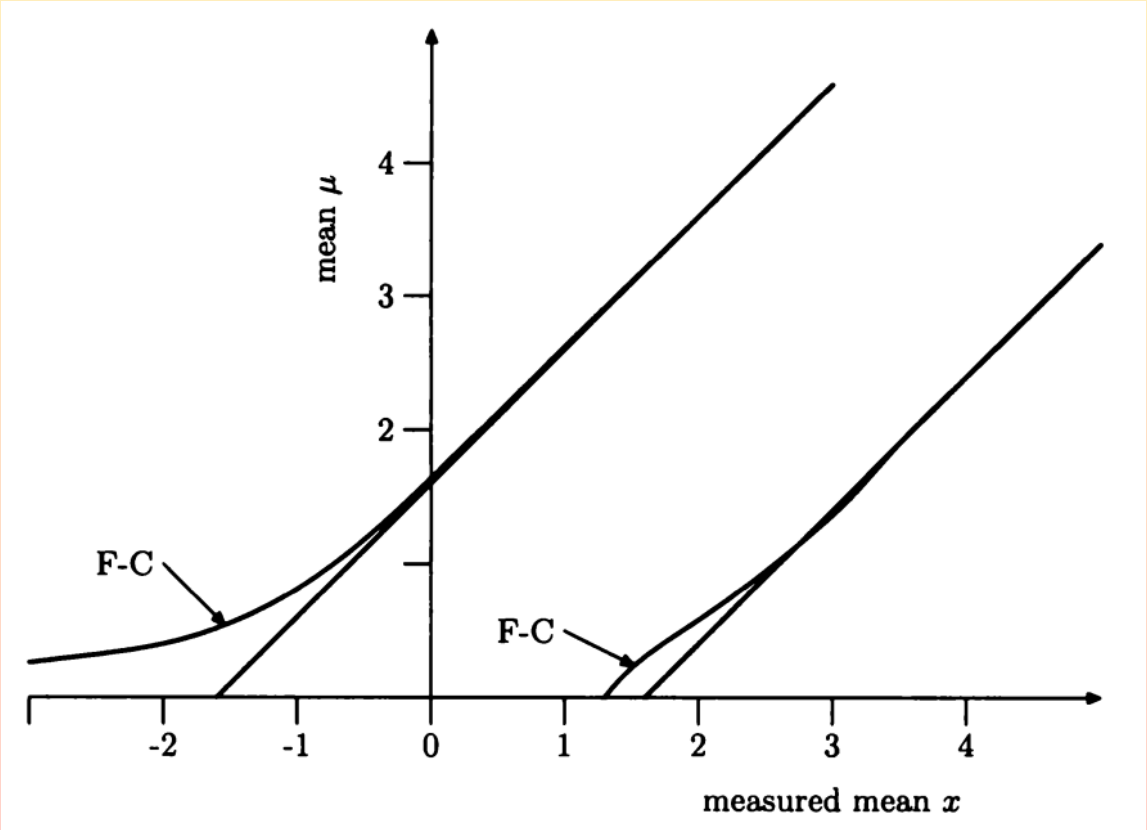


Plot from James, 2nd ed.

Unphysical values: Feldman-Cousins

- The Neyman construction results in guaranteed coverage, but choice still free on how to fill probability content
  - Different ordering principles are possible (e.g. central/upper/lower limits)
- Unified approach for determining interval for $\mu = \mu_0$: the likelihood ratio ordering principle
  - Include in order by largest $\ell(x) = \frac{P(x|\mu_0)}{P(x|\hat{\mu})}$
  - $\hat{\mu}$ value of μ which maximizes $P(x|\mu)$ within the physical region
  - $\hat{\mu}$ remains equal to zero for $\mu < 1.65$, yielding deviation w.r.t. central intervals

- Minimizes Type II error (likelihood ratio for simple test is the most powerful test)
- Solves the problem of empty intervals
- Avoids flip-flopping in choosing an ordering prescription



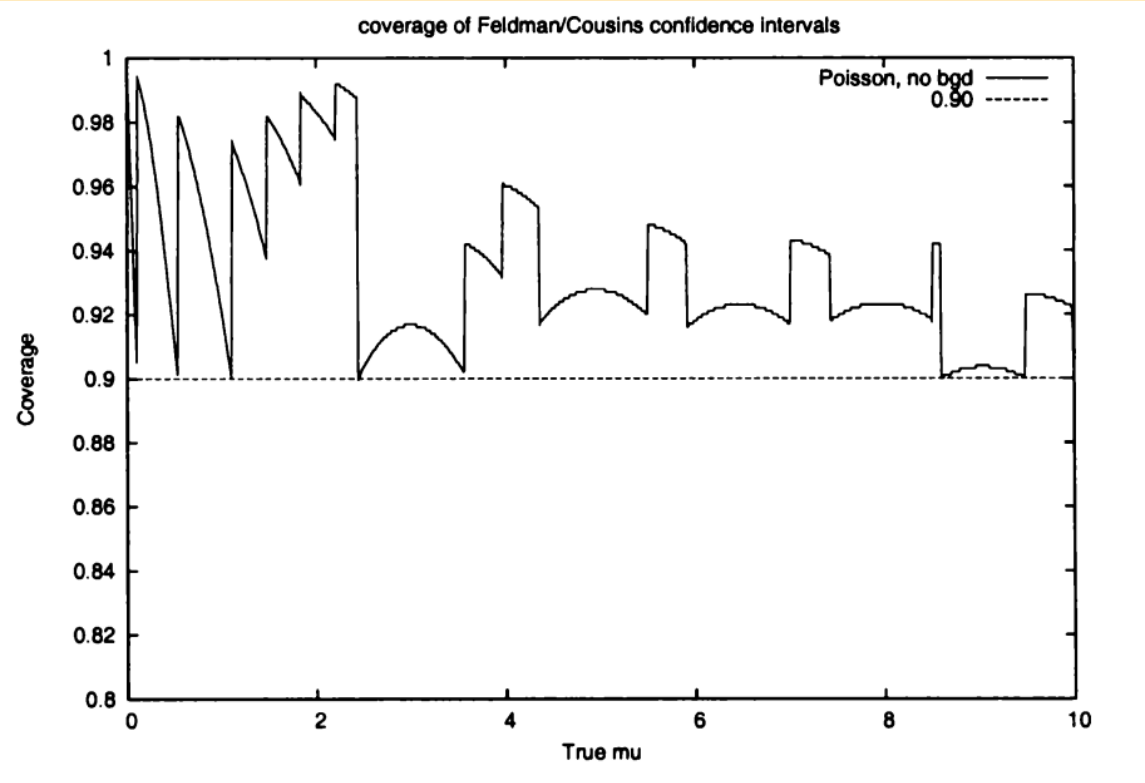
Plot from James, 2nd ed.

Feldman-Cousins in HEP

- The most typical HEP application of F-C is confidence belts for the mean of a Poisson distribution
- Discreteness of the problem affects coverage
- When performing the Neyman construction, will add discrete elements of probability
- The exact probability content won't be achieved, must accept overcoverage

$$\int_{x_1}^{x_2} f(x|\theta)dx = \beta \qquad \rightarrow \qquad \sum_{i=L}^{U} P(x_i|\theta) \geq \beta$$

- Overcoverage larger for small values of μ (but less than other methods)



Plot from James, 2nd ed.

- Often numerically identical to frequentist confidence intervals
  - Particularly in the large sample limit
- Interpretation is different: credible intervals
- Posterior density summarizes the complete knowledge about θ

$$\pi(\theta|\boldsymbol{X}) = \frac{\prod_{i=1}^{N} f(X_i, \theta)\pi(\theta)}{\int \prod_{i=1}^{N} f(X_i, \theta)\pi(\theta) d\theta}$$

- Sometimes you may want to summarize the prior with estimates of its location and of its dispersion
  - For the location, you can use mode or median (see tomorrow's lecture)
- An interval $[\theta_L, \theta^U]$ with content β defined by $\int_{\theta_L}^{\theta^U} \pi(\theta|\boldsymbol{X}) d\theta = \beta$
- Bayesian statement! $P(\theta_L < \theta < \theta^U) = \beta$
  - Again, non unique
- Issues with empty intervals don't arise, though, because the prior takes care of defining the physical region in a natural way!
  - But this implies that central intervals cannot be seamlessly converted into upper limits
  - Need the notion of shortest interval
  - Issue of the metric (present in frequentist statistic) solved because here the preferred metric is defined by the prior

Bayesian intervals and coverage

- What about computing the frequentist coverage for Bayesian intervals?
- Question time: Coverage Bayes

Bayesian intervals and coverage

- What about computing the frequentist coverage for Bayesian intervals?
- Question time: Coverage Bayes
- Even if you are not interested in frequentist methods, it can be useful! Certainly it doesn't hurt
- Knowing the sampling properties of a method can always give insights or work as a cross-check of the method
- Particularly given that typically Bayesian and frequentist answers tend to converge in the high-N limit
  - Except for hypothesis tests, we'll find out later today

Image from the Statistical Statistics Memes Facebook Page

Test of Hypotheses

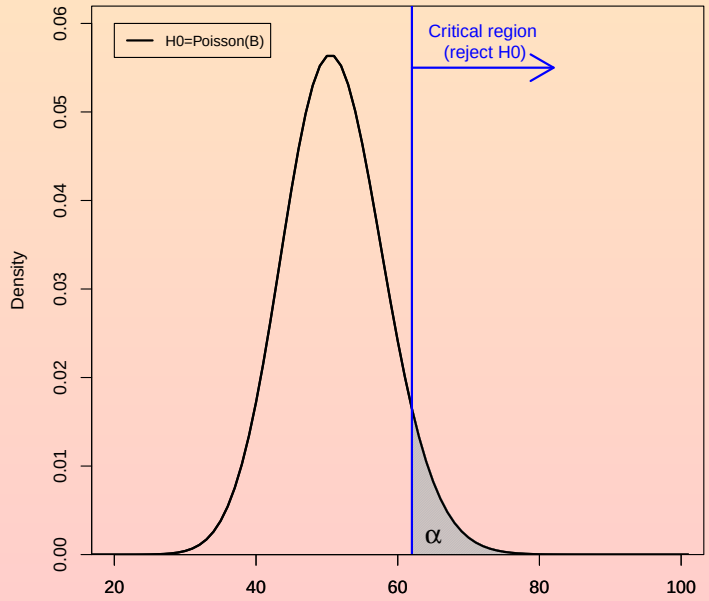
What is an hypothesis...

- Is our hypothesis compatible with the experimental data? By how much?
- Hypothesis: a complete rule that defines probabilities for data.
  - An hypothesis is simple if it is completely specified (or if each of its parameters is fixed to a single value)
  - An hypothesis is complex if it consists in fact in a family of hypotheses parameterized by one or more parameters
- "Classical" hypothesis testing is based on frequentist statistics
  - An hypothesis—as we do for a parameter $\vec{\theta}_{true}$—is either true or false. We might improperly say that $P(H)$ can only be either 0 or 1
  - The concept of probability is defined only for a set of data $\vec{x}$
- We take into account probabilities for data, $P(\vec{x}|H)$
  - For a fixed hypotesis, often we write $P(\vec{x}; H)$, skipping over the fact that it is a conditional probability
  - The size of the vector $\vec{x}$ can be large or just 1, and the data can be either continuos or discrete.

- The hypothesis can depend on a parameter
  - Technically, it consists in a family of hypotheses scanned by the parameter
  - We use the parameter as a proxy for the hypothesis, $P(\vec{x};\theta) := P(\vec{x};H(\theta))$.
- We are working in frequentist statistics, so there is no $P(H)$ enabling conversion from $P(\vec{x}|\theta)$ to $P(\theta|\vec{x})$.
- Statistical test
  - A statistical test is a proposition concerning the compatibility of H with the available data.
  - A binary test has only two possible outcomes: either accept or reject the hypothesis

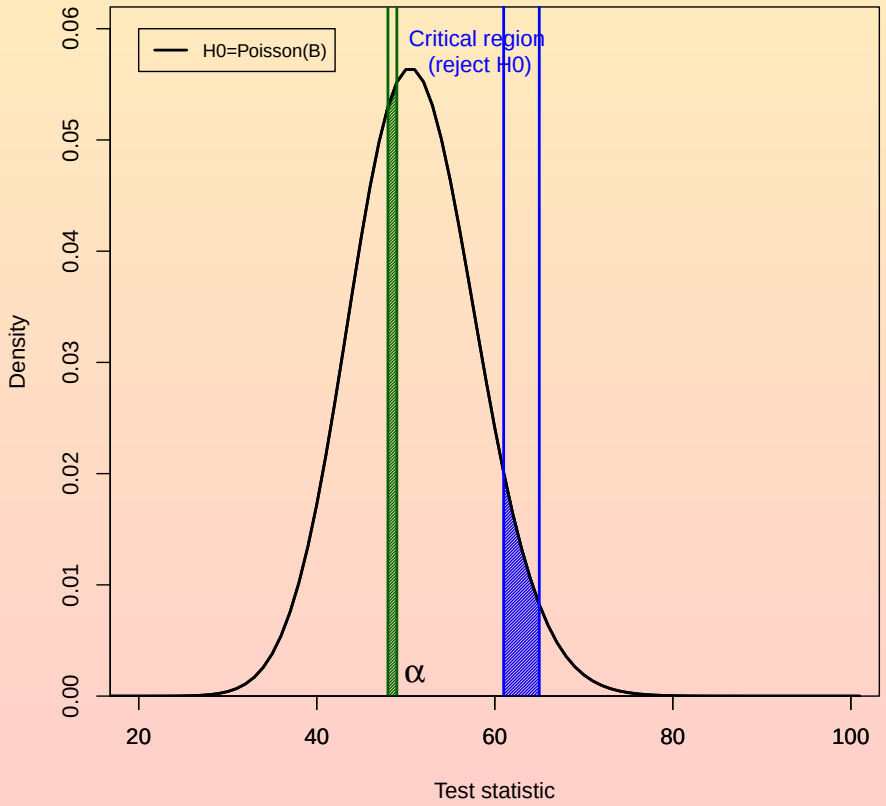
Testing an hypothesis H_0...

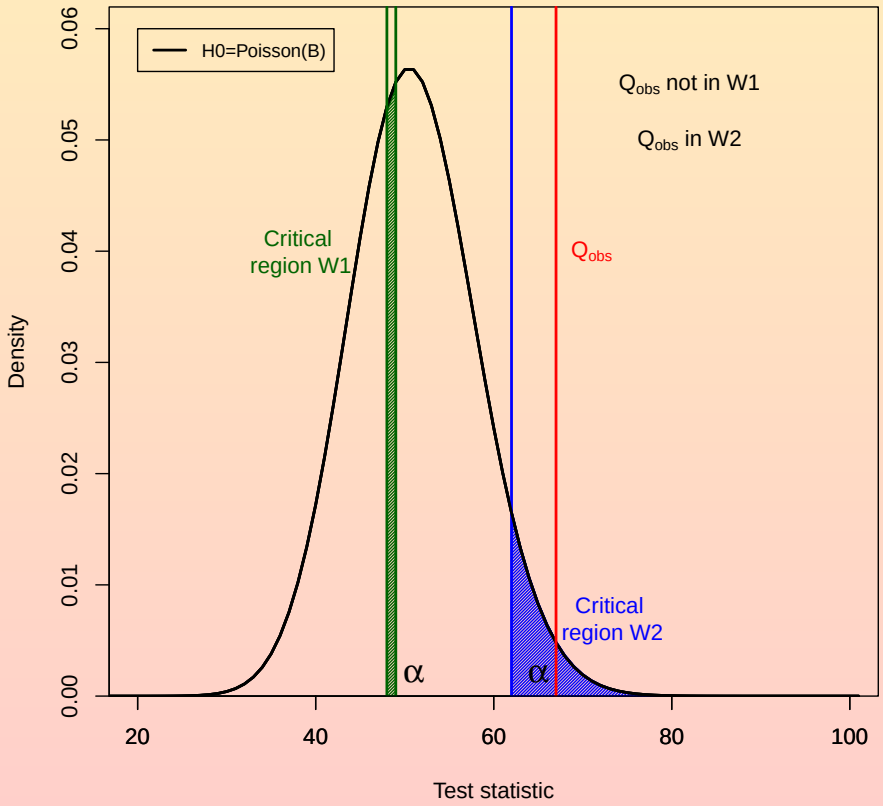
- H_0 is normally the hypothesis that we assume true in absence of further evidence
- Let $\mathbf{X}$ be a function of the observations (called "*test statistic*")
- Let W be the space of all possible values of $\mathbf{X}$, and divide it into
  - A critical region w: observations X falling into w are regarded as suggesting that H_0 is NOT true
  - A region of acceptance $W - w$
- The size of the critical region is adjusted to obtain a desired *level of significance* α
  - Also called *size of the test*
  - $P(X \in w | H_0) = \alpha$
  - α is the (hopefully small) probability of rejecting H_0 when H_0 is actually true
- Once $\mathcal{W}$ is defined, given an observed value $\vec{x}_{obs}$ in the space of data, we define the test by saying that we reject the hypothesis H_0 if $\vec{x}_{obs} \in W$.
- If $\vec{x}_{obs}$ is inside the critical region, then H_0 is rejected; in the other case, H_0 is accepted
  - In this context, accepting H_0 does not mean demonstrating its truth, but simply not rejecting it
- Choosing a small α is equivalent to giving a priori preference to H_0!!!



...while introducing some spice in it

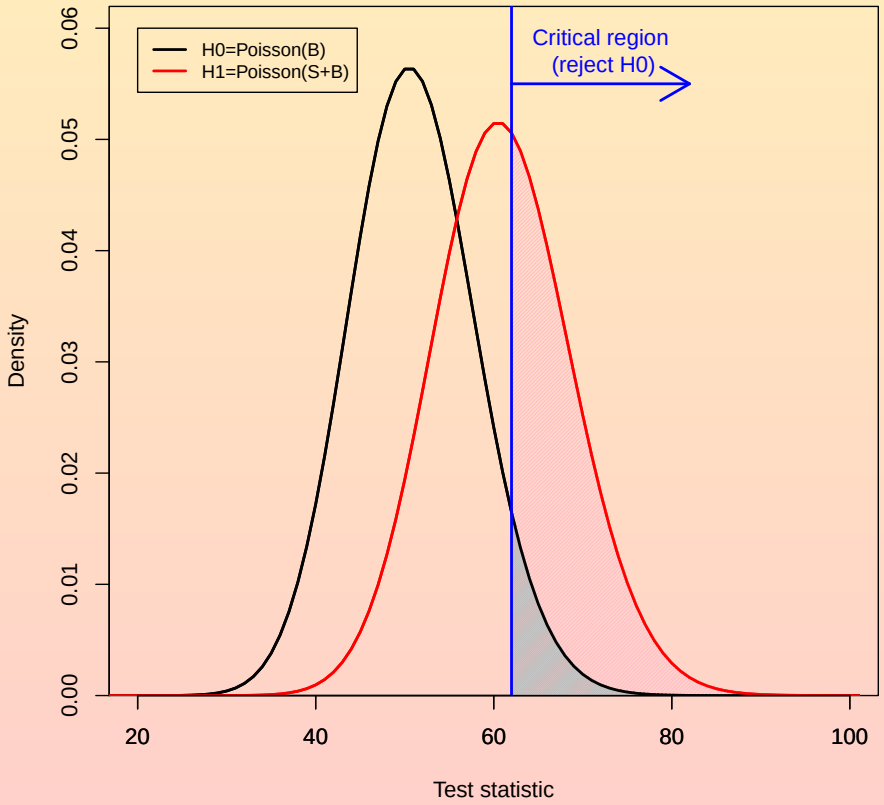
- The definition of $\mathcal{W}$ depends only on its area α, without any other condition
  - Any other area of area α can be defined as critical region, independently on how it is placed with respect to $\vec{x}_{obs}$
  - In particular, for an infinite number of choices of $\mathcal{W}$, the point $\vec{x}_{obs}$—which beforehand was situated outside of $\mathcal{W}$—is now included inside the critical region
  - In this condition, the result of the test switches from accept H_0 to reject H_0
- To remove or at least reduce this arbitrariness in the choice of $\mathcal{W}$, we introduce the alternative hypothesis, H_1

