$\mathcal{N}$ovel $\mathcal{A}$spects of $\mathcal{S}$cattering $\mathcal{E}$quations

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Outline

Backgrounds & Motivations

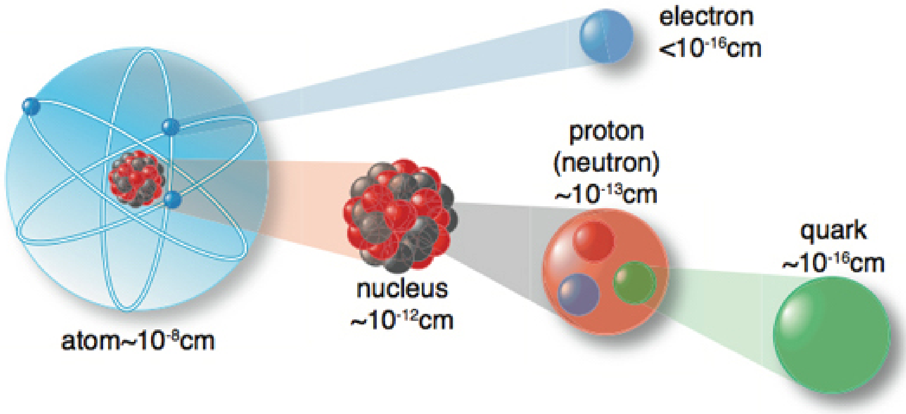
A mini introduction to the scattering equations

My PhD work

- Solving the scattering equations analytically
- Solving the scattering equations numerically

Conclusions & Outlook

Particles & Fields



- Ultimately, our (visible) universe is made of tiny particles, quarks and leptons
- Interactions between subatomic particles are described by quantum field theory (QFT)

Particles & Fields

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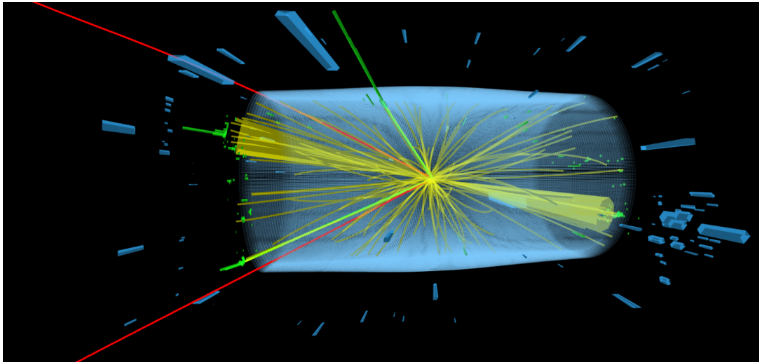
- The Standard Model of particle physics is a QFT
  - ▶ Quantum unification of electromagnetic, weak and strong forces
  - ▶ All matter is made up of quarks and leptons; the vector boson carries the interaction
  - ▶ The SM is now complete after the discovery of the Brout-Englert-Higgs boson

Scattering Amplitudes



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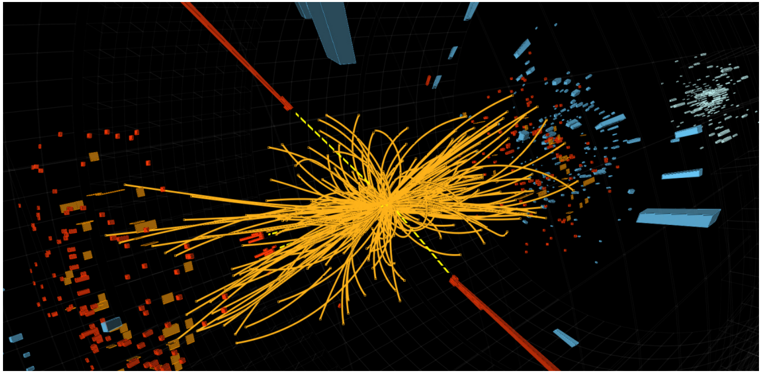
- Scattering of elementary particles is fundamental to our ability to unravel laws of nature
- Scattering amplitudes = probability amplitudes for the scattering of quantum particles
- They allow us to make predictions for physical observables in collider experiments
- Calculating scattering amplitudes efficiently is extremely important for collider physics
  - ▶ understand the properties of the BEH boson more precisely
  - ▶ search for new physics beyond the Standard Model

Scattering Amplitudes



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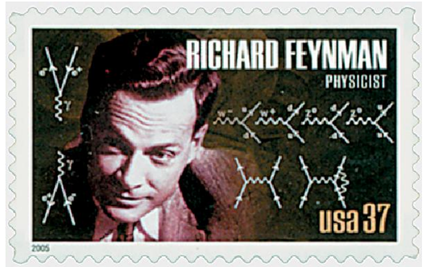
- Amplitudes contain a remarkably rich mathematical structure
- A good understanding of the mathematical structures of amplitudes may lead to
  - new methods to perform computations
  - a deeper understanding of quantum field theory
- From both experimental (phenomenological) and theoretical sides, calculating amplitudes efficiently is important!