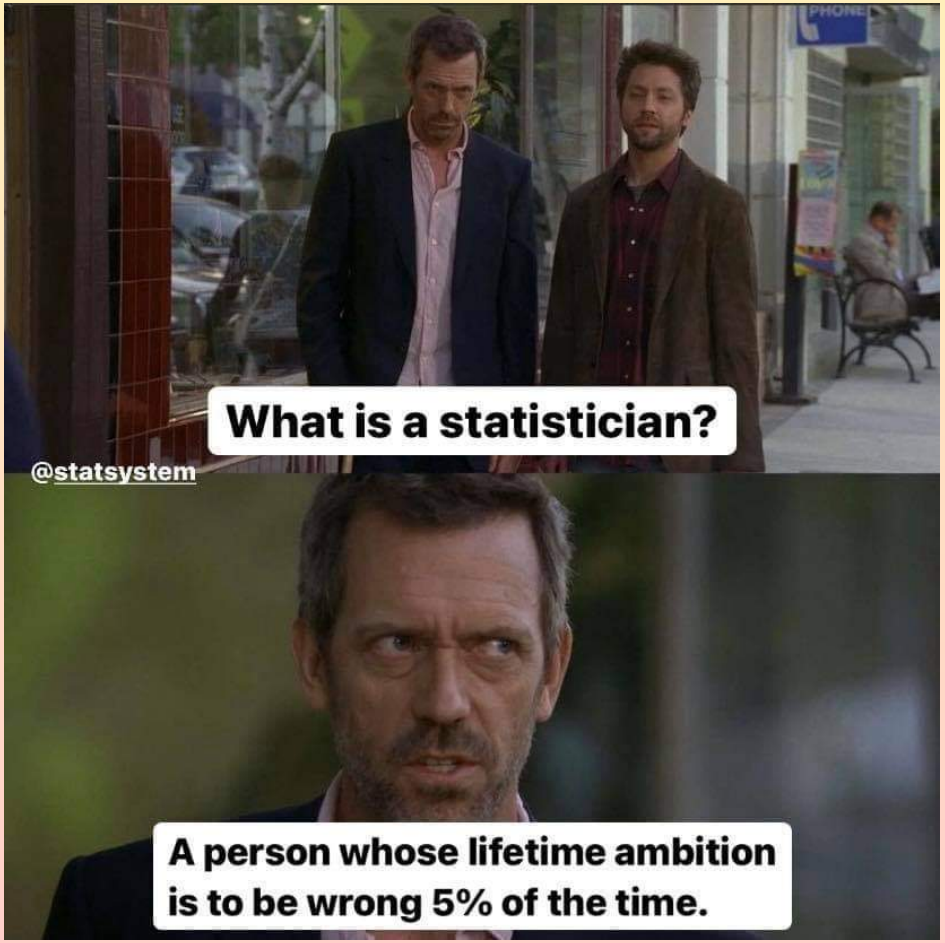




Choose reasonable regions

- Choose a critical region so that $P(\vec{x} \in \mathcal{W}|H_0)$ is α under H_0, and as large as possible under H_1
- Choice of regions is somehow arbitrary, and many choices are not more justified than others
- In Physics, after ruling out an hypothesis we aim at substituting it with one which explains better the data
  - Often H_1 becomes the new H_0, e.g. from (H_0:noHiggs, H_1 =Higgs) to (H_1:Higgs , H_1:otherNewPhysics)
  - We can use our expectations about reasonable alternative hypotheses to design our test to exlude H_0





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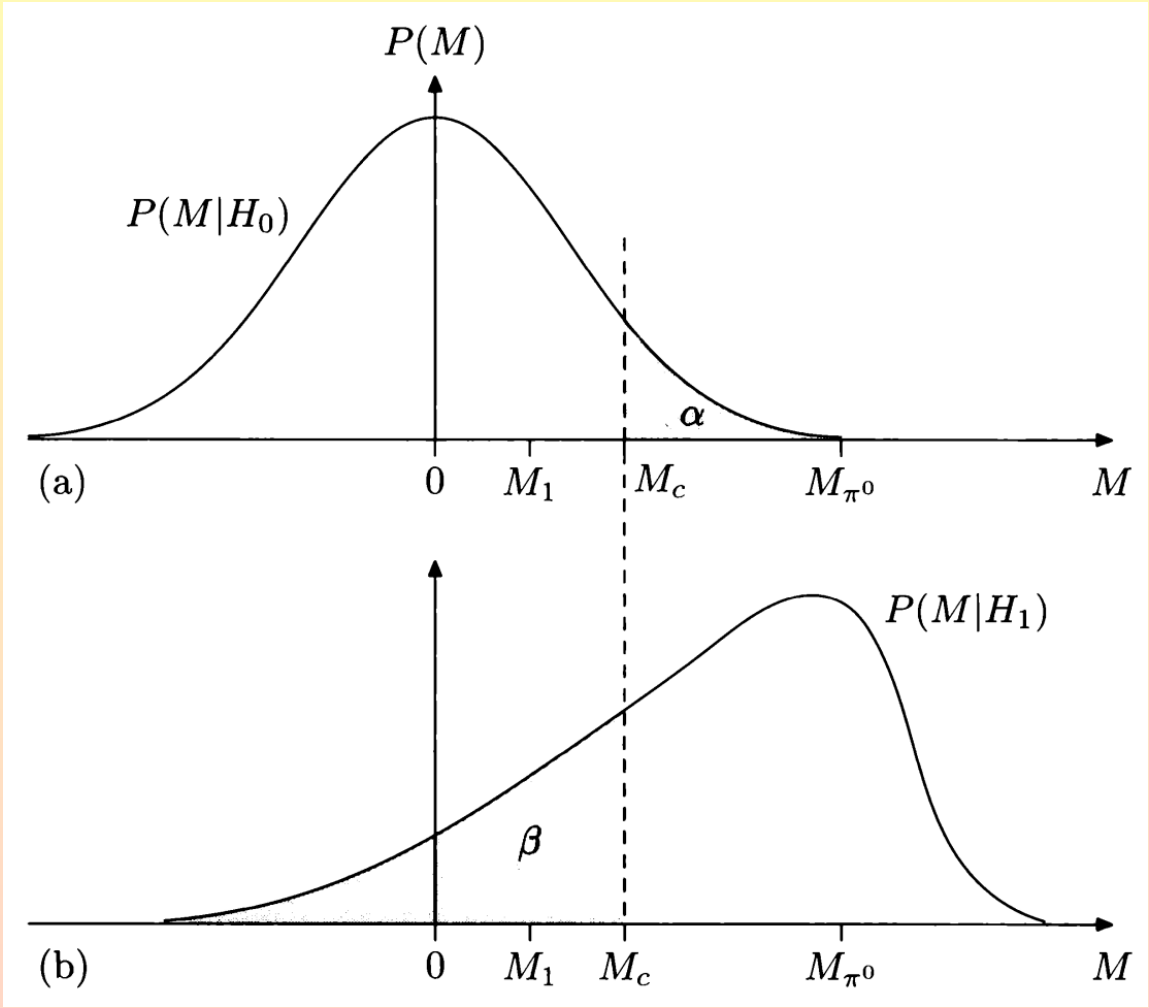
A small example

- H_0: $pp \to pp$ elastic scattering
- H_1: $pp \to pp\pi^0$
- Compute the missing mass M (as total rest energy of unseen particles)
- Under H_0, $M = 0$
- Under H_1, $M = 135\ MeV$



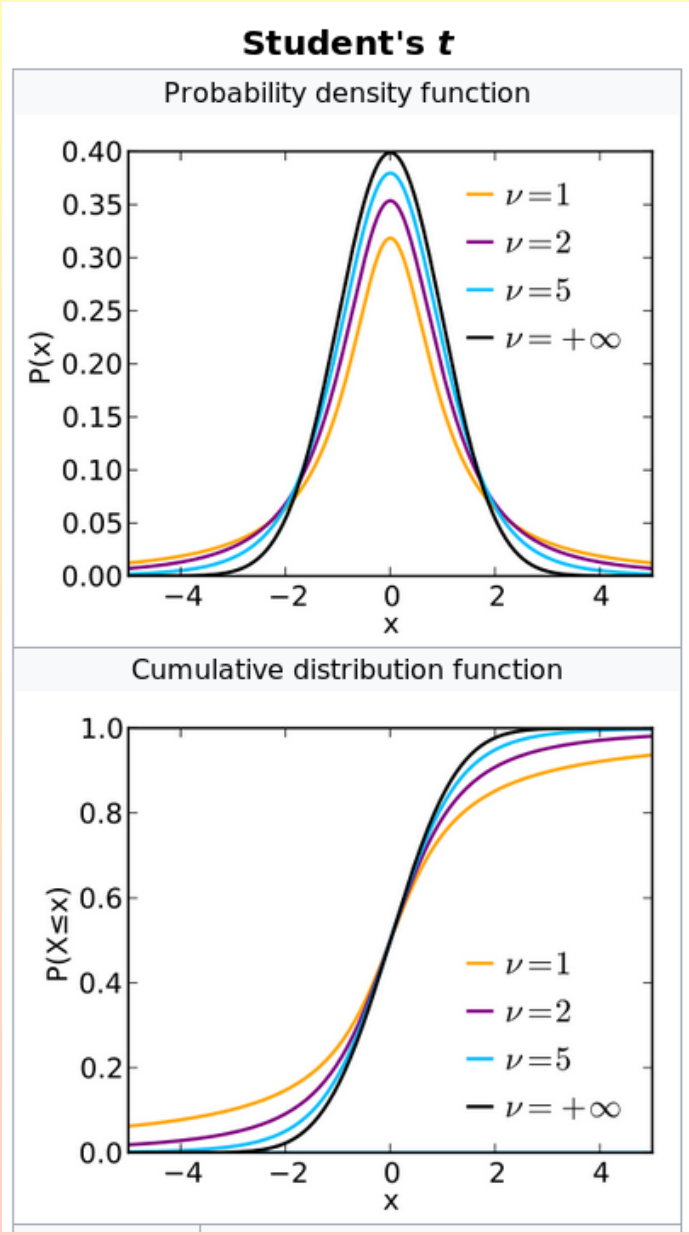
	Choose H_0	Choose H_1
H_0 is true	$1-\alpha$	α (Type I error)
H_1 is true	β (Type II error)	$1-\beta$ (power)

Plot from James, 2nd ed.

A longer example

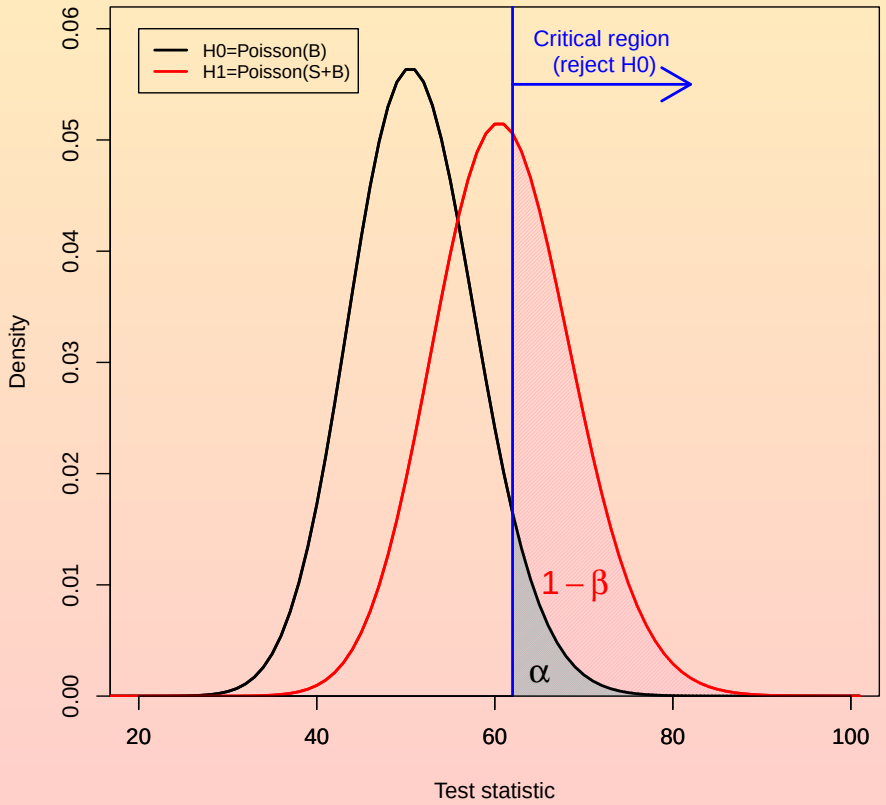
- Student's t distribution
- Test the mean!
- Will not run it this afternoon, you can check it at home hyptest.ipynb

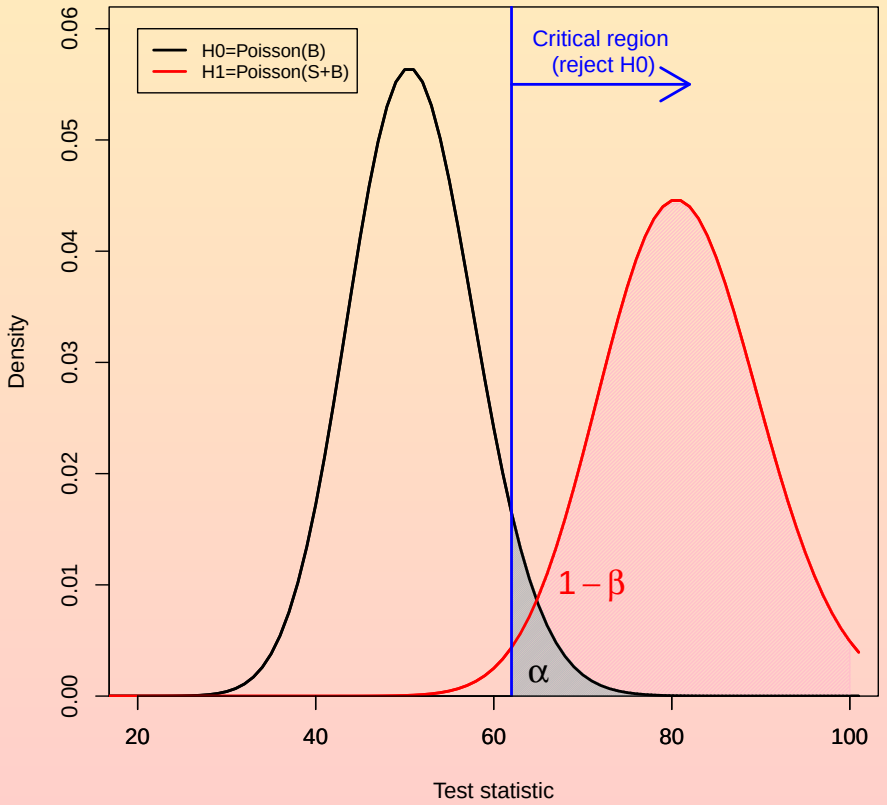
PDF	$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$



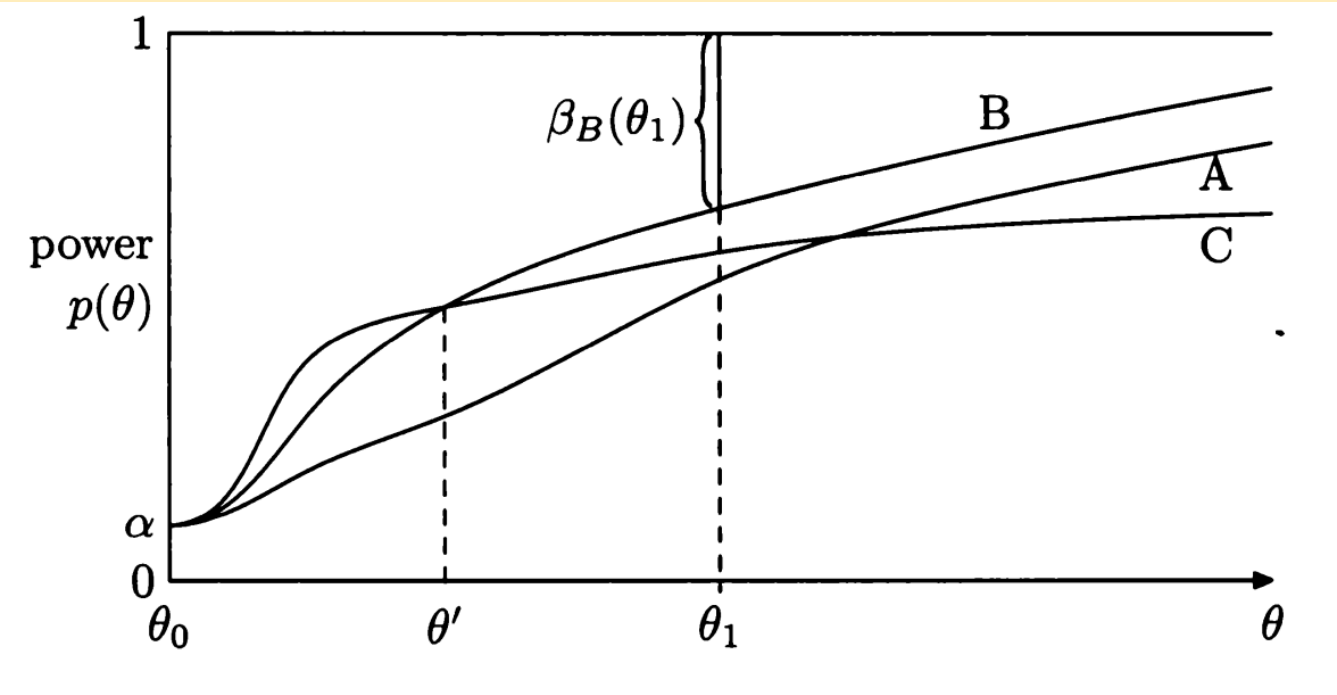
Basic hypothesis testing – 4

- The usefulness of the test depends on how well it discriminates against the alternative hypothesis
- The measure of usefulness is the *power of the test*
  - $P(X \in w|H_1) = 1 - \beta$
  - Power $(1 - \beta)$ is the probabiliity of X falling into the critical region if H_1 is true
  - $P(X \in W - w|H_1) = \beta$
  - β is the probability that X will fall into the acceptance region if H_1 is true
- NOTE: some authors use β where we use $1 - \beta$. Pay attention, and live with it.





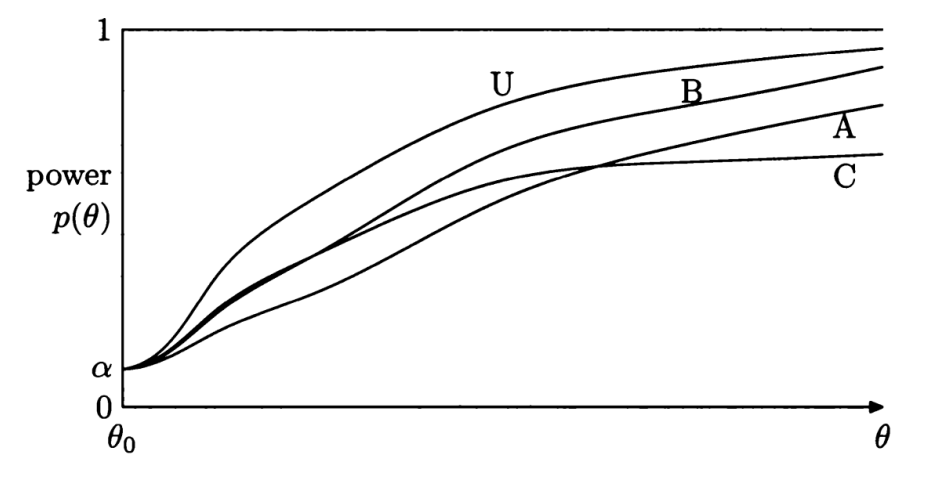
- For parametric (families of) hypotheses, the power depends on the parameter
  - $H_0 : \theta = \theta_0$
  - $H_1 : \theta = \theta_1$
  - Power: $p(\theta_1) = 1 - \beta$
- Generalize for all possible alternative hypotheses: $p(\theta) = 1 - \beta(\theta)$
  - For the null, $p(\theta_0) = 1 - \beta(\theta_0) = \alpha$

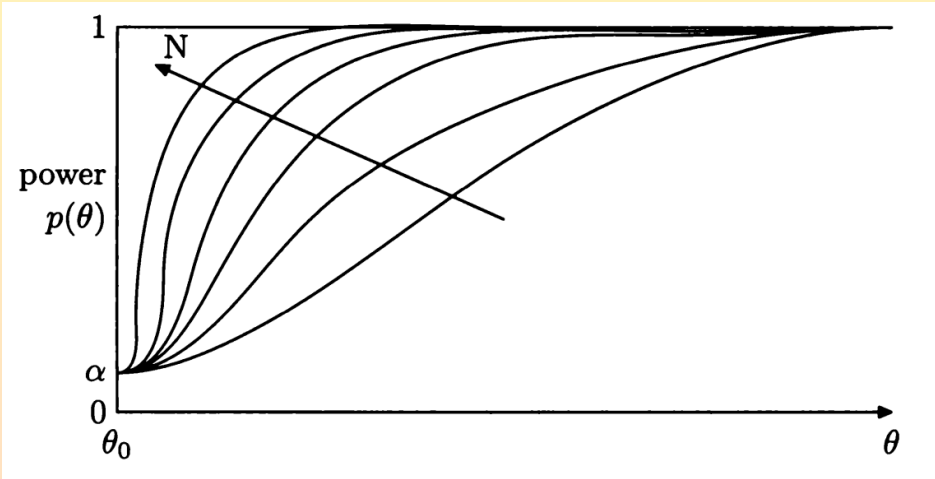


Plot from James, 2nd ed.