Feynman diagrams

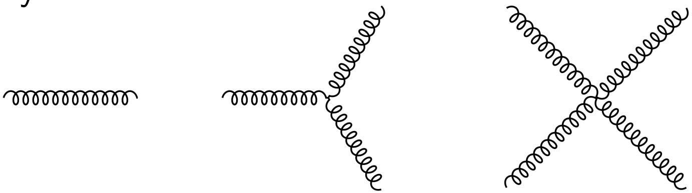
Traditionally, we compute scattering amplitudes using Feynman diagrams perturbatively.



- A standard textbook method for calculating amplitudes
- A systematic procedure for calculations
  - ▶ Non-abelian gauge theory:

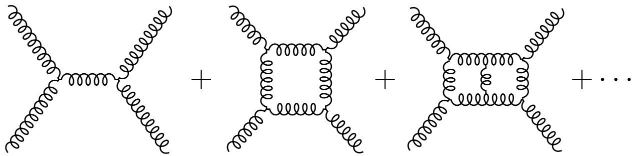
$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \sim (\partial A)^2 + (\partial A) A^2 + A^4$$

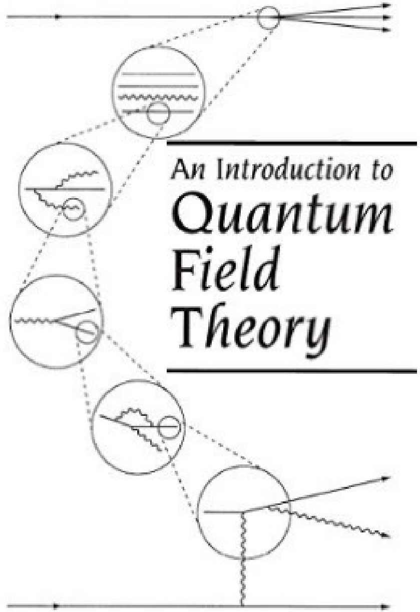
  - ▶ Feynman rules:



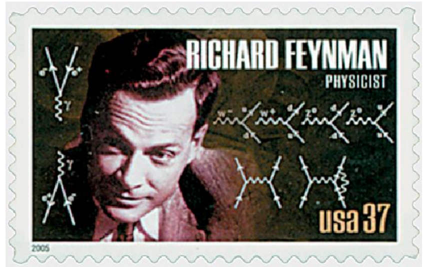
  - ▶ Feynman diagrams

$\mathcal{A}(gg \to gg) =$





Feynman diagrams



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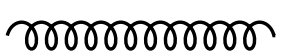
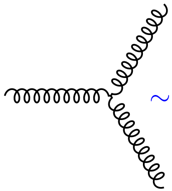
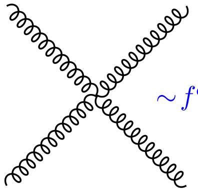


$\frac{\eta^{\mu\nu}\delta^{ab}}{p^2}$

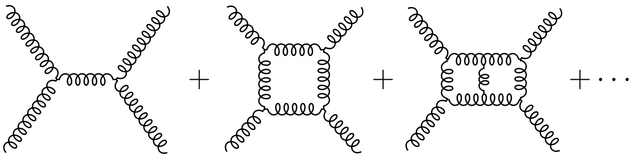


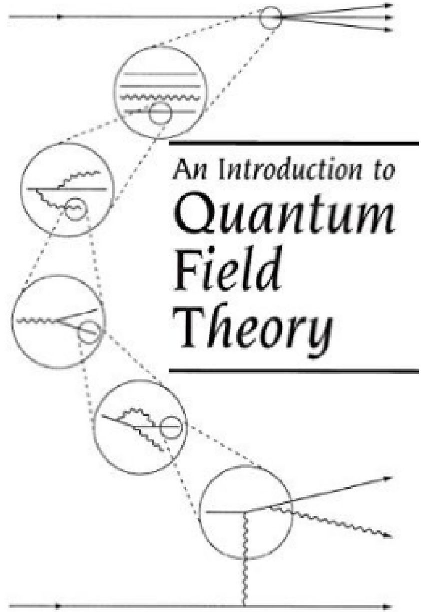
$\sim f^{abc}\eta^{\mu\nu}p^{\alpha}$



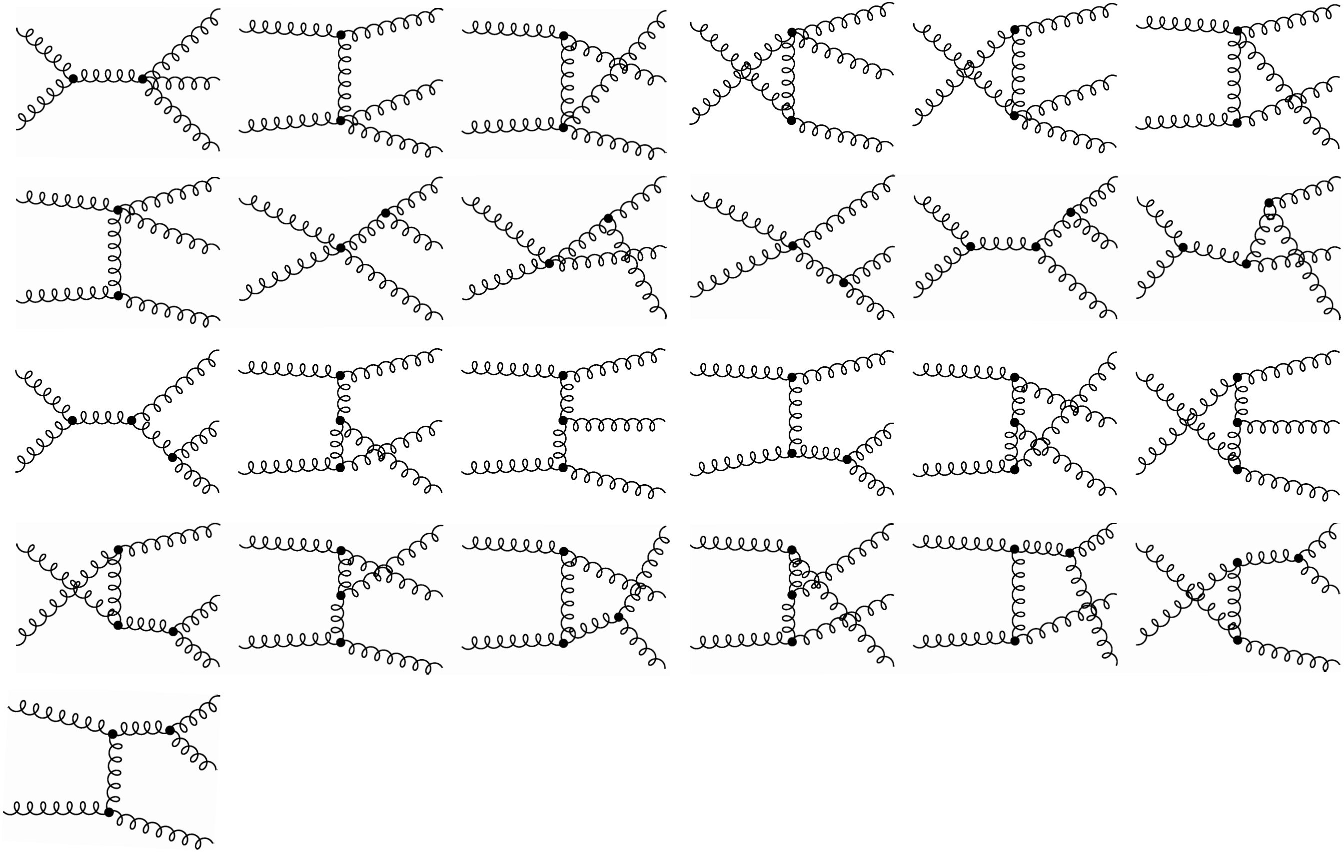
$\sim f^{abe}f^{ecd}\eta^{\mu\nu}\eta^{\alpha\beta}$