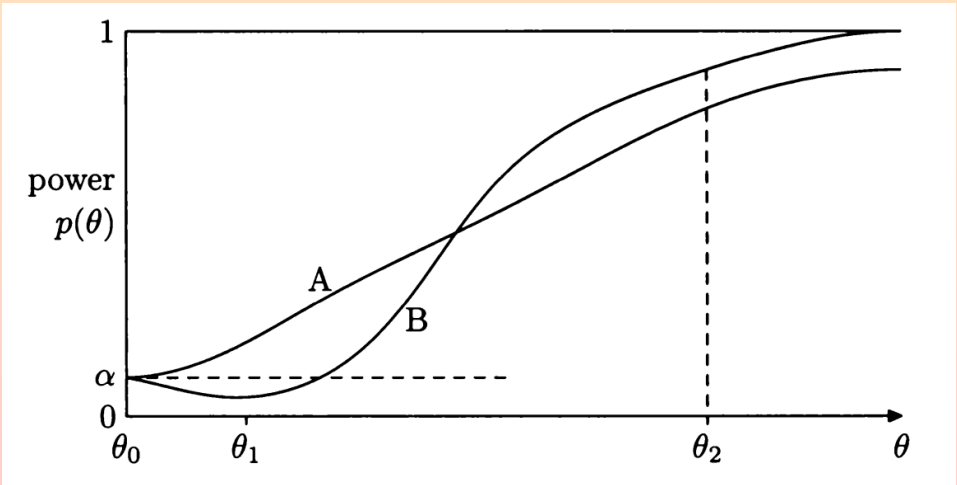
Properties of tests

- More powerful test: a test which at least as powerful as any other test for a given θ
- Uniformly more powerful test: a test which is the more powerful test for any value of θ
  - A less powerful test might be preferrable if more robust than the UMP[1]
- If we increase the number of observations, it makes sense to require consistency
  - The more observations we add, the more the test distinguishes between the two hypotheses
  - Power function tends to a step function for $N \to \infty$





- Biased test: $argmin(p(\theta)) \neq \theta_0$
- More likely to accept H_0 when it is false than when it is true
- Big no-no for θ_0 vs θ_1]
- Still useful (larger power) for θ_0 vs θ_2



Plot from James, 2nd ed.

[1] Robust: a test with low sensitivity to unimportant changes of the null hypothesis

Play with Type I (α) and Type II (β) errors freely

Image from the Statistical Statistics Memes Facebook Page

Play with Type I (α) and Type II (β) errors freely

- Comparing only based on the power curve is asymmetric w.r.t. α
- For each value of $\alpha = p(\theta_0)$, compute $\beta = p(\theta_1)$, and draw the curve
  - Unbiased tests fall under the line $1 - \beta = \alpha$
  - Curves closer to the axes are better tests
- Ultimately, though, choose based on the cost function of a wrong decision
  - Bayesian decision theory

$$h(\mathbf{X}|\theta, \phi, \psi) = \theta f(\mathbf{X}|\phi) + (1 - \theta) g(\mathbf{X}, \psi)$$

d_0 : No choice is possible; results are ambiguous

d_1, ϕ^* : Family was $f(\mathbf{X}|\phi)$, with $\phi = \phi^*$

d_2, ψ^* : Family was $g(\mathbf{X}|\psi)$, with $\psi = \psi^*$.

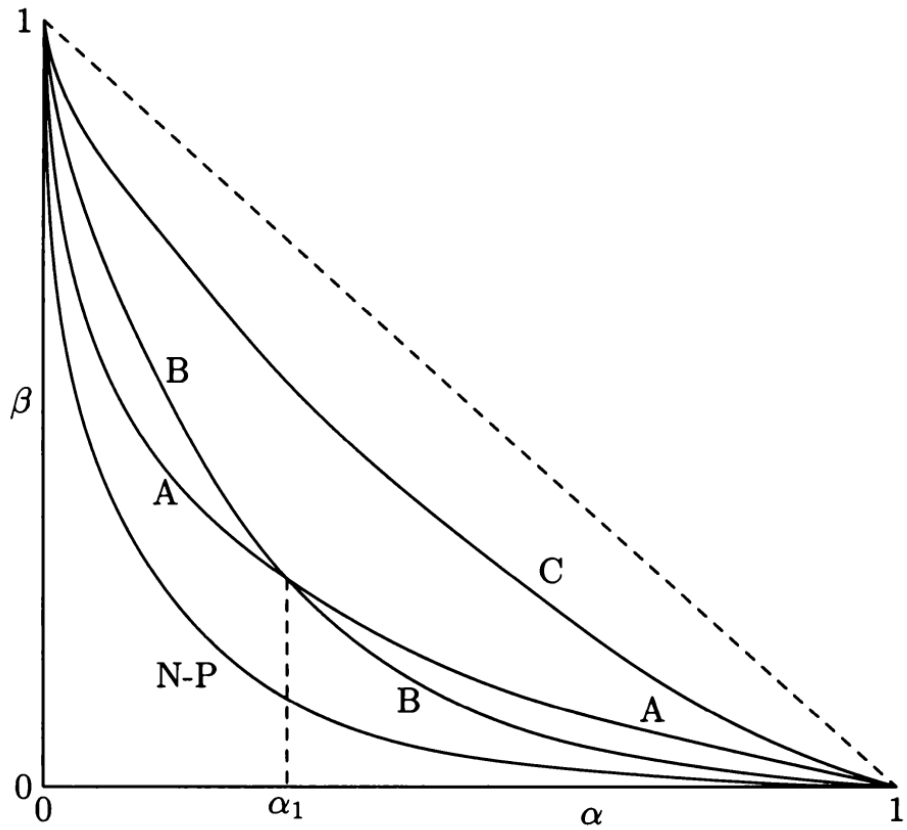


Table 10.4. A cost function.

Decisions	True state of nature	
	$\theta = \theta_1 = 1, \phi$	$\theta = \theta_2 = 0, \psi$
d_0	β_1	β_2
d_1, ϕ^*	$\alpha_1(\phi^* - \phi)^2$	γ_1
d_2, ψ^*	γ_2	$\alpha_2(\psi^* - \psi)^2$

Plot from James, 2nd ed.

- Testing simple hypotheses H_0 vs H_1, find the best critical region
- Maximize power curve $1-\beta = \int_{w_\alpha} f(\mathbf{X}|\theta_1)d\mathbf{X}$, given $\alpha = \int_{w_\alpha} f(\mathbf{X}|\theta_0)d\mathbf{X}$
- The best critical region w_α consists in the region satisfying the likelihood ratio equation

$$\ell(\mathbf{X}, \theta_0, \theta_1) := \frac{f(\mathbf{X}|\theta_1)}{f(\mathbf{X}|\theta_0)} \geq c_\alpha$$

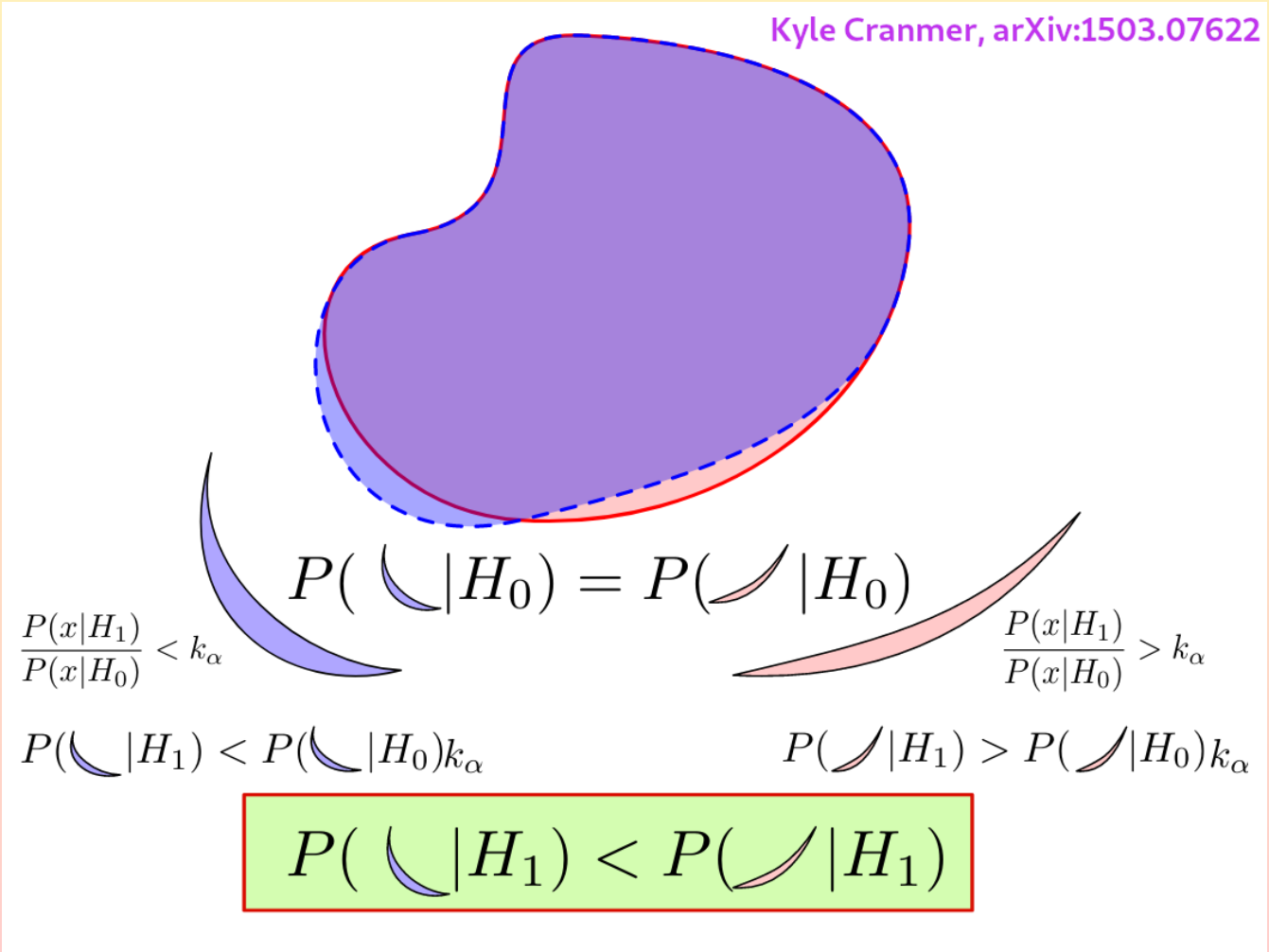
- The criterion, called Neyman-Pearson test, is therefore
  - If $\ell(\mathbf{X}, \theta_0, \theta_1) > c_\alpha$ then choose H_1
  - If $\ell(\mathbf{X}, \theta_0, \theta_1) \leq c_\alpha$ then choose H_0
- The likelihood ratio must be calculable for any $\mathbf{X}$
  - The hypotheses must therefore be completely specified simple hypotheses
  - For complex hypotheses, ℓ is not necessarily optimal

- We want to prove that $\ell(\mathbf{X}, \theta_0, \theta_1) := \frac{f(\mathbf{X}|\theta_1)}{f(\mathbf{X}|\theta_0)} \geq c_\alpha$ gives the best acceptance region

Image from Evan Vucci, Shutterstock, meme is mine

Demonstrating the Neyman-Pearson lemma

- We want to prove that $\ell(\mathbf{X}, \theta_0, \theta_1) := \frac{f(\mathbf{X}|\theta_1)}{f(\mathbf{X}|\theta_0)} \geq c_\alpha$ gives the best region
  - Critical region from NP (red contour), demonstrate that any other region (blue contour) has less power
  - Take out a wedge region and add it e.g. to the other side
  - Regions must have equal area under H_0 (tests with same size)
  - Being on different sides of the red contour, under H_1 data is less likely in the added region than in the removed one
  - Less probability to reject the null $\rightarrow$ test based on the new contour is less powerful!

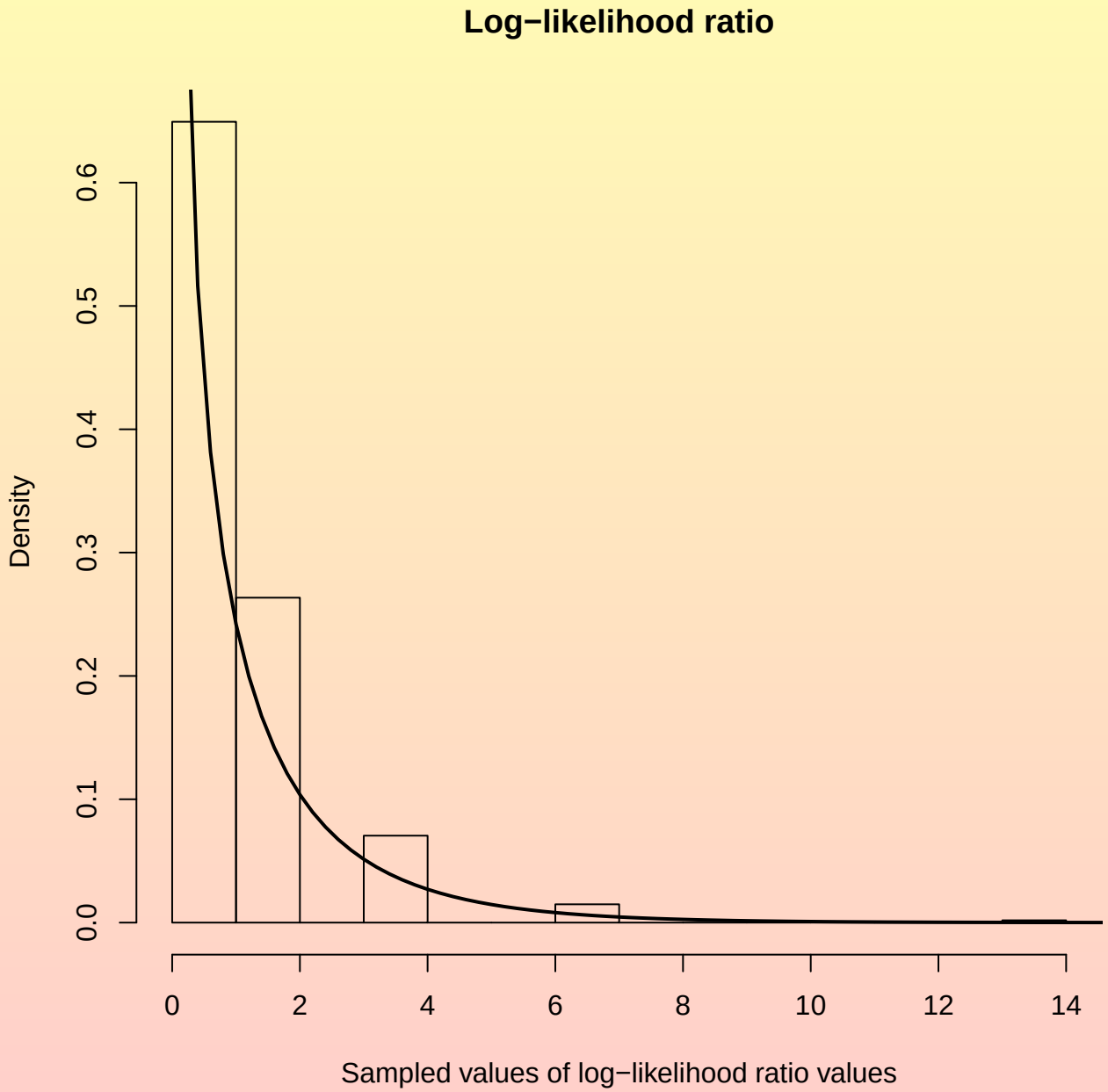


- The likelihood ratio is commonly used
- As any test statistic in the market, in order to select critical regions based on confidence levels it is necessary to know its distribution
  - Run toys to find its distribution (very expensive if you want to model extreme tails)
  - Find some asymptotic condition under which the likelihood ratio assumes a simple known form
- Wilks theorem: when the data sample size tends to ∞, the likelihood ratio tends to $\chi^2(N - N_0)$
  - Exercise yesterday afternoon

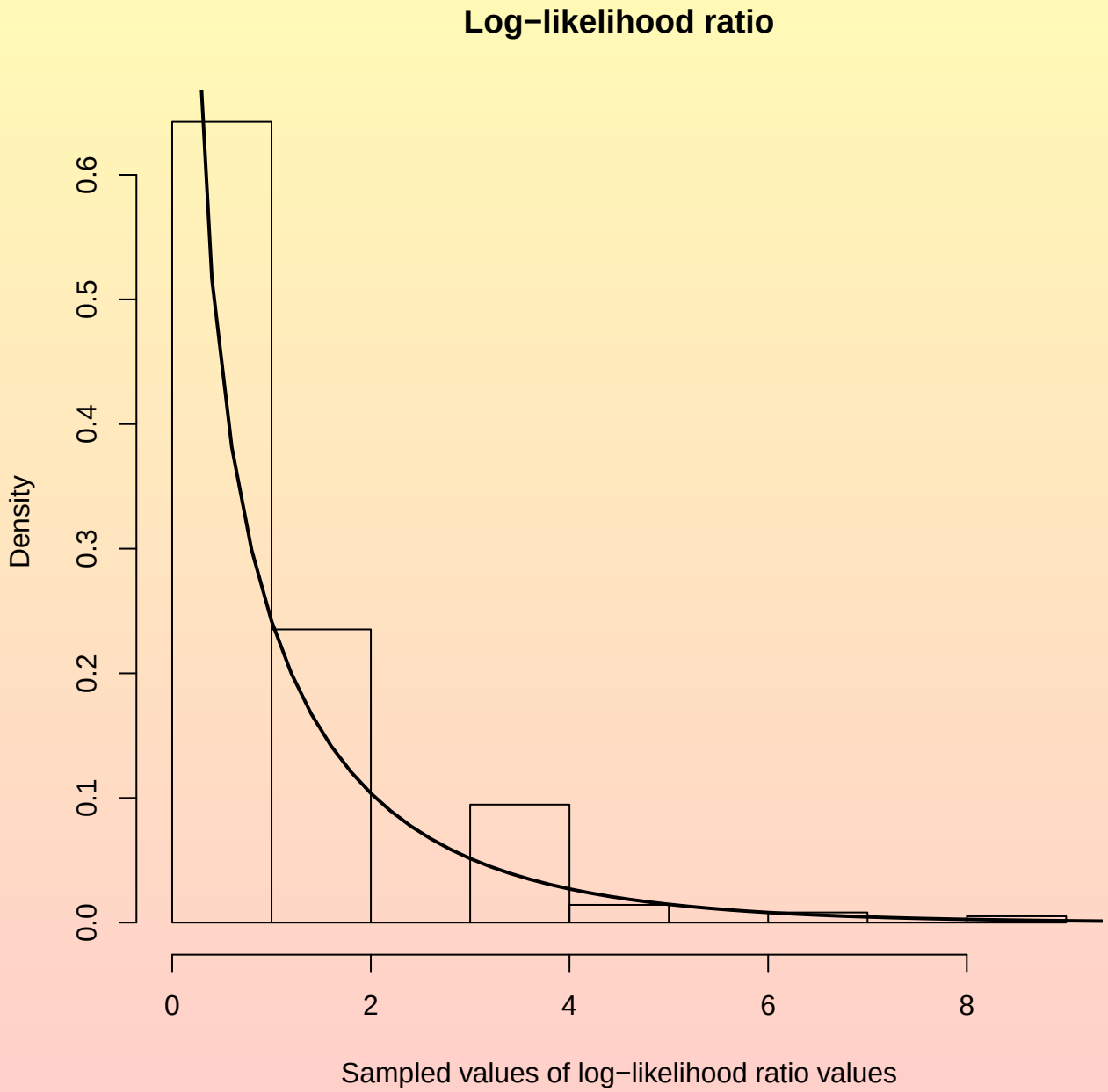
We can summarize in the

Theorem: If a population with a variate x is distributed according to the probability function $f(x, \theta_1, \theta_2 \cdots \theta_h)$, such that optimum estimates $\bar{\theta}_i$ of the θ_i exist which are distributed in large samples according to (3), then when the hypothesis H is true that $\theta_i = \theta_{0i}$, $i = m + 1, m + 2, \cdots h$, the distribution of $-2 \log \lambda$, where λ is given by (2) is, except for terms of order $1/\sqrt{n}$, distributed like χ^2 with $h - m$ degrees of freedom.