  - ▶ Feynman diagrams

$\mathcal{A}(gg \to gg) =$  $+ \cdots$

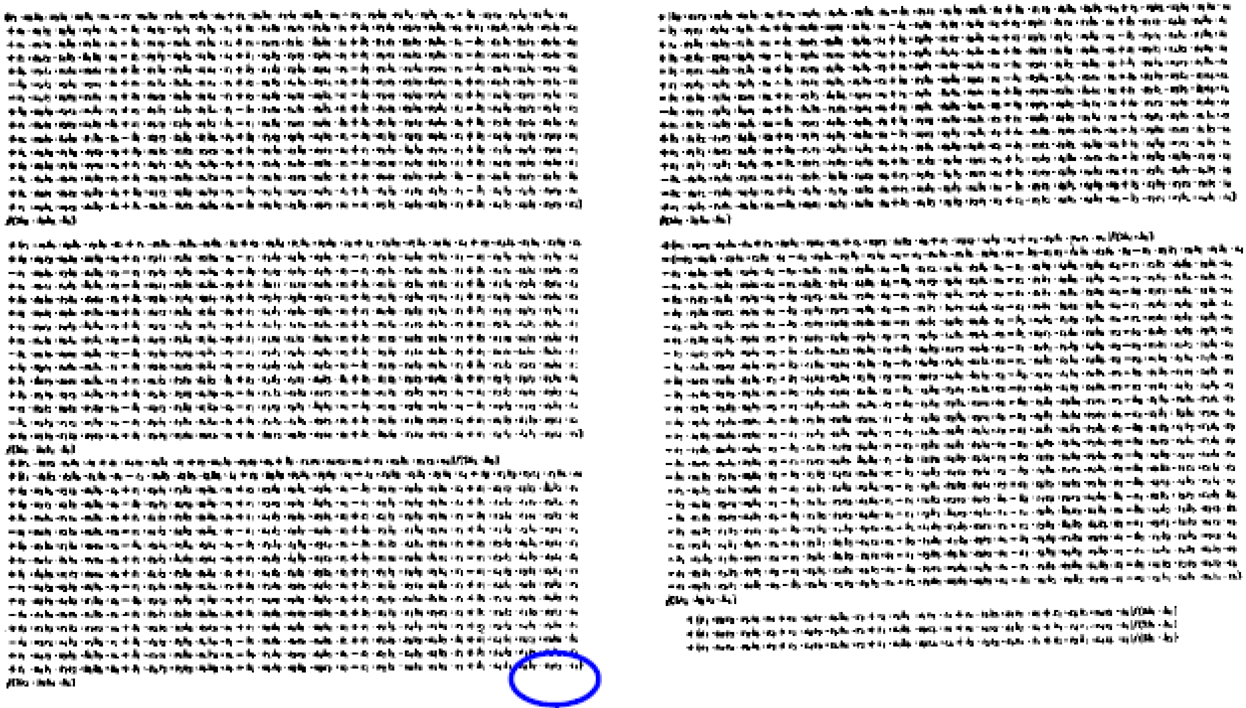


Example: $gg \to ggg$



Example: $gg \to ggg$

Brute force calculation:



[taken from Zvi Bern's talk]

$$(k_1 \cdot k_4)(\varepsilon_2 \cdot k_1)(\varepsilon_1 \cdot \varepsilon_3)(\varepsilon_4 \cdot \varepsilon_5)$$

Example: $gg \to ggg$

But the final result is extremely simple

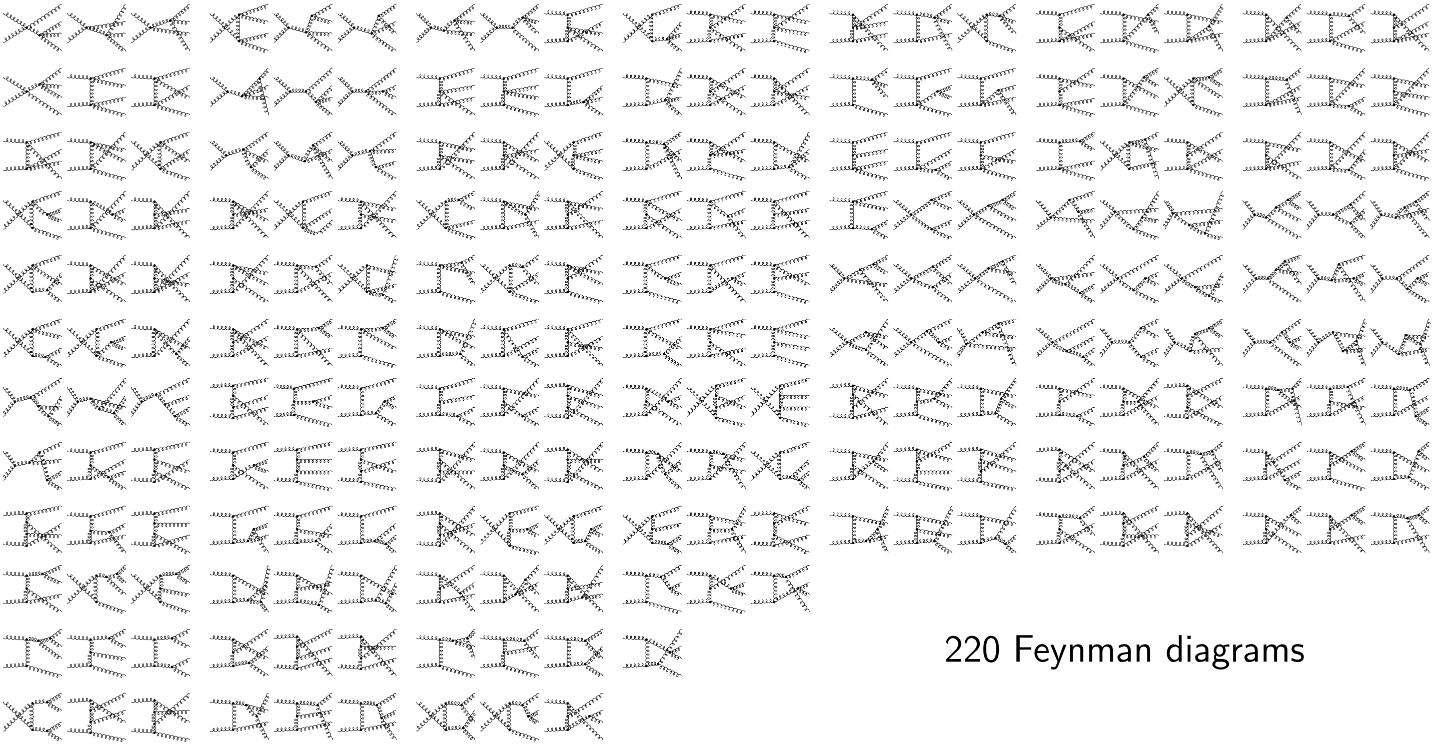
$$\mathcal{A}_5(g^{\pm}g^{-} \to g^{+}g^{+}g^{+}) = 0$$

$$k_\mu \sigma^\mu_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$$
$$\langle i\, j \rangle \equiv \lambda_{i\alpha} \lambda_j^\alpha$$
$$[i\, j] \equiv \tilde{\lambda}_{i\dot{\alpha}} \tilde{\lambda}_j^{\dot{\alpha}}$$

$$\mathcal{A}_5(g^{+}g^{+} \to g^{+}g^{+}g^{+}) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}$$

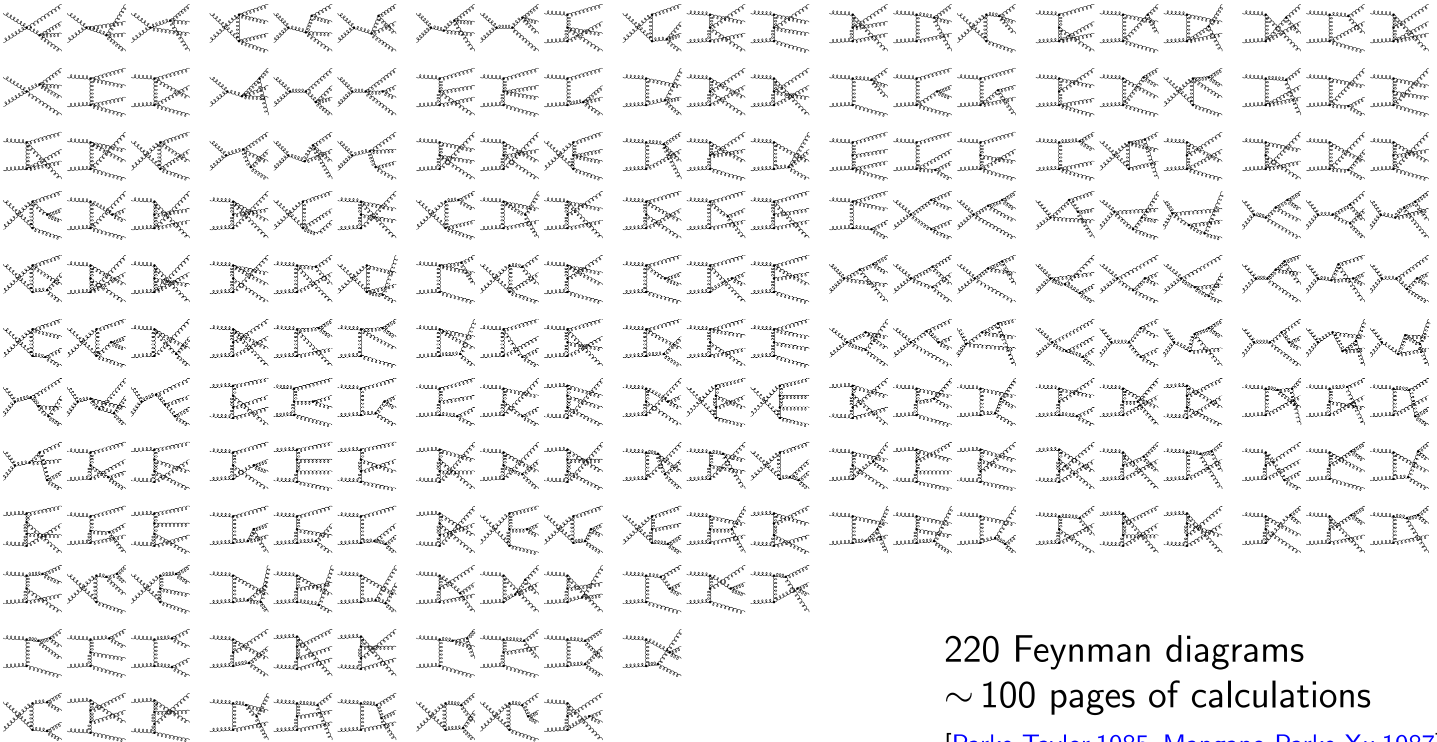
$$\mathcal{A}_5(g^{+}g^{-} \to g^{-}g^{+}g^{+}) = \frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}$$