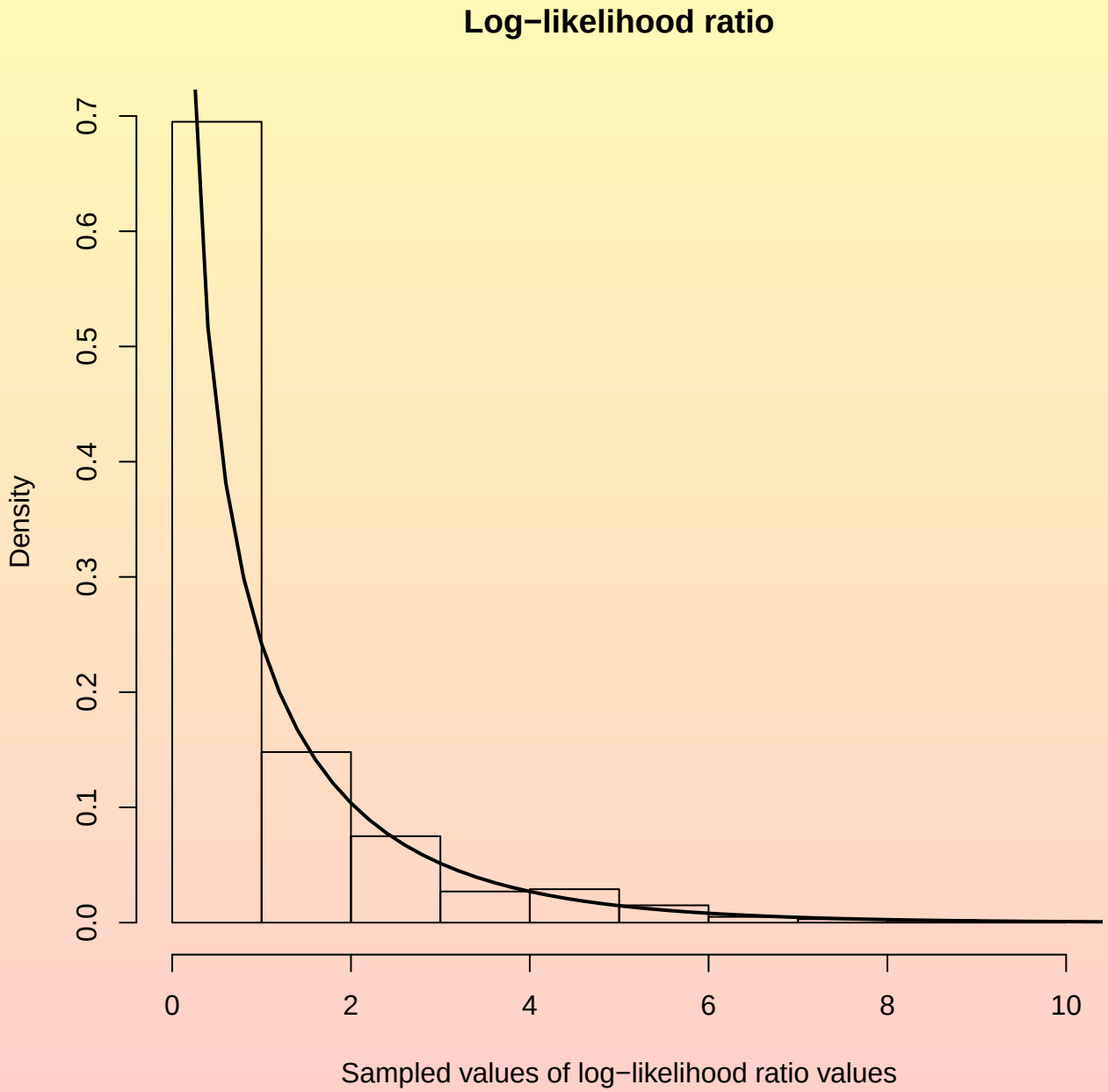
Verifying the Wilks theorem: N=2



Verifying the Wilks theorem: N=10



Verifying the Wilks theorem: N=100



Bayesian model selection — two models...

- The parameter θ might be predicted by two models M_0 and M_1: $P(\theta|\vec{x},M) = \frac{P(\vec{x}|\theta,M)P(\theta|M)}{P(\vec{x}|M)}$
  - A step further than yesterday in writing down the Bayes theorem: now multiple conditioning
  - $P(\vec{x}|M) = \int P(\vec{x}|\theta,M)P(\theta|M)d\theta$: *Bayesian evidence* or *model likelihood*
- Posterior for M_0: $P(M_0|\vec{x}) = \frac{P(\vec{x}|M_0)\pi(M_0)}{P(\vec{x})}$
- Posterior for M_1: $P(M_1|\vec{x}) = \frac{P(\vec{x}|M_1)\pi(M_1)}{P(\vec{x})}$
- The *odds* indicate relative preference of one model over the other
- Posterior odds: $\frac{P(M_0|\vec{x})}{P(M_1|\vec{x})} = \frac{P(\vec{x}|M_0)\pi(M_0)}{P(\vec{x}|M_1)\pi(M_1)}$
  - Posterior odds = Bayes Factor $\times$ prior odds
- $B_{01} := \frac{P(\vec{x}|M_0)}{P(\vec{x}|M_1)}$
- Various slightly different scales for the Bayes Factor
  - Interesting: deciban, unit supposedly theorized by Turing (according to IJ Good) as *the smallest change of evidence human mind can discern*

Jeffreys

K	dHart	bits	Strength of evidence
$< 10^0$	0	—	Negative (supports M_2)
10^0 to $10^{1/2}$	0 to 5	0 to 1.6	Barely worth mentioning
$10^{1/2}$ to 10^1	5 to 10	1.6 to 3.3	Substantial
10^1 to $10^{3/2}$	10 to 15	3.3 to 5.0	Strong
$10^{3/2}$ to 10^2	15 to 20	5.0 to 6.6	Very strong
$> 10^2$	> 20	> 6.6	Decisive

Kass and Raftery

$\log_{10} K$	K	Strength of evidence
0 to 1/2	1 to 3.2	Not worth more than a bare mention
1/2 to 1	3.2 to 10	Substantial
1 to 2	10 to 100	Strong
> 2	> 100	Decisive

Trotta

|lnB|	relative odds	favoured model's probability	Interpretation
< 1.0	< 3:1	< 0.750	not worth mentioning
< 2.5	< 12:1	0.923	weak
< 5.0	< 150:1	0.993	moderate
> 5.0	> 150:1	> 0.993	strong

Images from Wikipedia and from Roberto Trotta, Chair Lemaitre Lectures 2018

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Bayesian model comparison of 193 models
Higgs inflation as reference model

$\ln(\mathcal{E}/\mathcal{E}_{\rm HI})$

Martin,RT+14

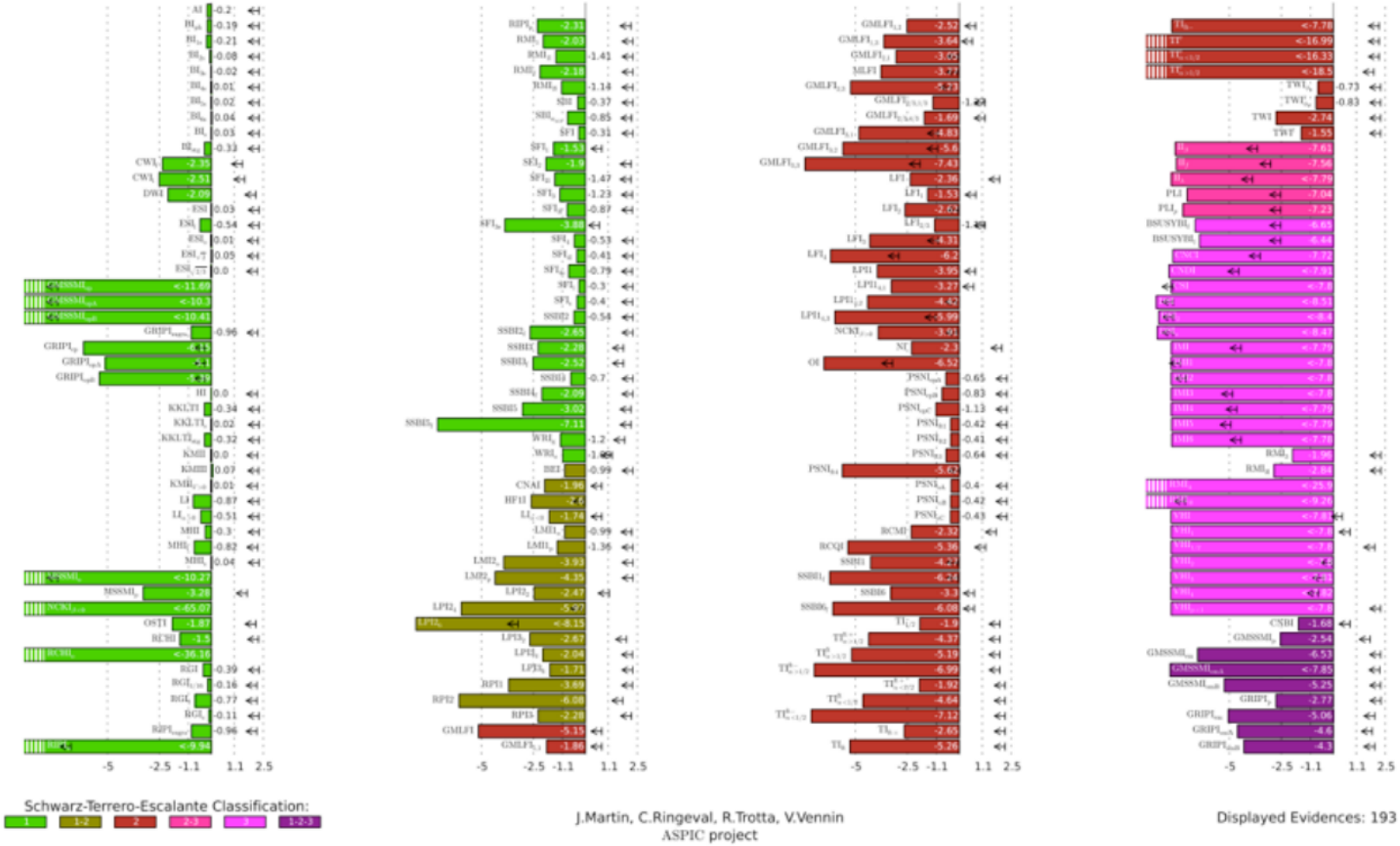

Image from Roberto Trotta, Chair Lemaitre Lectures 2018

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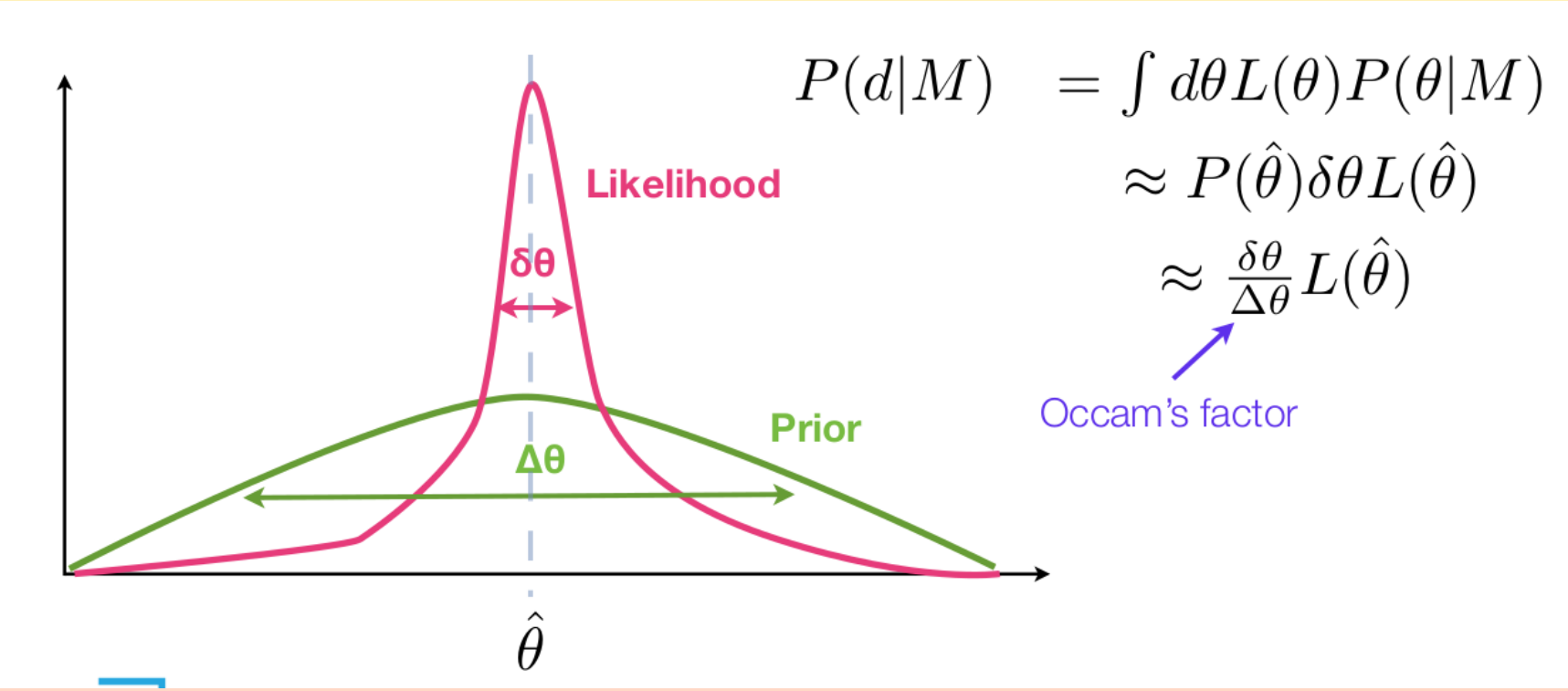
- The Bayes Factor also takes care of penalizing excessive model complexity
- Highly predictive models are rewarded, broadly-non-null priors are penalized



From Roberto Trotta, Chair Lemaitre Lectures 2018

Bayes vs p-values: the Jeffreys-Lindley paradox

- Data X (N data sampled from $f(x|\theta)$)<
  - H_0:$\theta = \theta_0$. Prior: π_0 (non-zero for point mass, Dirac's δ, counting measure)
  - H_1: $\theta ! = \theta_0$. Prior: $\pi_1 = 1 - \pi_0$ (usual Lebesgue measure)
- Conditional on H_1 being true:
  - Prior probability density $g(\theta)$
  - If $f(x|\theta) \sim Gaus(\theta, \sigma^2)$, then the sample mean $\bar{X} \sim Gaus(\theta, \sigma_{tot} = \sigma/N)$
- Likelihood ratio of H_0 to best fit for H_1: $\lambda = \frac{\mathcal{L}(\theta_0)}{\mathcal{L}(\hat{\theta})} = exp(-Z^2/2) \propto \frac{\sigma_{tot}}{\tau} B_{01}$; $Z := \frac{\hat{\theta}-\theta_0}{\sigma_{tot}}$
  - λ disfavours the null hypothesis for large significances (small p-values), independent of sample size
  - B_{01} includes σ_{tot}/τ (Ockham Factor, penalizing H_1 for imprecise determination of θ), sample dependent!
- For arbitrarily large Z (small p-values), λ disfavours H_0, while there is always a N for which B_{01} favours H_0 over H_1

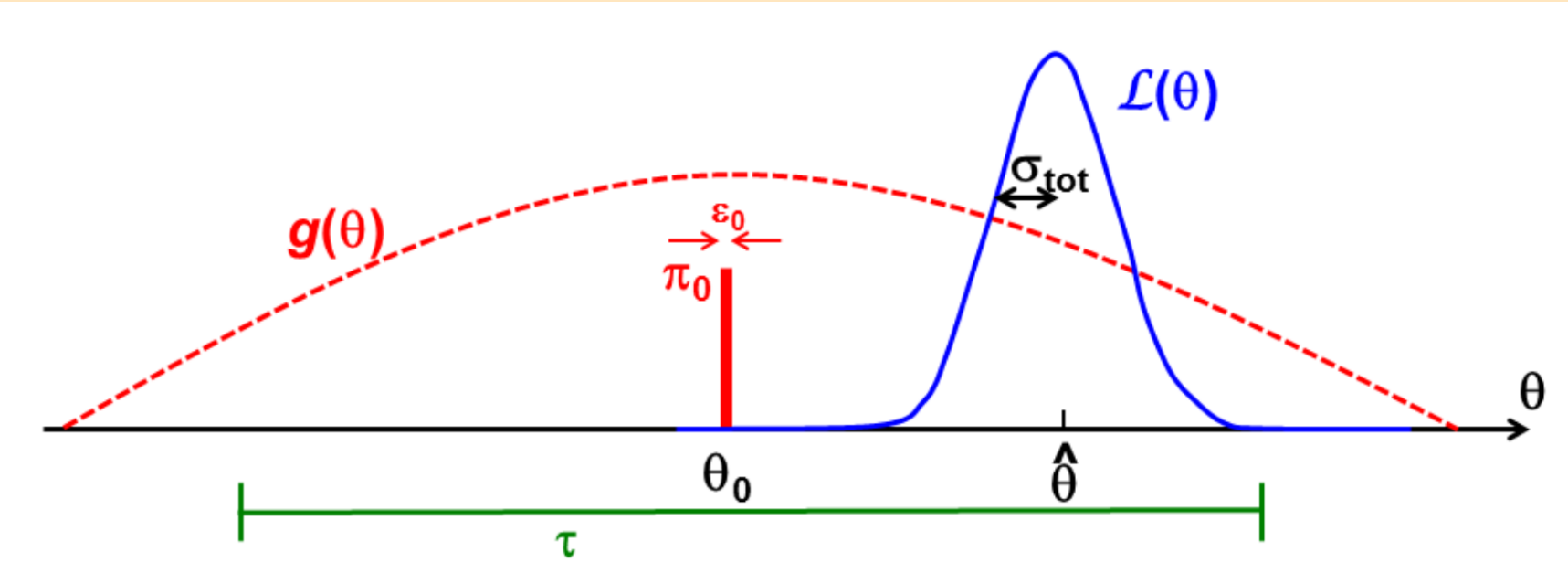


Image from Cousins, doi:10.1007/s11229-014-0525-z

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AMERICAN STATISTICAL ASSOCIATION RELEASES STATEMENT ON STATISTICAL SIGNIFICANCE AND *P*-VALUES

Provides Principles to Improve the Conduct and Interpretation of Quantitative Science

March 7, 2016

The American Statistical Association (ASA) has released a "Statement on Statistical Significance and *P*-Values" with six principles underlying the proper use and interpretation of the *p*-value [http://amstat.tandfonline.com/doi/abs/10.1080/00031305.2016.1154108#.Vt2XIOaE2MN]. The ASA releases this guidance on *p*-values to improve the conduct and interpretation of quantitative science and inform the growing emphasis on reproducibility of science research. The statement also notes that the increased quantification of scientific research and a proliferation of large, complex data sets has expanded the scope for statistics and the importance of appropriately chosen techniques, properly conducted analyses, and correct interpretation.

doi:10.1080/00031305.2016.1154108

Unprecedented policy statement

In February 2014, George Cobb, Professor Emeritus of Mathematics and Statistics at Mount Holyoke College, posed these questions to an ASA discussion forum:

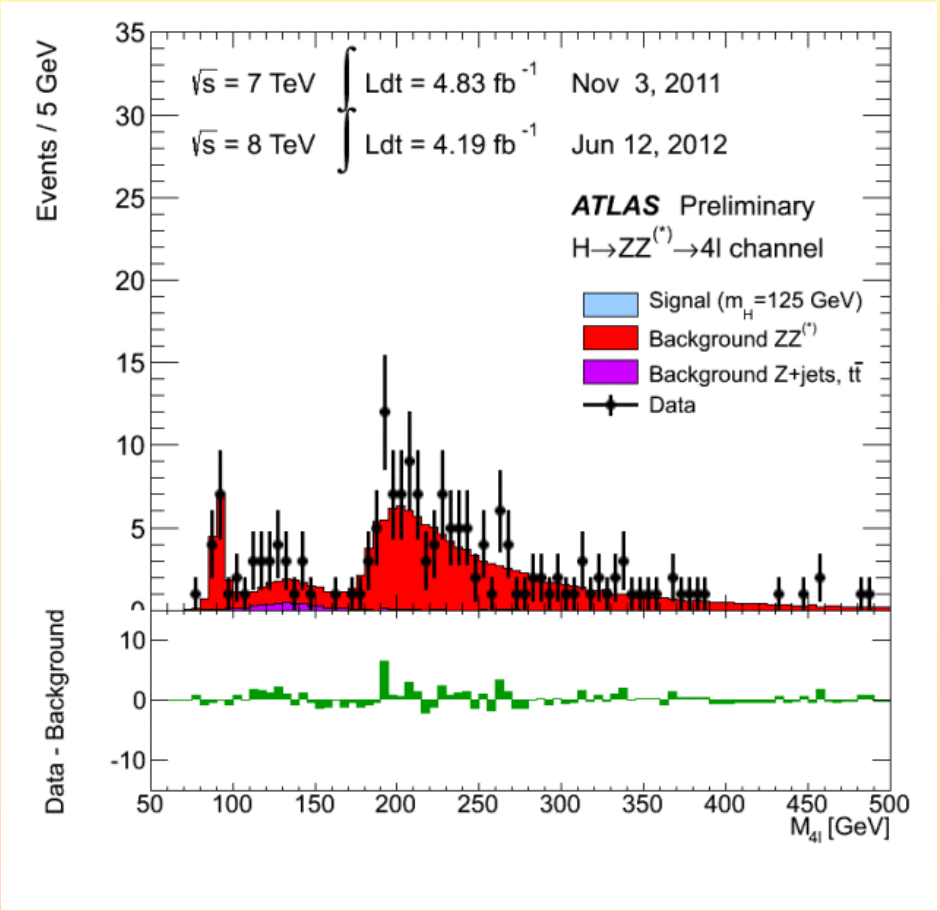
> Q: Why do so many colleges and grad schools teach $p = 0.05$?
> A: Because that's still what the scientific community and journal editors use.
> Q: Why do so many people still use $p = 0.05$?
> A: Because that's what they were taught in college or grad school.