Six gluons



220 Feynman diagrams

Six gluons

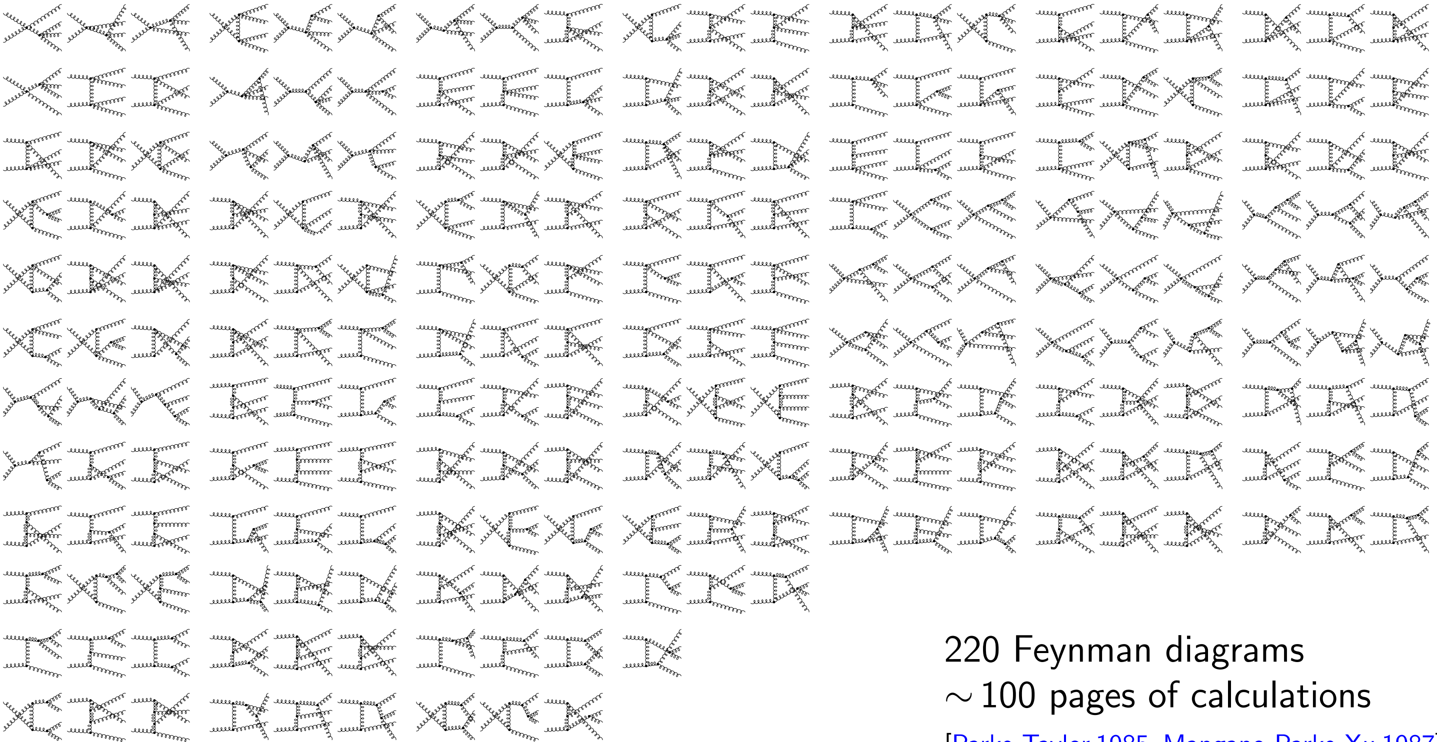


220 Feynman diagrams

$\sim$100 pages of calculations

[Parke-Taylor 1985, Mangano-Parke-Xu 1987]

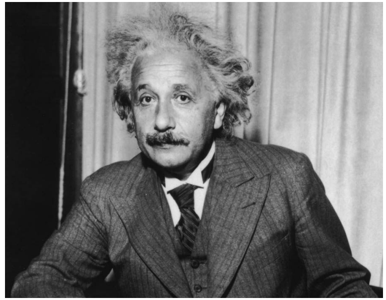
Six gluons



220 Feynman diagrams
∼100 pages of calculations
[Parke-Taylor 1985, Mangano-Parke-Xu 1987]

The final result still can be one-line!

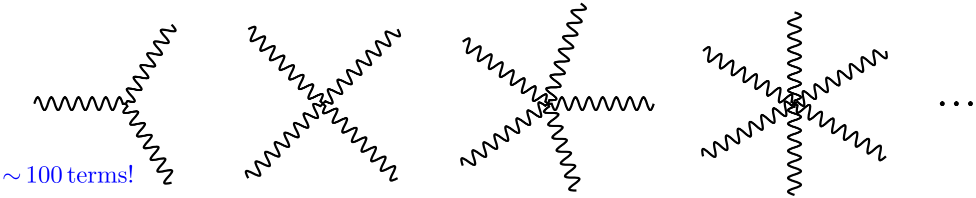
Einstein gravity



• Einstein's theory provides an elegant geometric description of the fundamental interaction of gravitation.

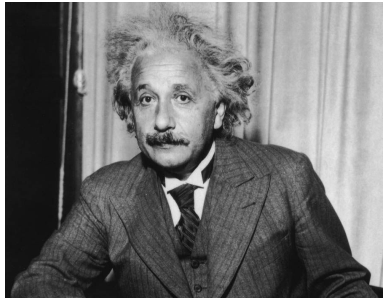
• QFT: Einstein gravity is equivalent to the QFT of a massless, self-interacting, spin-2 particle (graviton).

• Feynman rules:



• It is unimaginable to calculate graviton amplitudes via Feynman diagrams.

Einstein gravity



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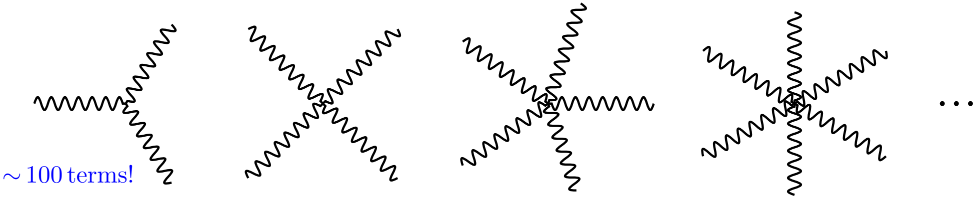


• It is unimaginable to calculate graviton amplitudes via Feynman diagrams. But the final results can be still extraordinarily simple! [Kawai-Lewellen-Tye 1986]

$$\mathcal{M}_3^{(\text{GR})} \sim \mathcal{A}_3^{(\text{YM})} \times \mathcal{A}_3^{(\text{YM})}, \qquad \mathcal{M}_4^{(\text{GR})} \sim \mathcal{A}_4^{(\text{YM})} \times \mathcal{A}_4^{(\text{YM})}$$

• Gravity as a double copy of Yang-Mills

$$\mathcal{M}_n^{(\text{GR})} \sim \mathcal{A}_n^{(\text{YM})} \times \mathcal{A}_n^{(\text{YM})}$$

Reformulate the S-matrix

- The Feynman diagram method is often impractical (for high-multiplicity processes)!
  - The number of diagrams grows rapidly with the number of external legs, e.g.,

$\mathrm{gg} \to n\,\mathrm{g}$	2	3	4	5	6	7	8
♯ of diagrams	4	25	220	2 485	34 300	559 405	10 525 900

  - Each diagram has many terms
  - Individual Feynman diagrams is unphysical
  - The result of adding the contributions of many diagrams can be extraordinarily simple!

- Substantial simplifications and hidden structures of amplitudes are invisible in FDs

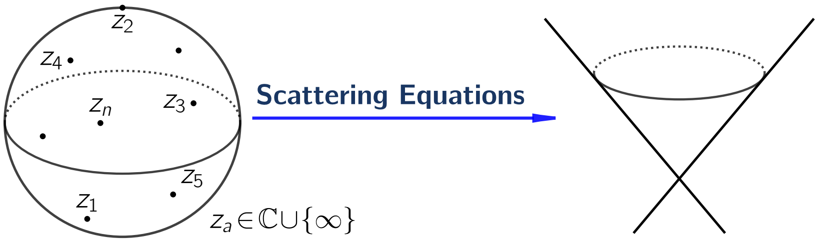
- It has motivated theorists to look for a better way to calculate amplitudes, many novel techniques have been developed, such as
  - Unitarity methods
  - On-shell recursions
  - **Scattering equations**

Introduction to Scattering Equations

Scattering equations

- The **scattering equations** link the scattering data with a space of n complex variables

[Cachazo-He-Yuan 2013]



$$f_a(z,k) = \sum_{b \neq a} \frac{k_a \cdot k_b}{z_a - z_b} = 0, \quad a = 1, \ldots, n$$

- This system has a redundancy, only $(n-3)$ out of n equations are independent.

- The total number of independent solutions is $(n-3)!$.

Cachazo-He-Yuan formalism

Based on the scattering equations, Cachazo, He and Yuan proposed a compact formula for tree-level scattering amplitudes in 2013 [Cachazo-He-Yuan 2013]

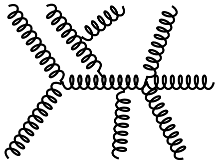
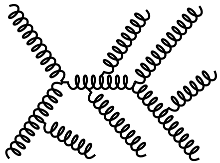
$$\mathcal{A}_n \sim \underbrace{\oint_{\mathcal{C}} \frac{d^n z_a}{\prod'_a f_a(z,k)}}_{\textbf{universal!}} \mathcal{I}_n(z,k)$$

$$f_a = \sum_{b \neq a} \frac{k_a \cdot k_b}{z_a - z_b}$$

- The contour $\mathcal{C}$ is entirely determined by the zeros of the scattering equations
- The scattering equations, $f_a = 0$, are universal for all theories.
- The integrand $\mathcal{I}_n$ encodes dynamics of the specific theory.
- This formula is valid for many massless QFTs, such as Yang-Mills, Einstein gravity.

Advantages

- Summing many many Feynman diagrams is equal to a one-line formula! E.g.,

$\mathcal{A}_8(gg \to gggggg) =$  $+$  $+$ 34288 diagrams

In scattering equations: summing $\sim$35K diagrams is equal to summing $\sim$100 residues.

- In 4D, the scattering eqs can be naturally decomposed into "smaller parts" (helicity sectors).

$$\mathcal{A}_n(g^+ g^+ \to g^+ \cdots g^+) = \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle \langle 2\,3\rangle \cdots \langle n1\rangle}$$

- An elegant description for the mysterious relation between different theories, e.g.

$$\text{Gravity} = \text{Yang-Mills}^2: \quad \mathcal{I}_n^{\text{Gravity}} \sim \mathcal{I}_n^{\text{YM}} \times \mathcal{I}_n^{\text{YM}}$$

- Powerful to reveal mathematical structures behind amplitudes, e.g. soft and Regge limits.