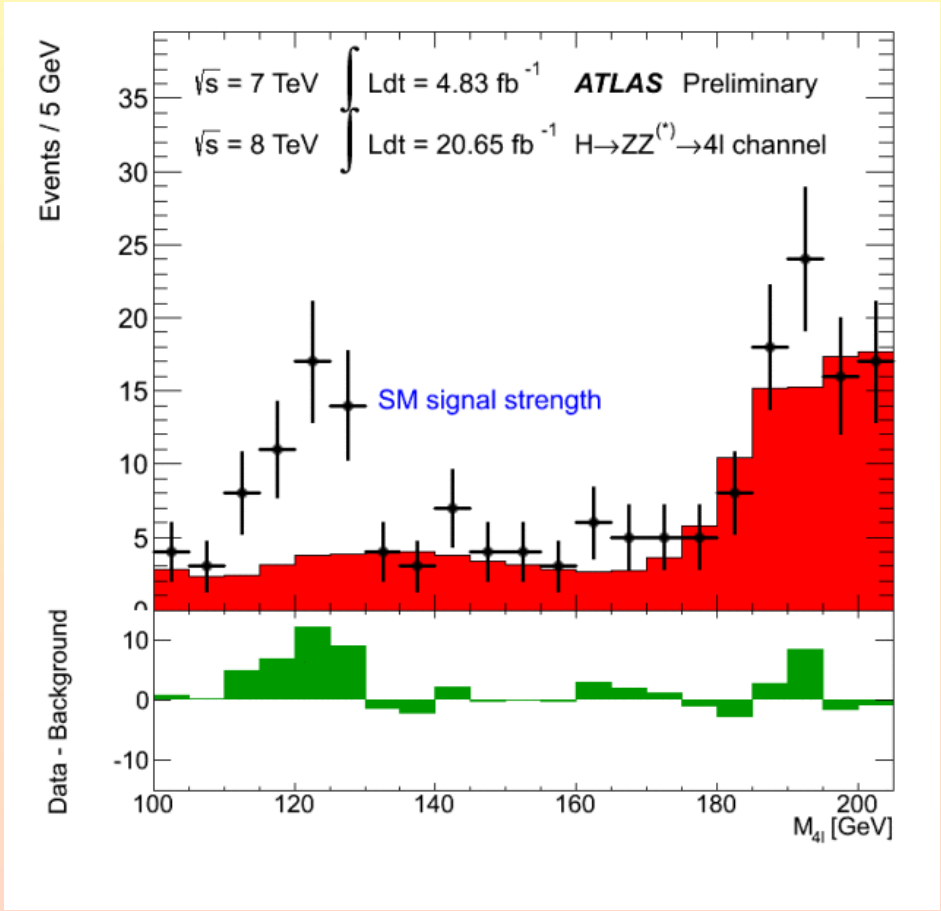
Cobb's concern was a long-worrisome circularity in the sociology of science based on the use of bright lines such as $p < 0.05$: "We teach it because it's what we do; we do it because it's what we teach." This concern was brought to the attention of the ASA Board.

Of course, it was not simply a matter of responding to some articles in print. The statistical community has been deeply concerned about issues of *reproducibility* and *replicability* of scientific conclusions. Without getting into definitions and distinctions of these terms, we observe that much confusion and even doubt about the validity of science is arising. Such doubt can lead to radical choices, such as the one taken by the editors of *Basic and Applied Social Psychology*, who decided to ban *p*-values (null hypothesis significance testing) (Trafimow and Marks 2015). Misunderstanding or misuse of statistical inference is only one cause of the "reproducibility crisis" (Peng 2015), but to our community, it is an important one.

When the ASA Board decided to take up the challenge of developing a policy statement on *p*-values and statistical significance, it did so recognizing this was not a lightly taken step. The ASA has not previously taken positions on specific matters of statistical practice. The closest the association has come

I have an excess, do I?



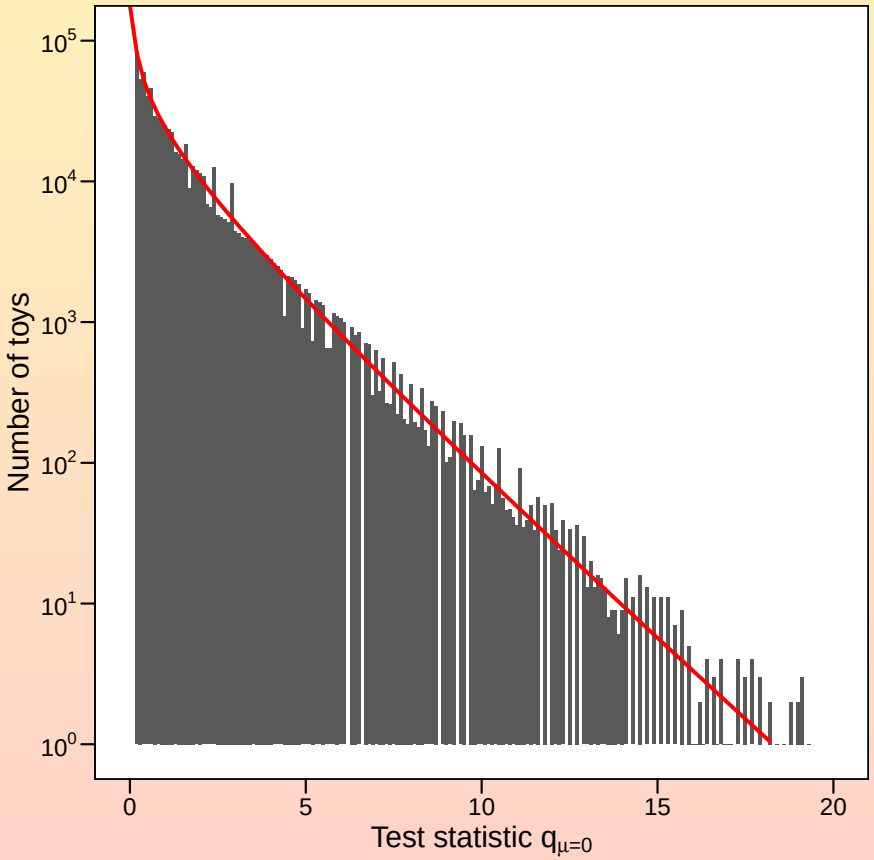


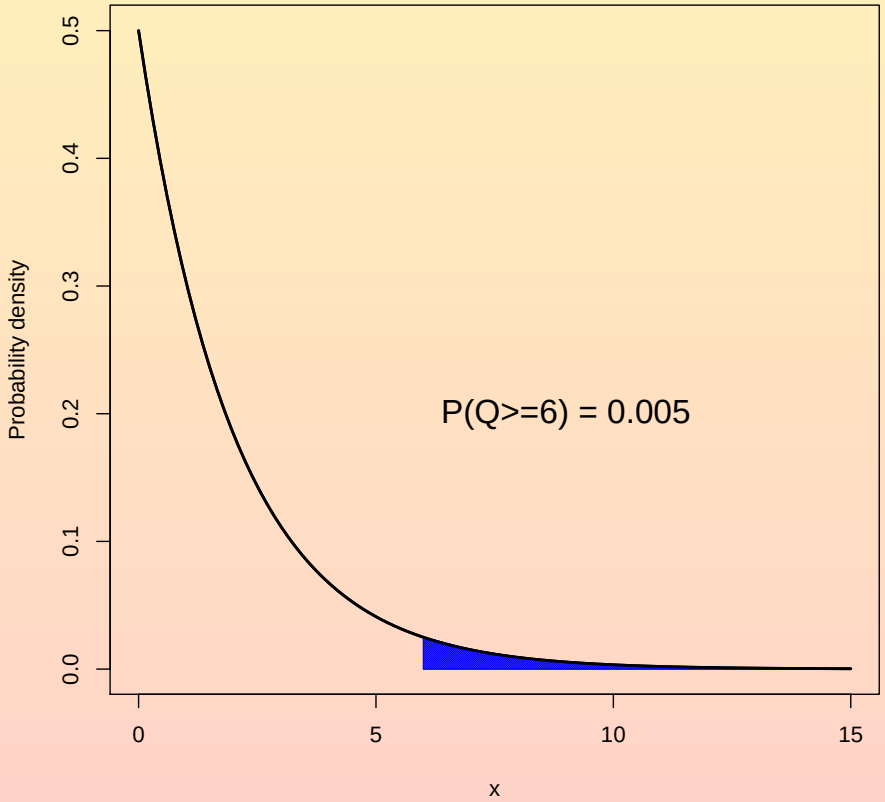
Plot from https://cds.cern.ch/record/2230893

Look only at the null hypothesis!

- Probability of obtaining a fluctuation with test statistic q_{obs} or larger, under the null hypothesis H_0
  - Distribution of test statistic under H_0 either with toys or asymptotic approximation (if N_{obs} is large, then $q \sim \chi^2(1)$)

Distribution of $q_{\mu=0}$ for H(μ=0)

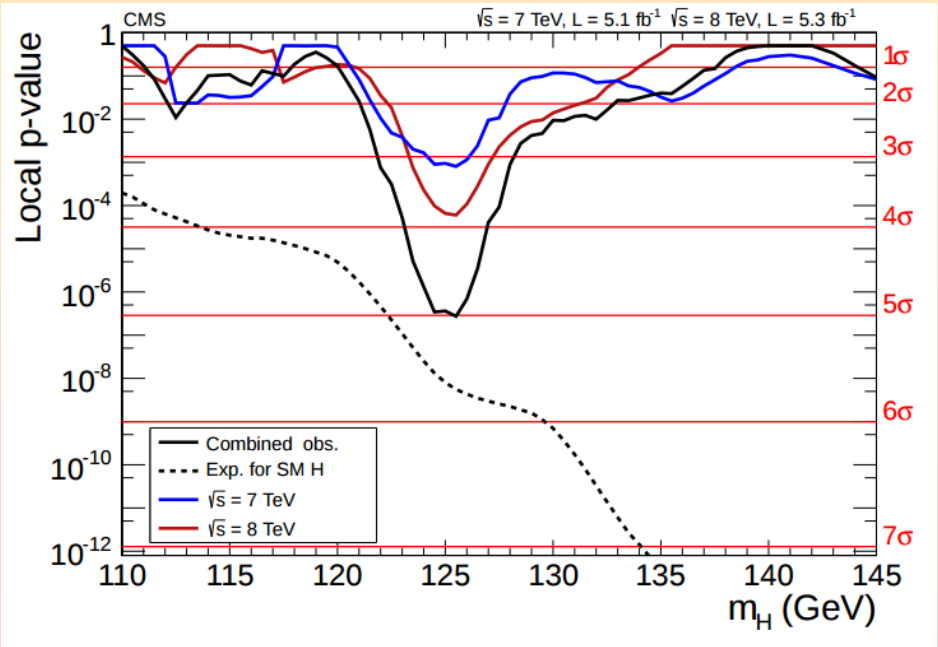


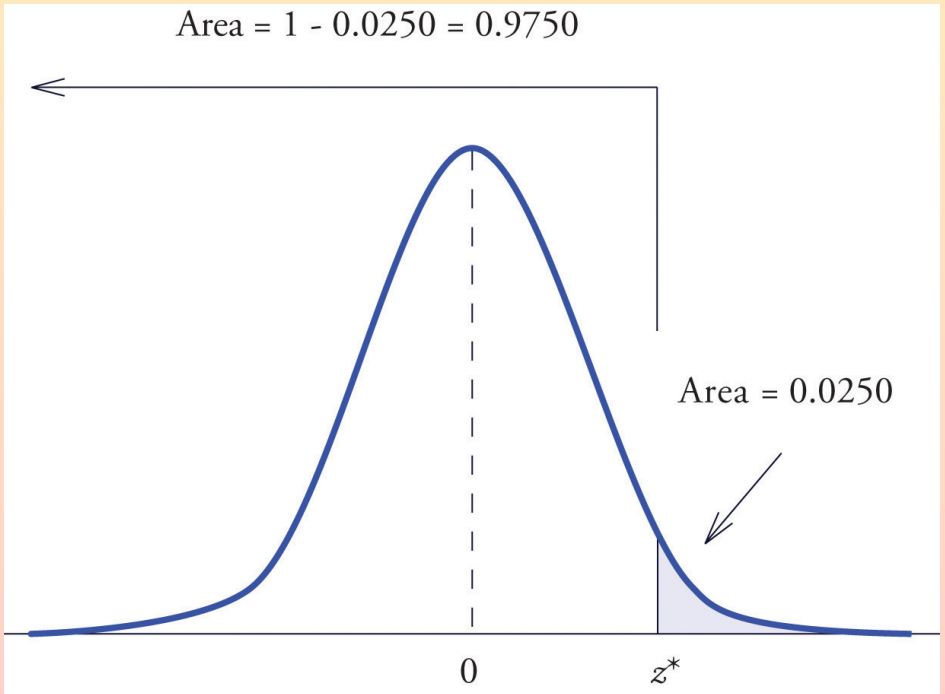


Plots from Vischia—in preparation with Springer

And the sigmas?

- Just an artifact to convert p-values to easy-to-remember $\mathcal{O}(1)$ numbers
  - 1σ: $p = 0.159$
  - 3σ: $p = 0.00135$
  - 5σ: $p = 0.000000285$
- No approximation involved, just a change of units to gaussian variances: one-sided tail area $\frac{1}{2\pi}\int_x^\infty e^{-\frac{t^2}{2}}dt = p$
  - p-value must be **flat** under the null, or interpretation is invalidated
- HEP: usually interested in one-sided deviations (upper fluctuations)
  - Most other disciplines interested in two-sided effects (e.g. 2σ: $p_{2sided} = 0.05$)





Left: ATLAS Collaboration, Right: https://saylordotorg.github.io/

1. P-values can indicate how incompatible the data are with a specified statistical model.
2. P-values do not measure the probability that the studied hypothesis is true, or the probability that the data were produced by random chance alone.
3. Scientific conclusions and business or policy decisions should not be based only on whether a p-value passes a specific threshold.
   - The widespread use of "statistical significance" (generally interpreted as $p \leq 0.05$) as a license for making a claim of a scientific finding (or implied truth) leads to considerable distortion of the scientific process.
4. Proper inference requires full reporting and transparency
5. A p-value, or statistical significance, does not measure the size of an effect or the importance of a result.
6. By itself, a p-value does not provide a good measure of evidence regarding a model or hypothesis.
   - ...supplement or even replace p-values with other approaches. These include methods that emphasize estimation over testing, such as confidence, credibility, or prediction intervals; Bayesian methods; alternative measures of evidence, such as likelihood ratios or Bayes Factors; and other approaches such as decision-theoretic modeling and false discovery rates.

doi:10.1080/00031305.2016.1154108

Responses to ASA statement: redefine pvalue threshold or not use it at all

- Benjamin *et al.* (doi:/10.31234/osf.io/mky9j) proposed to switch to lower threshold ($p < 0.005$) and not use it as criterion for publication

> **One Sentence Summary:** We propose to change the default *P*-value threshold for statistical significance for claims of new discoveries from 0.05 to 0.005.

- Wagenmakers (doi:/10.3758/BF03194105) proposed to switch to Bayesian criteria

A practical solution to the pervasive problems of *p* values

ERIC-JAN WAGENMAKERS
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In the field of psychology, the practice of *p* value null-hypothesis testing is as widespread as ever. Despite this popularity, or perhaps because of it, most psychologists are not aware of the statistical peculiarities of the *p* value procedure. In particular, *p* values are based on data that were never observed, and these hypothetical data are themselves influenced by subjective intentions. Moreover, *p* values do not quantify statistical evidence. This article reviews these *p* value problems and illustrates each problem with concrete examples. The three problems are familiar to statisticians but may be new to psychologists. A practical solution to these *p* value problems is to adopt a model selection perspective and use the Bayesian information criterion (BIC) for statistical inference (Raftery, 1995). The BIC provides an approximation to a Bayesian hypothesis test, does not require the specification of priors, and can be easily calculated from SPSS output.

- Gelman (statmodeling.stat.columbia.edu) proposes to not limit ourselves to a single summary statistic or threshold
  - "I put much of the blame on statistical education, for two reasons"
  - "First [...] we typically focus on the choice of sample size, not on the importance of valid and reliable measurements."
  - "Second, it seems to me that statistics is often sold as a sort of alchemy that transmutes randomness into certainty, an *uncertainty laundering* [...] Just try publishing a result with p = 0.20"
  - "In summary, I agree with most of the ASA's statement on p-values but I feel that the problems are deeper, and that the solution is not to reform p-values or to replace them with some other statistical summary or threshold, but rather to move toward a greater acceptance of uncertainty and embracing of variation."

- It seems so: The Bayer Study (https://www.nature.com/articles/nrd3545)

Published: 31 August 2011

Reliability of 'new drug target' claims called into question

Asher Mullard

Nature Reviews Drug Discovery **10**, 643–644(2011) | Cite this article

841 Accesses | **68** Citations | **69** Altmetric | Metrics

Bayer halts nearly two-thirds of its target-validation projects because in-house experimental findings fail to match up with published literature claims, finds a first-of-a-kind analysis on data irreproducibility.

- "Irreproducibility was high both when Bayer scientists applied the same experimental procedures as the original researchers and when they adapted their approaches to internal needs (for example, by using different cell lines)."
- "High-impact journals did not seem to publish more robust claims, and, surprisingly, the confirmation of any given finding by another academic group did not improve data reliability."

The funny bit

- Ioannidis (doi:/10.1371/journal.pmed.0020124) identifies several causes mostly linked to scientists' own biases
  - Investigator prejudice, incorrect statistical methods, competition in hot fields, publishing bias

Comment on this paper

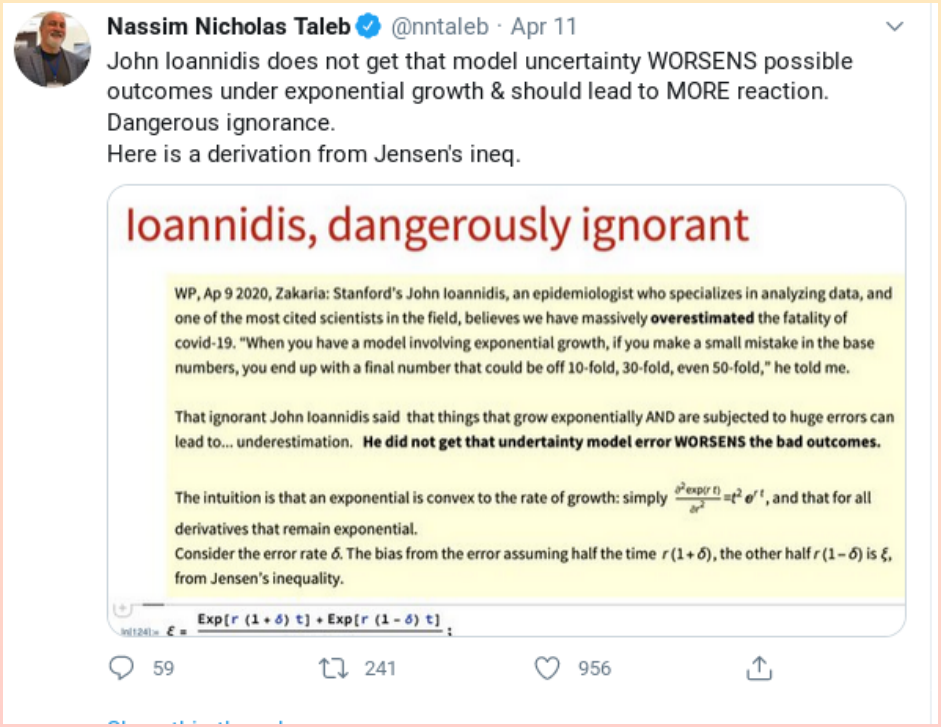
Population-level COVID-19 mortality risk for non-elderly individuals overall and for non-elderly individuals without underlying diseases in pandemic epicenters

John P. A. Ioannidis, Cathrine Axfors, Despina G. Contopoulos-Ioannidis

doi: https://doi.org/10.1101/2020.04.05.20054361

This article is a preprint and has not been certified by peer review [what does this mean?]. It reports new medical research that has yet to be evaluated and so should *not* be used to guide clinical practice.

- Then Ioannidis got accused of the same issues, just last month



How HEP protects itself

- Goal: seamless transition between exclusion, observation, discovery (historically for the Higgs)
  - Exclude Higgs as strongly as possible in its absence (in a region where we would be sensitive to its presence)
  - Confirm its existence as strongly as possible in its presence (in a region where we are sensitive to its presence)
  - Maintain Type I and Type II errors below specified (small) levels
- Identify observables, and a suitable test statistic Q
- Define rules for exclusion/discovery, i.e. ranges of values of Q leading to various conclusions
  - Specify the significance of the statement, in form of confidence level (CL)
- Confidence limit: value of a parameter (mass, xsec) excluded at a given confidence level CL
  - A confidence limit is an upper(lower) limit if the exclusion confidence is greater(less) than the specified CL for all values of the parameter below(above) the confidence limit
- The resulting intervals are neither frequentist nor bayesian!

- Counting experiment: observe n events
- Assume they come from Poisson processes: $n \sim Pois(s+b)$, with known b
- Set limit on s given n_{obs}
- Exclude values of s for which $P(n \leq n_{obs}|s+b) \leq \alpha$ (guaranteed coverage $1-\alpha$)
- $b = 3$, $n_{obs} = 0$
  - Exclude $s+b \leq 3$ at 95%CL
  - Therefore excluding $s \leq 0$, i.e. **all** possible values of s (can't distinguish b-only from very-small-s)
- Zech: let's condition on $n_b \leq n_{obs}$ (n_b unknown number of background events)
  - For small n_b the procedure is more likely to undercover than when n_b is large, and the distribution of n_b is independent of s
  - $P(n \leq n_{obs}|n_b \leq n_{obs}, s+b) = \ldots = \frac{P(n \leq n_{obs}|s+b)}{P(n \leq n_{obs}|b)}$

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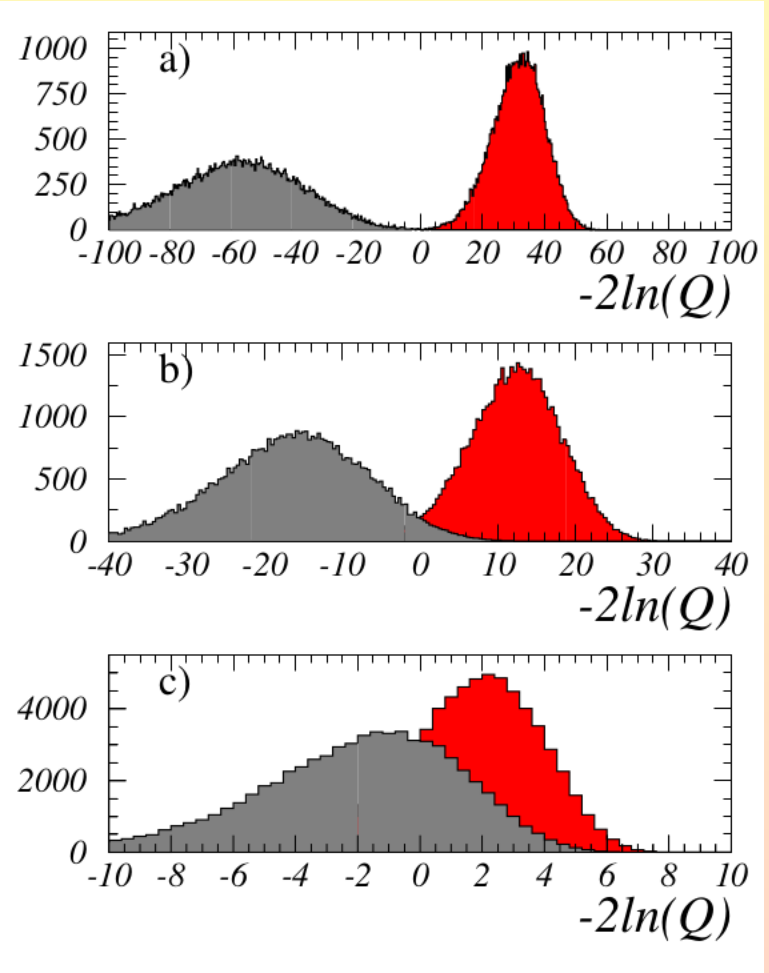
Get your confidence levels right

- Find a monotonic Q for increasing signal-like experiments (e.g. likelihood ratio)
- $CL_{s+b} = P_{s+b}(Q \leq Q_{obs})$
  - Small values imply poor compatibility with $S + B$ hypothesis, favouring B-only
- $CL_b = P_b(Q \leq Q_{obs})$
  - Large (close to 1) values imply poor compatibility with B-only, favouring $S + B$
- What to do when the estimated parameter is unphysical?
  - The same issue solved by Feldman-Cousins
  - If there is also underfluctuation of backgrounds, it's possible to exclude even zero events at 95%CL!
  - It would be a statement about future experiments
  - Not enough information to make statements about the signal
- Normalize the $S + B$ confidence level to the B-only confidence level!



Plot from Read, CERN-open-2000-205

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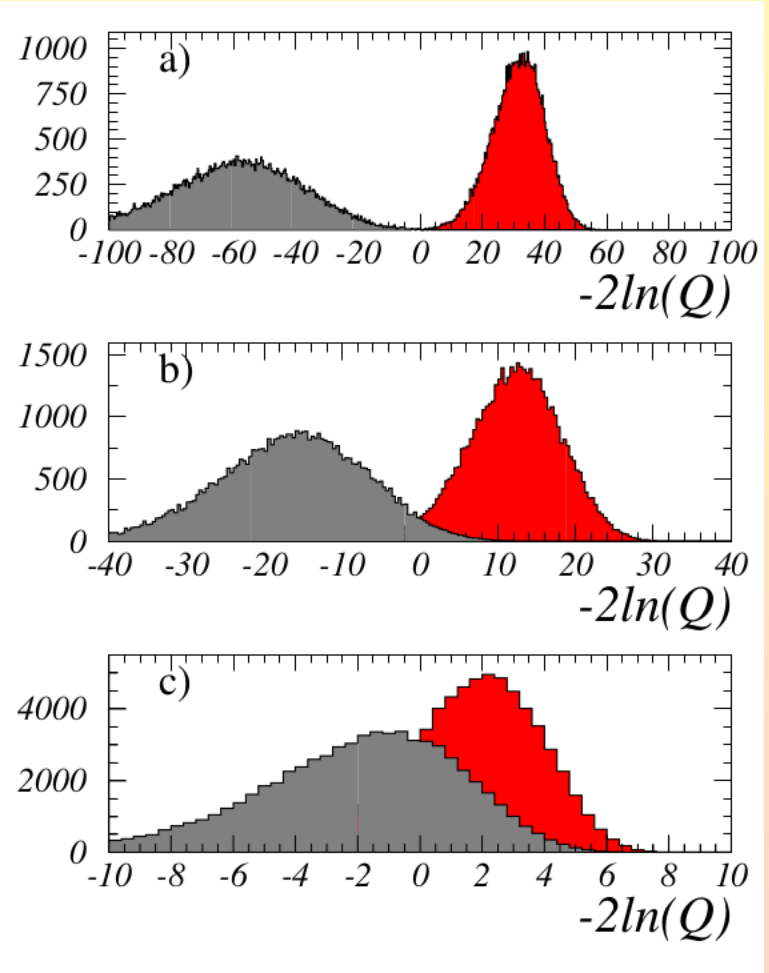
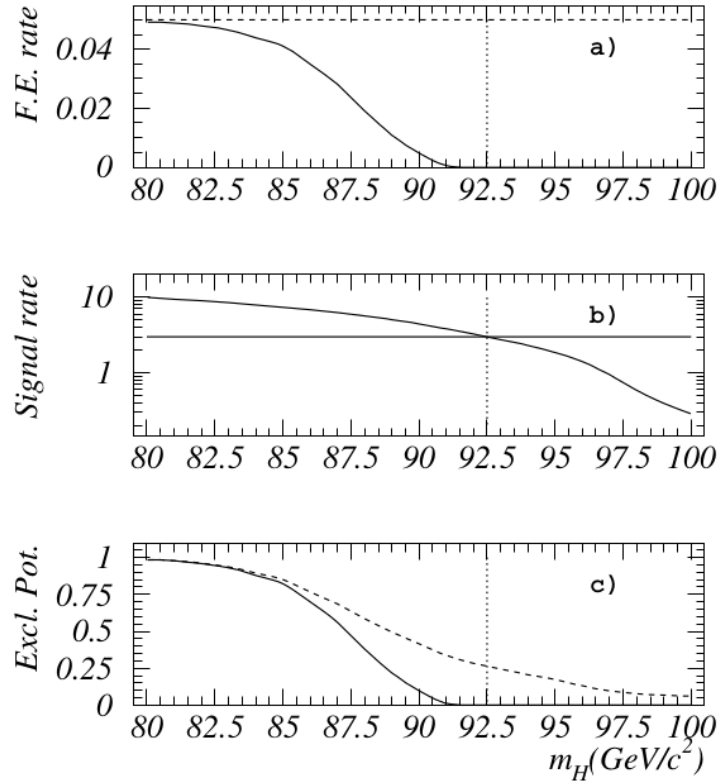


Plot from Read, CERN-open-2000-205

Avoid issues at low signal rates

- $CL_s := \frac{CL_{s+b}}{CL_b}$
- Exclude the signal hypothesis at confidence level CL if $1 - CL_s \leq CL$
- Ratio of confidences is not a confidence
  - The hypotetical false exclusion rate is generally less than the nominal $1 - CL$ rate
  - CL_s and the actual false exclusion rate grow more different the more $S + B$ and B p.d.f. become similar
- CL_s increases coverage, i.e. the range of parameters that can be exclude is reduced
  - It is more conservative
  - Approximation of the confidence in the signal hypothesis that might be obtained if there was no background
- Avoids the issue of CL_{s+b} with experiments with the same small expected signal
  - With different backgrounds, the experiment with the larger background might have a better expected performance
- Formally corresponds to have $H_0 = H(\theta! = 0)$ and test it against $H_1 = H(\theta = 0)$
  - Test inversion!

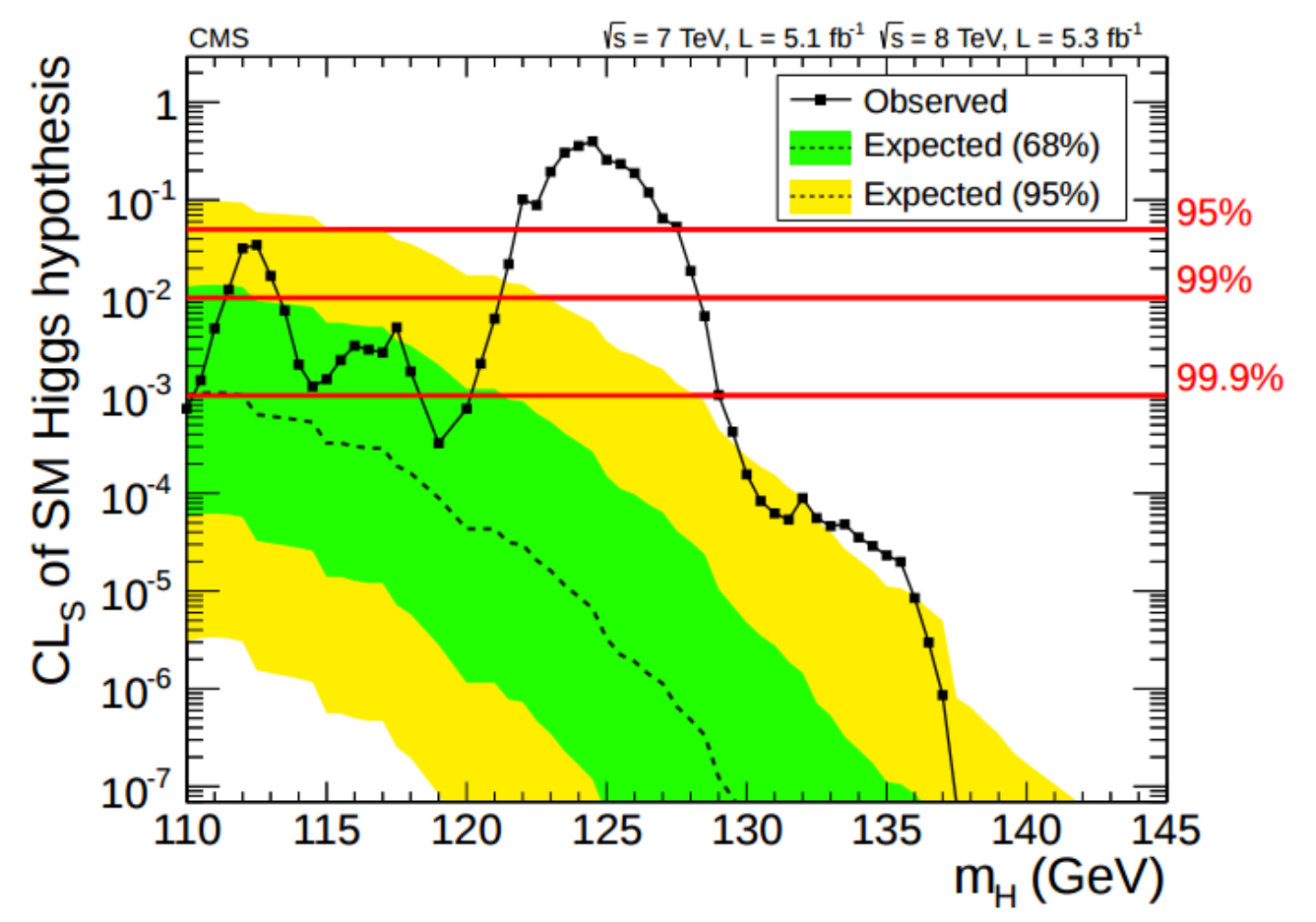


Dashed: CL_{s+b}
Solid: CL_s
$S < 3$: exclusion for a B-free search $\equiv 0$

Plot from Read, CERN-open-2000-205

That's what we used for the Higgs discovery!

- Apply the CL_s method to each Higgs mass point
- Green/yellow bands indicate the $\pm 1\sigma$ and $\pm 2\sigma$ intervals for the expected values under B-only hypothesis
  - Obtained by taking the quantiles of the B-only hypothesis



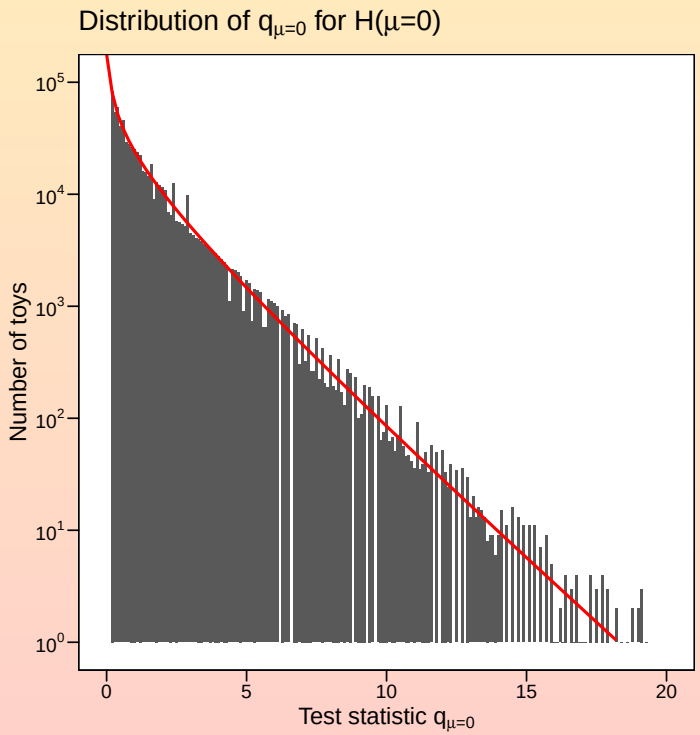
Plot from Higgs discovery paper

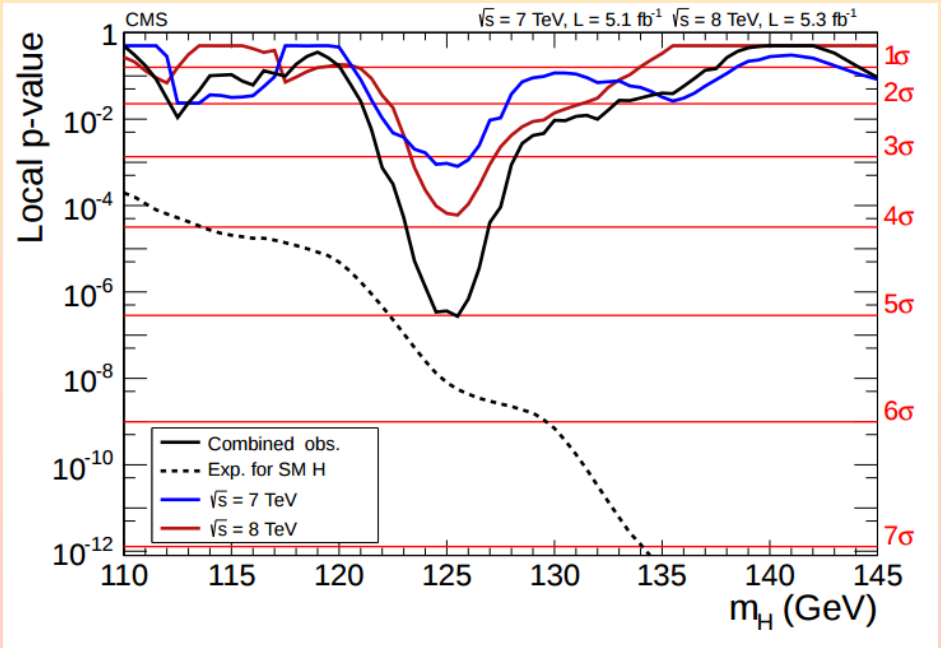
Deal with CLs!

- This afternoon we'll play with CLs!

Quantifying excesses

- Quantify the presence of the signal by using the background-only p-value
  - Probability that the background fluctuates yielding and excess as large or larger of the observed one
- For the mass of a resonance, $q_0 = -2log\frac{\mathcal{L}(data|0,\hat{\theta}_0)}{\mathcal{L}(data|\hat{\mu},\hat{\theta})}$, with $\hat{\mu} \geq 0$
  - Interested only in upwards fluctuation, accumulate downwards one to zero
- Use pseudo-data to generate background-only Poisson counts and nuisance parameters θ_0^{obs}
  - Use distribution to evaluate tail probability $p_0 = P(q_0 \leq q_0^{obs})$
  - Convert to one-sided Gaussian tail areas by inverting $p = \frac{1}{2}P_{\chi_1^2}(Z^2)$





Left plot by Pietro Vischia, right plot from ATL-PHYS-PUB-2011-011 and Higgs discovery paper

- Question time: Significance

Fluctuations in HEP? The proposal of a 5σ criterion

- Rosenfeld, 1968 (https://escholarship.org/uc/item/6zm2636q) *Are there any Far-out Mesons or Baryons?*
  - "In summary of all the discussion abouve, I conclude that each of our 150,000 annual histograms is capable of generating somewhere between 10 and 100 deceptive upward fluctuations [...] (we) should expect several 4σ and hundreds of 3σ fluctuations"

of 3σ fluctuations. What are the implications? To the theoretician or phenomenologist the moral is simple; wait for nearly 5σ effects. For the experimental group who have just spent a year of their time and perhaps a million dollars, the problem is harder. I suggest that they should go ahead and publish their tantalizing bump (or at least circulate it as a report.) But they should realize that any bump less than about 5σ constitutes only a call for a repeat of the experiment. If they, or somebody else, can double the number of counts, the number of standard deviations should increase by $\sqrt{2}$, and that will confirm the original effect.

My colleague Gerry Lynch has instead tried to study this problem "experimentally" using a "Las Vegas" computer program called Game. Game is played as follows. You wait until an unsuspecting "friend" comes to show you his latest 4σ peak. You draw a smooth curve through his data (based on the hypothesis that the peak is just a fluctuation), and punch this smooth curve as one of the inputs for game. The other input is his actual data. If you then call for 100 Las Vegas histograms, Game will generate them, with the actual data reproduced for comparison at some random page. You and your friend then go around the halls, asking physicists to pick out the most surprising histogram in the printout. Often it is one of the 100 phoneys, rather than the real "4σ" peak. Figure 3 shows two Game histograms, each one being one of the more interesting ones in a run of 100. The smooth curves drawn through them are of course absurd; they are supposed to be the background estimates of the inexperienced experimenter. But they do illustrate that a 2σ or 3σ fluctuation can easily be amplified to "4σ" or "5σ"; all it takes is a little enthusiasm.

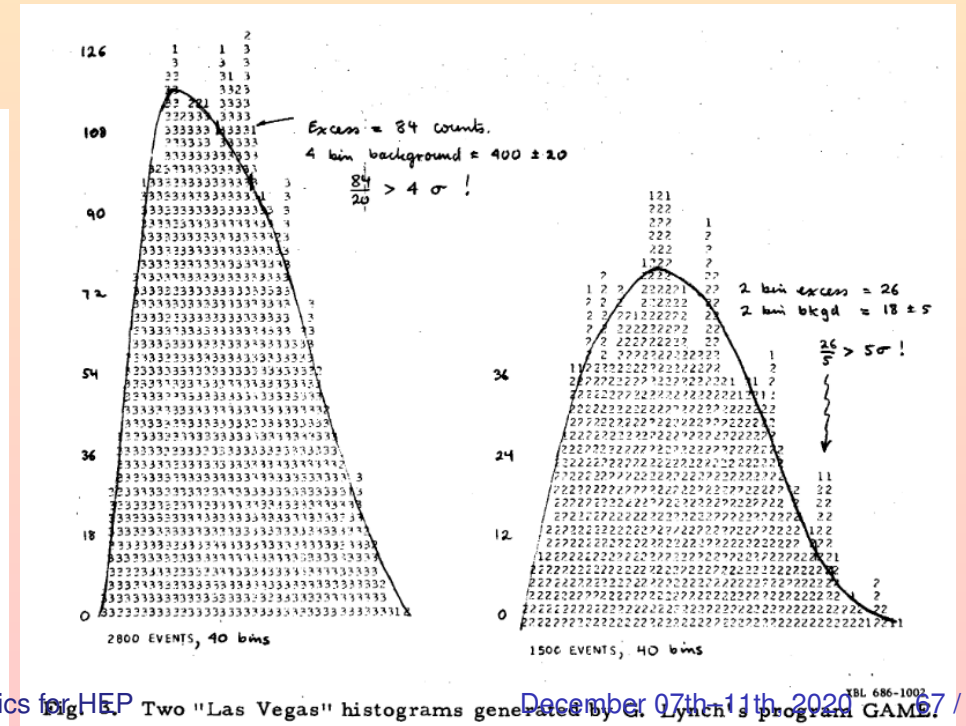
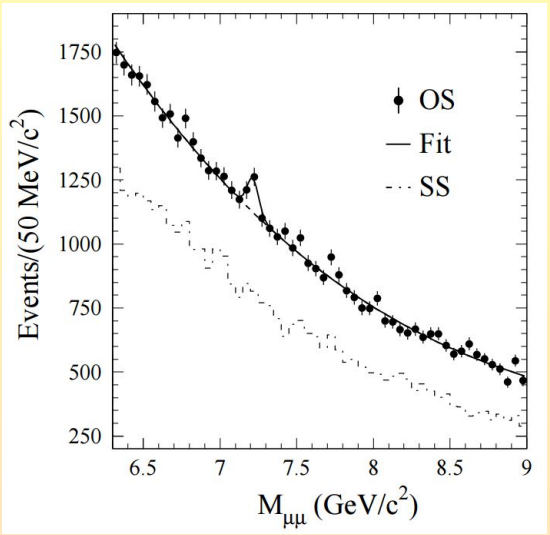
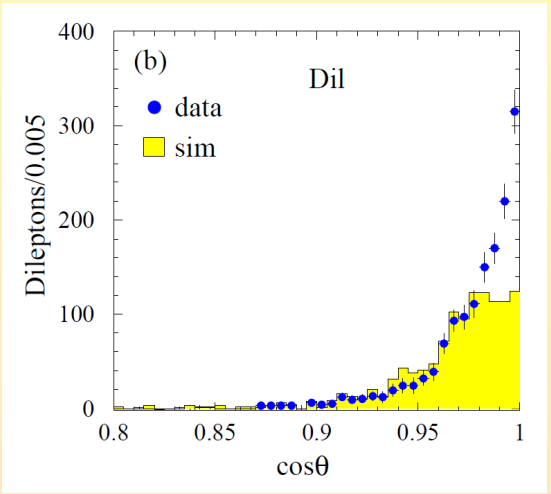


Fig. 3. Two "Las Vegas" histograms generated by G. Lynch's program GAME.

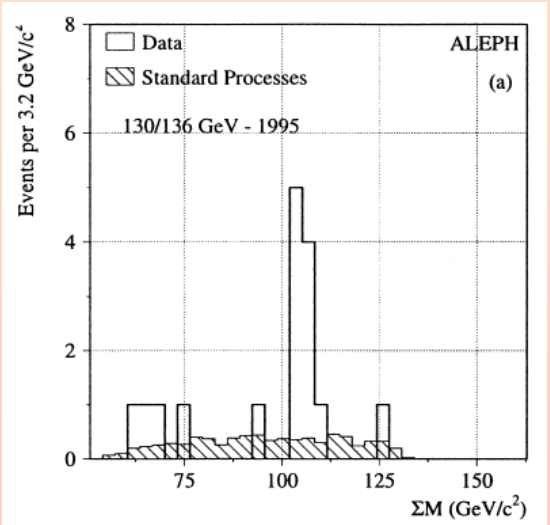
HEP has a history of unconfirmed effects

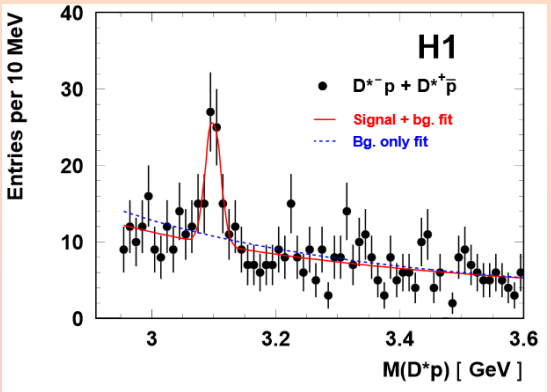
- 3.5σ (2005, CDF) in dimuon (candidate bottom squark, doi:/10.1103/PhysRevD.72.092003)





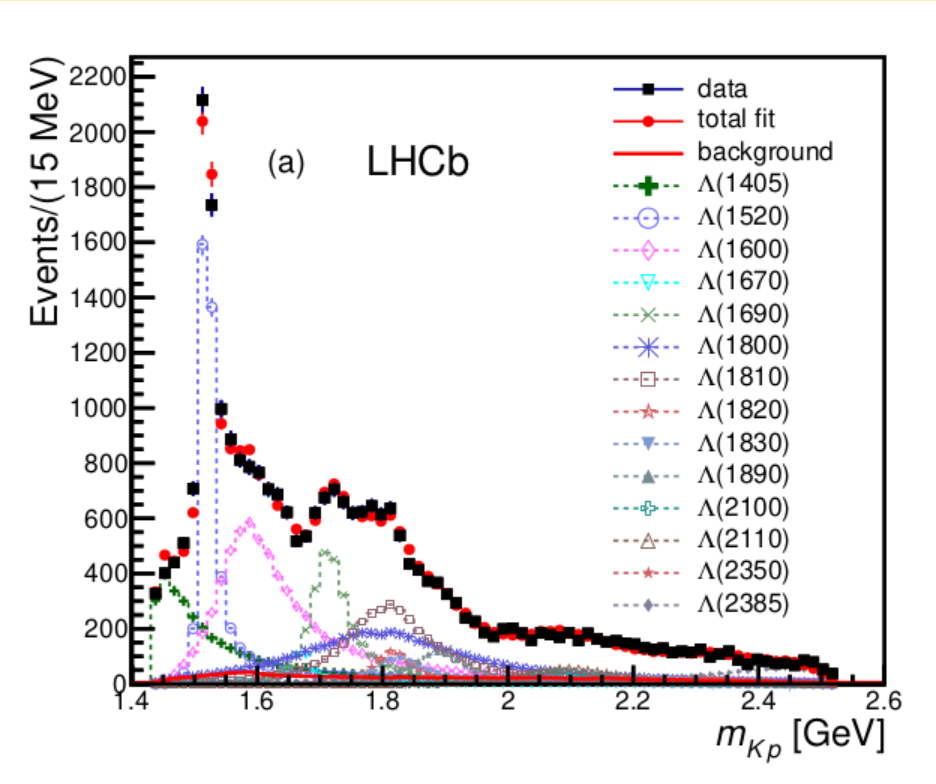
- $\sim 4\sigma$ (1996, Aleph) in four-jet (Higgs boson candidate, doi:/10.1007/BF02906976)
- 6σ (2004, H1) (narrow $\bar{c}$ baryon state, doi:/10.1016/j.physletb.2004.03.012)
  - H1 speaks of “Evidence”, not confirmed.

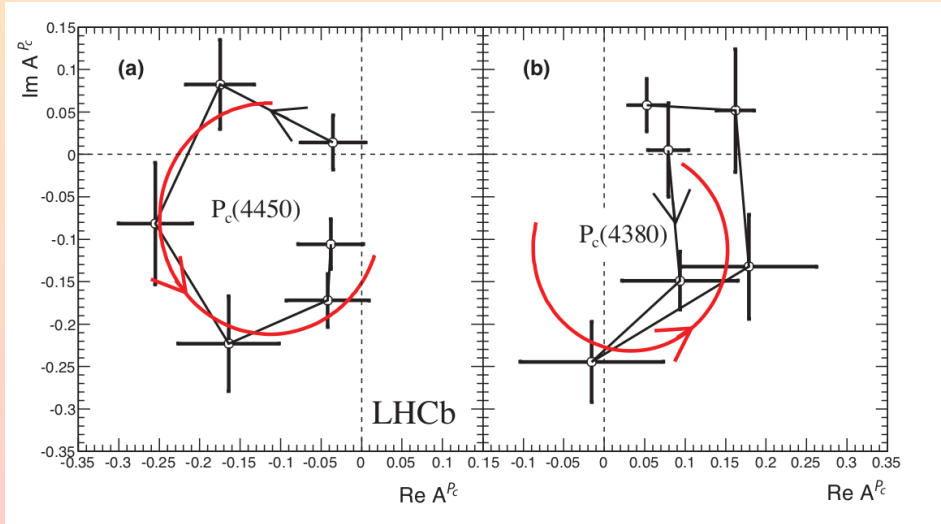




The revenge of the pentaquarks

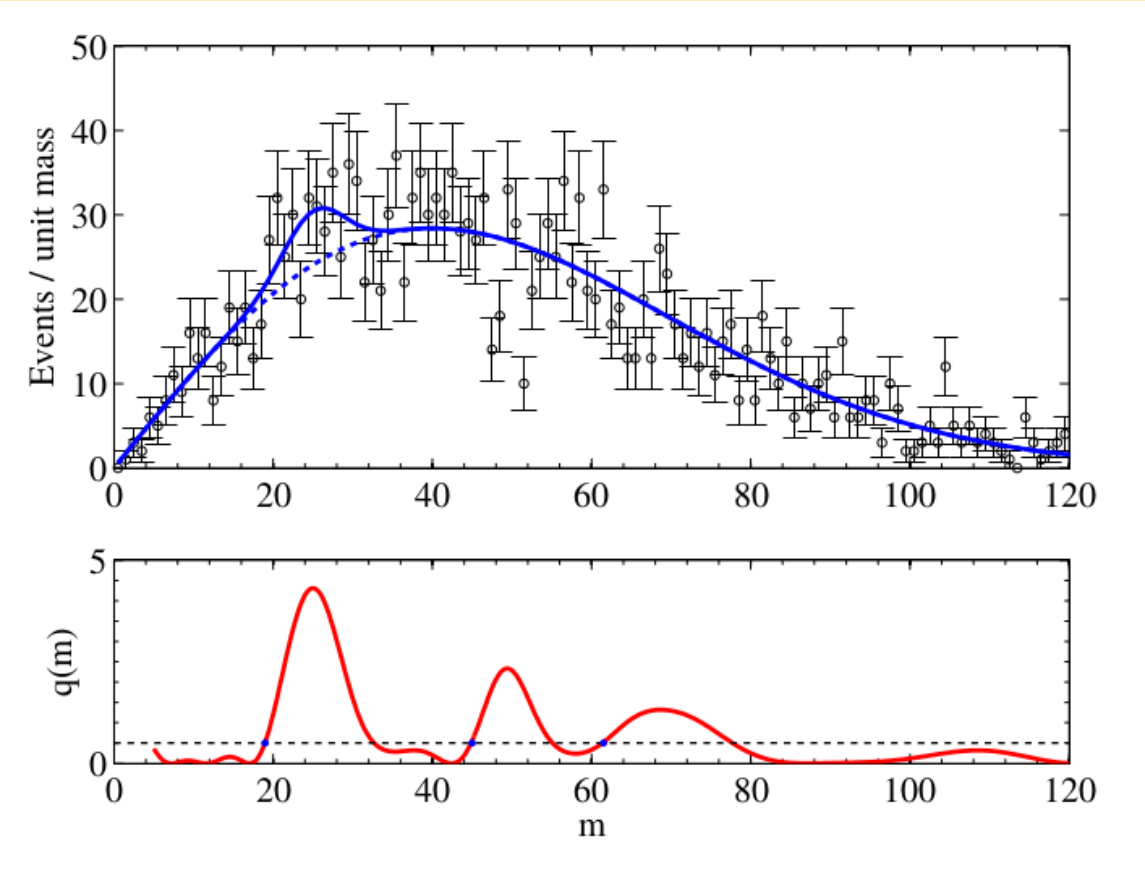
- 9σ and 12σ (2015, LHCb): pentaquarks! (doi:/10.1103/PhysRevLett.115.072001)
  - Several cross-checks (fit to mass spectrum, fit with non-resonant components, evolution of complex amplitute in Argand diagrams)
  - Mass measurement, soft statement: “Interpreted as resonant states they must have minimal quark content of $ccuud$, and would therefore be called charmonium-pentaquark states.
- One remark: quoting significances above about 5–6σ is meaningless
  - Asymptotic approximation not trustable (tail effects). Can run lots of toys but...
  - ...cannot possibly trust knowing your systematic uncertainties to that level





The Look-elsewhere effect — 1

- Searching for a resonance X of arbitrary mass
  - $H_0 =$ no resonance, the mass of the resonance is not defined (Standard Model)
  - $H_1 = H(M \neq 0)$, but there are infinite possible values of M
- Wilks theorem not valid anymore, no unique test statistic encompassing every possible H_1
- Quantify the compatibility of an observation with the B-only hypothesis
  - $q_0(\hat{m_X}) = \max_{m_X} q_0(m_X)$
  - Write a global p-value as $p_b^{global} := P(q_0(\hat{m_X}) > u) \leq \langle N_u \rangle + \frac{1}{2} P_{\chi^2_1}(u)$
  - u fixed confidence level
  - Crossings (Davis, Biometrika 74, 33–43 (1987)) , computable using pseudo-data (toys)



Plot from Gross-Vitells, 10.1140/epjc/s10052-010-1470-8

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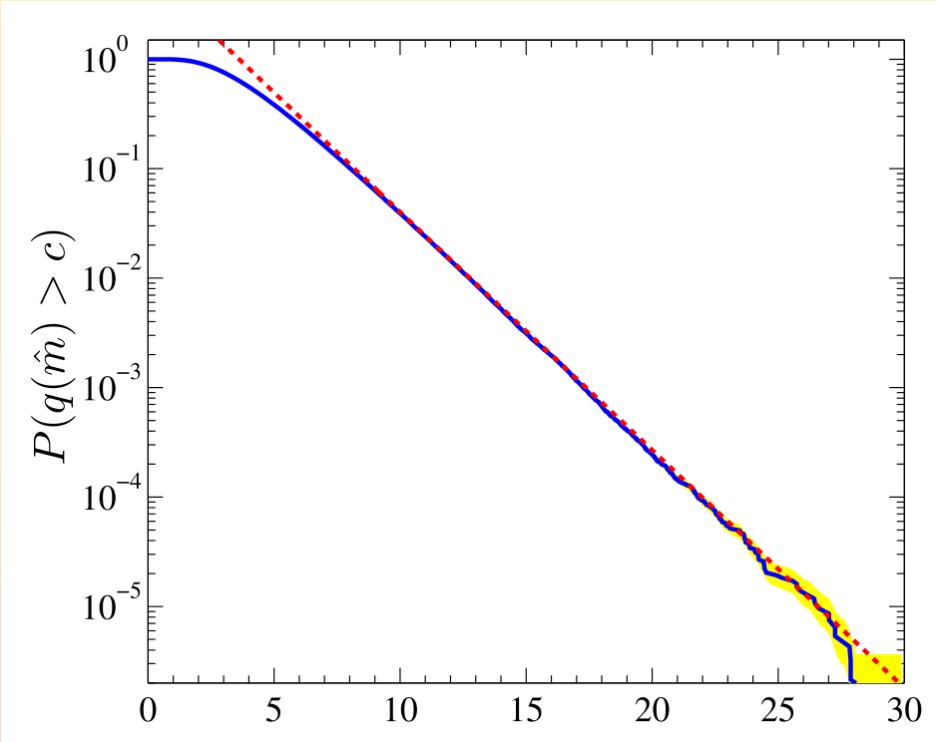
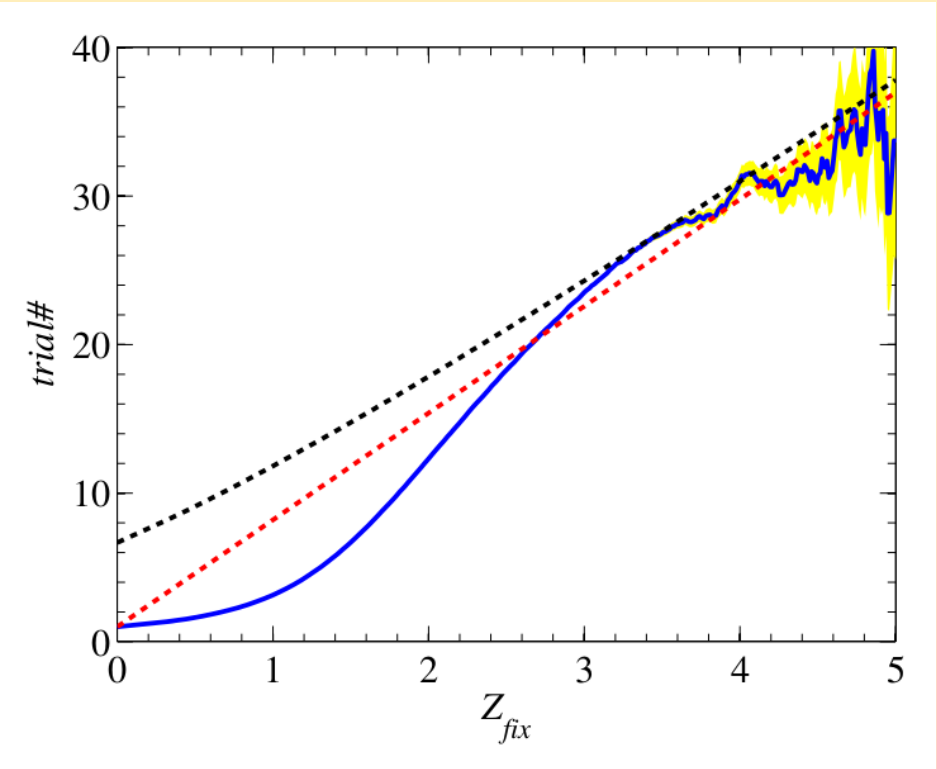
Institut de recherche
en mathématique et physique

- Ratio of local (excess right here) and global (excess anywhere) p-values: trial factor
- Asymptoticly linear in the number of search regions and in the fixed significance level
  - Dashed red lines: prediction based on the formula with upcrossings
  - Blue: 10^6 toys (pseudoexperiments)
- Here *asymptotic* means *for increasingly smaller tail probabilities*

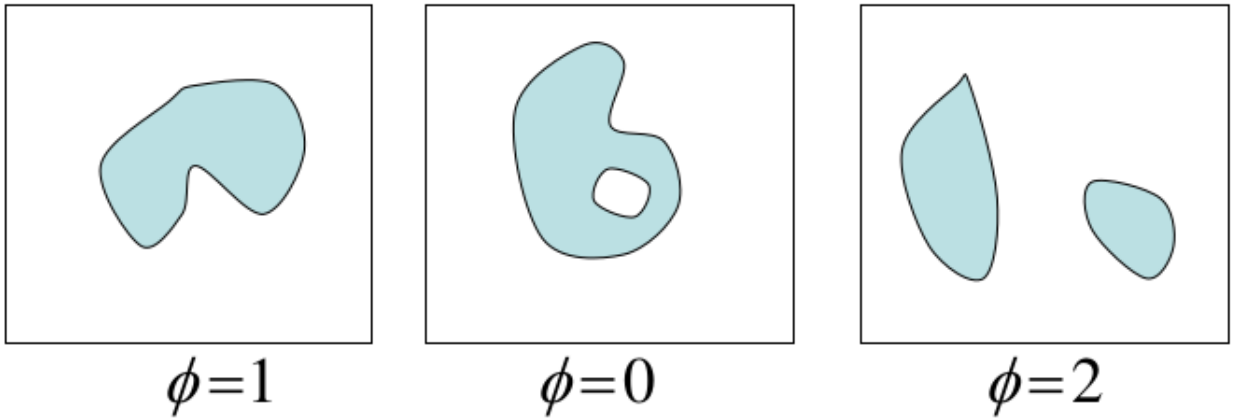


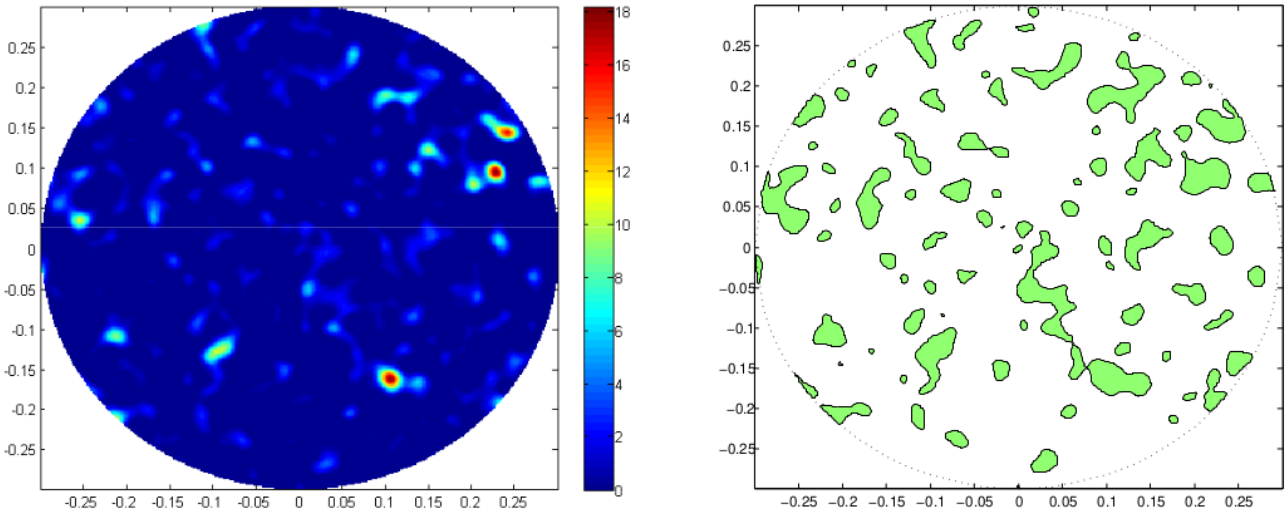


Plot from Gross-Vitells, 10.1140/epjc/s10052-010-1470-8

The Look-elsewhere effect, now also in 2D — 1

- Extension to two dimensions requires using the theory of random fields
  - Excursion set: set of points for which the value of a field is larger than a threshold u
  - Euler characteristics interpretable as number of disconnected regions minus number of holes

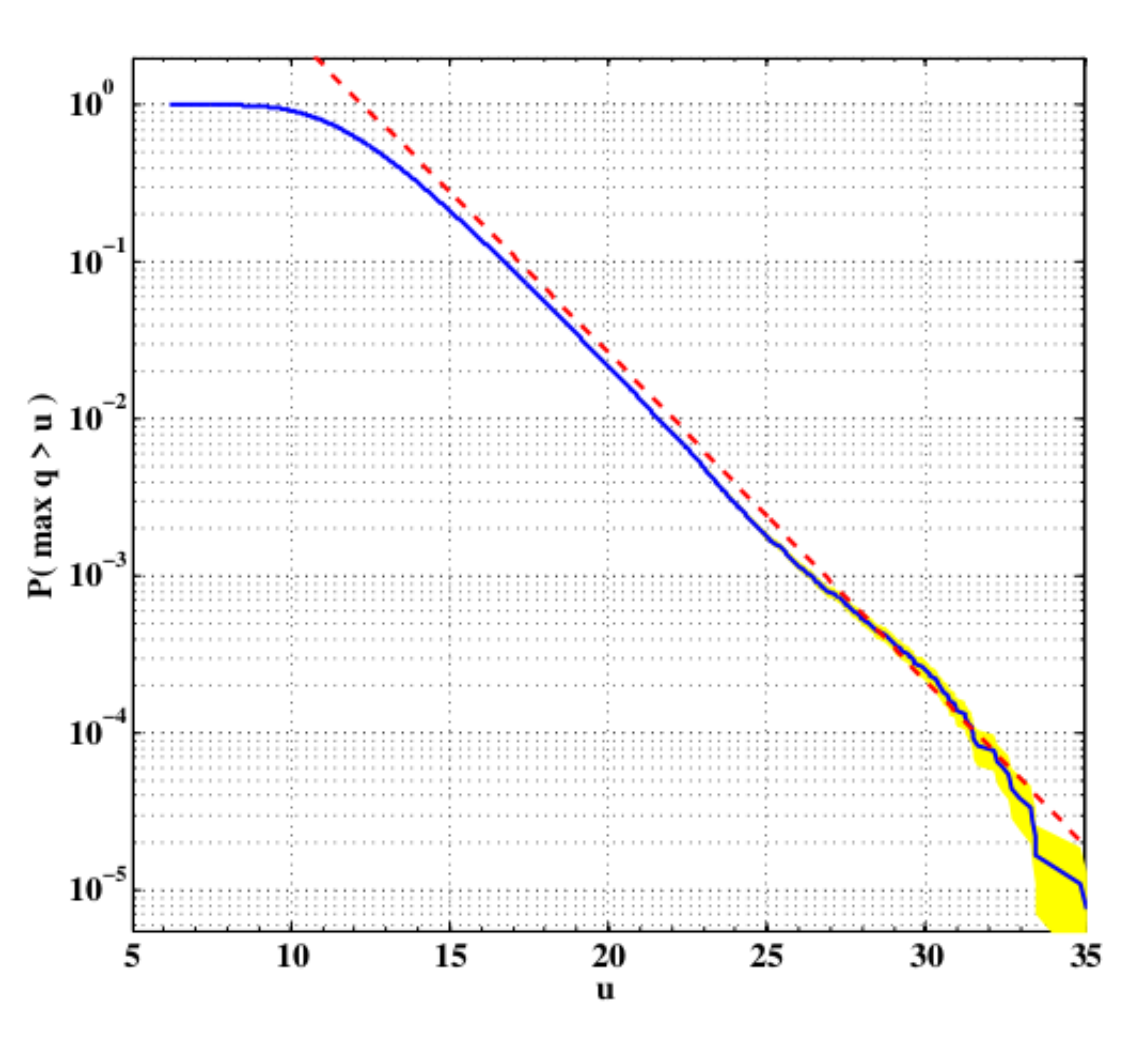




Plot from Gross-Vitells, 10.1016/j.astropartphys.2011.08.005

The Look-elsewhere effect, now also in 2D — 2

- Asymptoticity holds also for the 2D effect, as desired
  - Dashed red lines: prediction based on the formula with upcrossings
  - Blue: $200k$ toys (pseudoexperiments)

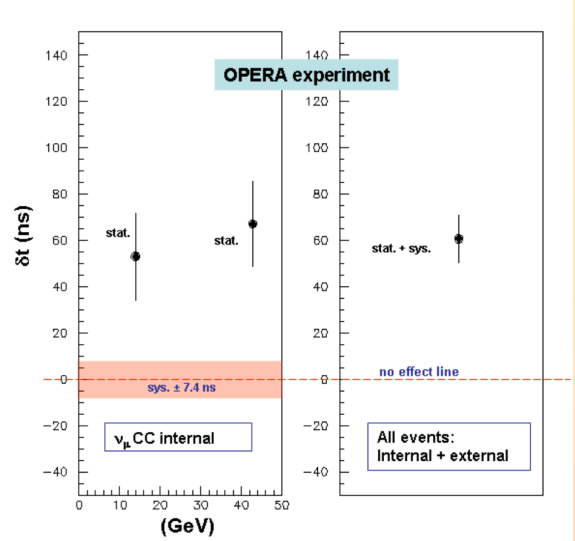


Plot from Gross-Vitells, 10.1016/j.astropartphys.2011.08.005

When there is no LEE, you still need to make sure your systematics are right

- In 2011 OPERA (arXiv:1109.4897v1) reported superluminal neutrino speed, with 6.0σ significance...

An early arrival time of CNGS muon neutrinos with respect to the one computed assuming the speed of light in vacuum of (60.7 ± 6.9 (stat.) ± 7.4 (sys.)) ns was measured. This anomaly corresponds to a relative difference of the muon neutrino velocity with respect to the speed of light (v-c)/c = (2.48 ± 0.28 (stat.) ± 0.30 (sys.)) $\times 10^{-5}$.



- ...but they had a loose cable connector (doi:/10.1007/JHEP10(2012)093)

After several months of additional studies, with the new results reported in this paper, the OPERA Collaboration has completed the scrutiny of the originally reported neutrino velocity anomaly by identifying its instrumental sources and coming to a coherent interpretation scheme.

- Frequentist testing based on Type I and Type 2 error rates (D. Mayo "Statistical Inference as Severe Testing". Cambridge UP, 2018.)
  - Point-null avoided by considering $H_0 : \mu \leq \mu_0$ vs $H_1 : \mu > \mu_0$
- Generalize to test $\mu_1 = (\mu_0 + \gamma)$, $\gamma \geq 0$
- Severe interpretation of negative results (SIN)
  - When H_0 not rejected, define severity
    $SEV(\mu \leq \mu_1) = P(Q > Qobs; \mu \leq \mu_1 false) = P(Q > Qobs; \mu > \mu_1) > P(Q > Qobs; \mu = \mu_1)$
  - Low severity: your test is not capable of detecting a discrepancy even when if it existed, therefore when not detected is's poor evidence of its absence (low power)
  - High severity: your test is highly capable of detecting a discrepancy if it existed, therefore when not detected is a good indication of its absence (high power)
- Severe interpretation of rejection (SIR)
  - When H_0 rejected, define severity
    $SEV(\mu > \mu_1) = P(Q \leq Qobs; \mu > \mu_1 false) = P(Q \leq Qobs; \mu \leq \mu_1) > P(Q \leq Qobs; \mu = \mu_1)$
  - Low severity: if probability of higher-than-observed $Qobs$ is fairly high, then $Qobs$ not a good indication of effect
  - High severity: if probability of smaller-than-observed $Qobs$ is very high, then such a large $Qobs$ indicates a real effect
- Cousins (arXiv:2002.09713) seems to argue that current CL HEP practice is substantially equivalent to Mayo's severe testing
  - Very specific to HEP. Other disciplines should be worried, instead

- Box (https://www.jstor.org/stable/2286841) warns that any model is an approximation

2.3 Parsimony

Since all models are wrong the scientist cannot obtain a "correct" one by excessive elaboration. On the contrary following William of Occam he should seek an economical description of natural phenomena. Just as the ability to devise simple but evocative models is the signature of the great scientist so overelaboration and overparameterization is often the mark of mediocrity.

2.4 Worrying Selectively

Since all models are wrong the scientist must be alert to what is importantly wrong. It is inappropriate to be concerned about mice when there are tigers abroad.

Truth and models: HEP is special

- Cousins (doi:/10.1007/s11229-014-0525-z) notes HEP is in a privileged position when compared with social or medical sciences

5 HEP and belief in the null hypothesis

At the heart of the measurement models in HEP are well-established equations that are commonly known as "laws of nature". By some historical quirks, the current "laws" of elementary particle physics, which have survived several decades of intense scrutiny with only a few well-specified modifications, are collectively called a "model", namely the Standard Model (SM). In this review, I refer to the equations of

There is a deeper point to be made about core physics models concerning the difference between a model being a good "approximation" in the ordinary sense of the word, and the concept of a mathematical limit. The equations of Newtonian physics have been superseded by those of special and general relativity, but the earlier equations are not just approximations that did a good job in predicting (most) planetary orbits; they are the correct *mathematical limits* in a precise sense. The kinematic

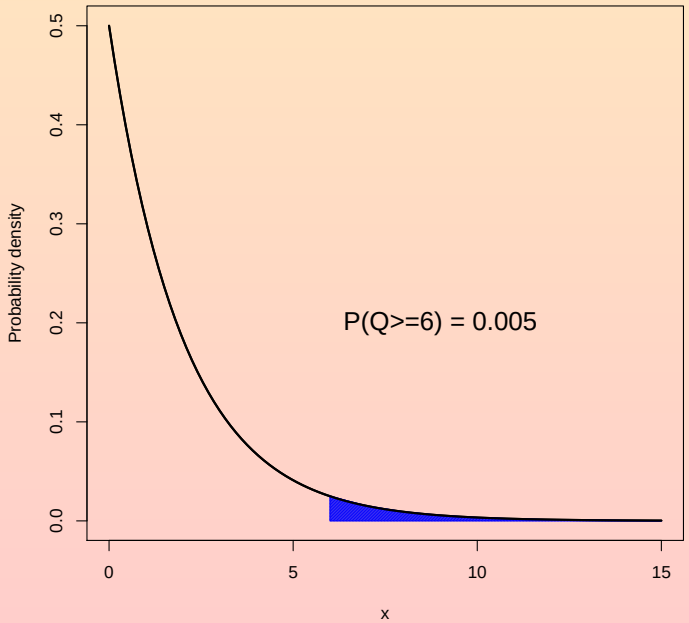
relationships. Nevertheless, whatever new physics is added, we also expect that the SM will remain a correct mathematical limit, or a correct effective field theory, within a more inclusive theory. It is in this sense of being the correct limit or correct effective field theory that physicists believe that the SM is "true", both in its parts and in the collective whole. (I am aware that there are deep philosophical questions about reality, and that this point of view can be considered "naive", but this is a point of view that is common among high energy physicists.)

- Others (Gelman, Raftery, Berger, Bernardo) argue that a point null is impossible (at most "small")

- I think a point or almost-point null is related to our simplifications rather than with a claim on reality
- Some disciplines deal with phenomena which cannot (yet) be explained from first principles
  - Maybe one day we will have a full quasi-deterministic model of a whole body or brain
  - Certainly so far most models are attempts at finding a functional form for the relationship between two variables
- Some disciplines (HEP) have to do with phenomena which can be explained from first principles
  - These principles are *reasonable* but not necessarily the best or the only possible ones
  - No guarantee that they reflect a universal truth
  - Arguing that the vast experimental agreement of the SM implies ground truth behaves based on our principles sounds a bit wishful thinking
  - What can be claimed is that the vast experimental agreement warrants the use of point or quasi-point nulls
- Box's view on models, and the Occam's Razor, should still lead considerations on model choices
  - A version of the Occam's Razor is even implemented in Bayesian model selection
- Still, to avoid interpreting fluctuations as real effects all disciplines should strive—when possible—to describe causal relationships rather than correlations

The χ^2 distribution: why degrees of freedom?

- Sample randomly from a Gaussian p.d.f., obtaining X_1 y X_2
- $Q = X_1^2 + X_2^2$ (or in general $Q = \sum_{i=1}^{N} X_i^2$) is itself a random variable
  - What is $P(Q \geq 6)$? Just integrate the $\chi^2(N = 2)$ distribution from 6 to ∞
- Depends only on N!
  - If we sample 12 times from a Gaussian and compute $Q = \sum_{i=1}^{12} X_i^2$, then $Q \sim \chi^2(N = 12)$
- Theorem: if $Z_1, ..., Z_N$ is a sequence of normal random variables, the sum $V = \sum_{i=1}^{N} Z_i^2$ is distributed as a $\chi^2(N)$
  - The sum of squares is closely linked to the variance $E[(X - \mu)^2] = E[X^2] - \mu^2$ from Eq. **??**
- The χ^2 distribution is useful for goodness-of-fit tests that check how much two distributions diverge point-by-point
- It is also the large-sample limit of many distributions (useful to simplify them to a single parameter)



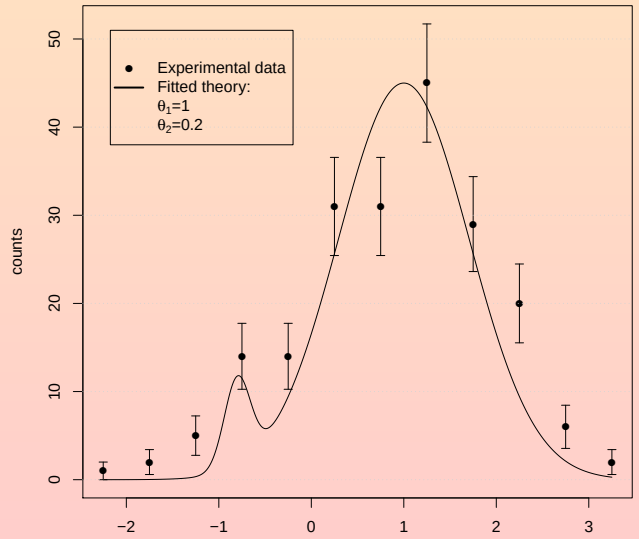
The χ^2 distribution: goodness-of-fit tests 1/

- Consider a set of M measurements $\{(X_i, Y_i)\}$
  - Suppose Y_i are affected by a random error representable by a gaussian with variance σ_i
- Consider a function $g(X)$ with predictive capacity, i.e. such that for each i we have $g(X_i) \sim Y_i$
- Pearson's χ^2 function related to the difference between the prediction and the experimental measurement in each point

$$\chi^2_P := \sum_{i=1}^{M} \left[\frac{Y_i - g(X_i)}{\sigma_i} \right]^2 \tag{1}$$

- Neyman's χ^2 is a similar expression under some assumptions
  - If the gaussian error on the measurements is constant, it can be factorized
  - If Y_i represent event counts $Y_i = n_i$, then the errors can be approximated with $\sigma_i \propto \sqrt{n_i}$

$$\chi^2_N := \sum_{i=1}^{M} \frac{\left(n_i - g(X_i)\right)^2}{n_i} \tag{2}$$



The χ^2 distribution: goodness-of-fit tests 2/

- If $g(X_i) \sim Y_i$ (i.e. $g(X)$ reasonably predicts the data), then each term of the sum is approximately 1
- Consider a function of $\chi^2_{N,P}$ and of the number of measurements M
  - $E[f(\chi^2_{N,P}, M)] = M$
  - The function is analytically a χ^2:

$$f(\chi^2, M) = \frac{2^{-\frac{M}{2}}}{\Gamma\left(\frac{N}{2}\right)} \chi^{N-2} e^{-\frac{\chi^2}{2}} \tag{3}$$

  - The cumulative of f is

$$1 - cum(f) = P(\chi^2 > \chi^2_{obs} | g(x) \text{ is the correct model}) \tag{4}$$

- Comparing χ^2 with the number of degrees of freedom M, we therefore have a criterion to test for goodness-of-fit
  - For a given M, the p.d.f. is known ($\chi^2(M)$) and the observed value can be computed and compared with it
  - Null hypothesis: there is no difference between prediction and observation (i.e. g fits well the data)
  - Alternative hypothesis: there is a significant difference between prediction and observation
  - Under the null, the sum of squares is distributed as a $\chi^2(M)$
  - p-values can be calculated by integration of the χ^2 distribution

$$\frac{\chi^2}{M} \sim 1 \Rightarrow \text{g(X) approximates well the data}$$

$$\frac{\chi^2}{M} >> 1 \Rightarrow \text{poor model (increases } \chi^2\text{), or statistically improbable fluctuation} \tag{5}$$

$$\frac{\chi^2}{M} << 1 \Rightarrow \text{overestimated } \sigma_i\text{, or fraudulent data, or statistically improbable fluctuation}$$

- $\chi^2(M)$ tends to a Normal distribution for $M \to \infty$
  - Slow convergence
  - It is generally not a good idea to substitute a χ^2 distribution with a Gaussian
- The goodness of fit seen so far is valid only if the model (the function $g(X)$) is fixed
- Sometimes the model has k free parameters that were not given and that have been fit to the data
- Then the observed value of χ^2 must be compared with $\chi^2(N')$, with $N' = N - k$ degrees of freedom
  - $N' = N - k$ are called reduced degrees of freedom
  - This however works only if the model is linear in the parameters
  - If the model is not linear in the parameters, when comparing χ^2_{obs} with $\chi^2(N-k)$ then the p-values will be deceptively small!
- Variant of the χ^2 for small datasets: the G-test
  - $g = 2\sum O_{ij} ln(O_{ij}/E_{ij})$
  - It responds better when the number of events is low (Petersen 2012)

Backup