Novel Aspects Of Scattering Equations

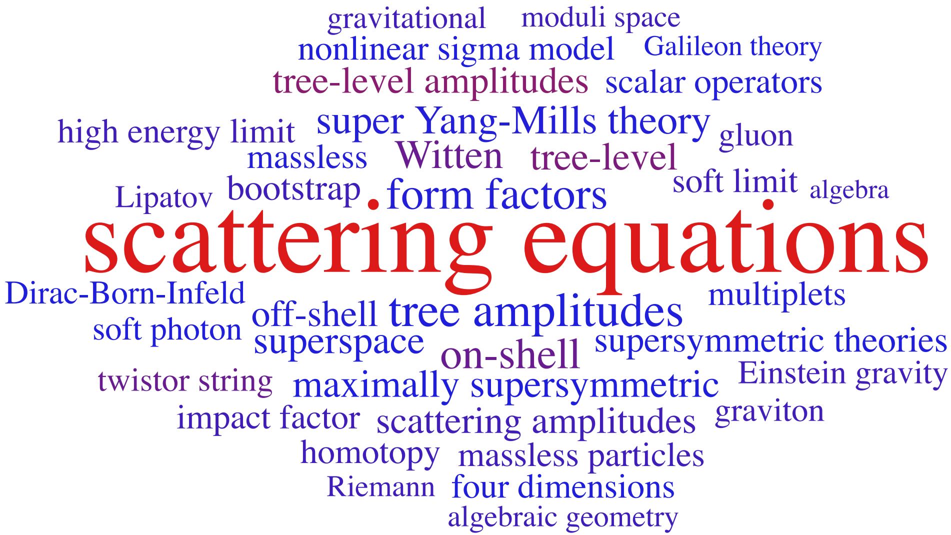
Zhengwen Liu

Août 2019

Thèse présentée en vue de l'obtention
du grade de docteur en sciences

Faculté des sciences
UCLouvain

My PhD thesis



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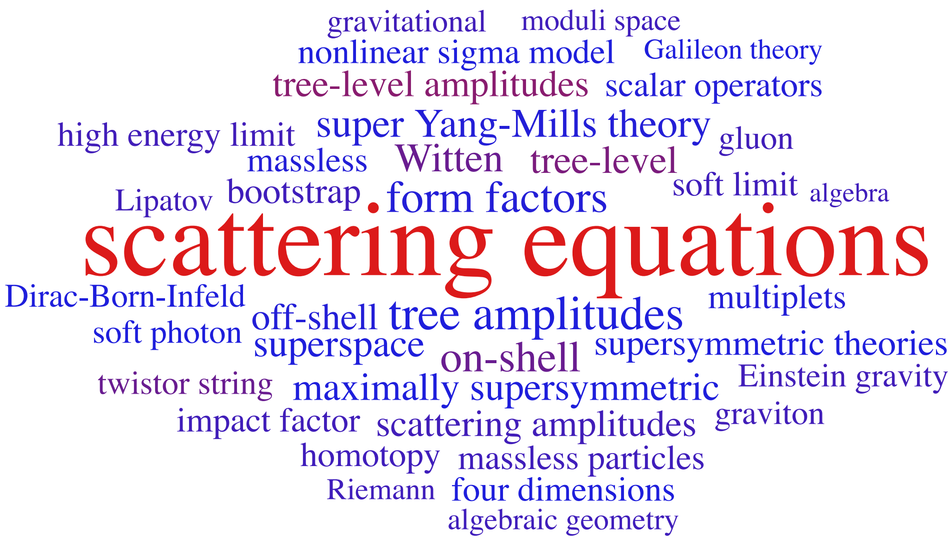


My PhD work:

- New formulas for tree amplitudes in EFTs
- Extend the scattering equation to form factors
- Solving the scattering equations in Regge kinematics analytically
- Solving the scattering equations by homotopy continuation numerically

Solving the scattering equations

- A tree amplitude = a sum over the $(n-3)!$ independent solutions of the scattering equations.

$$f_a(z,k) = \sum_{b\neq a} \frac{k_a \cdot k_b}{z_a - z_b} = 0, \quad a = 1, \dots, n$$

- It is easily understandable that solving scattering equations is crucially important!
- However, solving scattering equations is very challenging due to their complexity.
  - When $n > 6$, solving the scattering equations analytically is impossible!

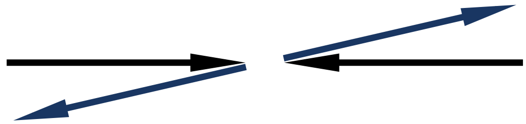
n	4	5	6	7	8	9	10	11	12
$(n-3)!$	1	2	6	24	120	720	5040	40 320	362 880

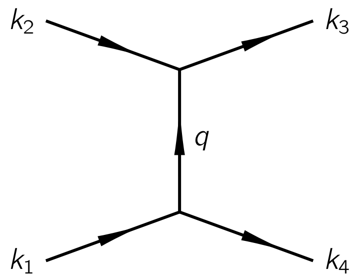
  - Only several very special solutions are previously known!

- My work has made it possible to solve the scattering equations
  - in a special kinematic regime for any multiplicity n analytically
  - for a high multiplicity via numerical algebraic geometry

Multi-Regge Kinematics

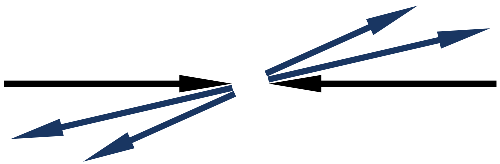
- Regge limit:

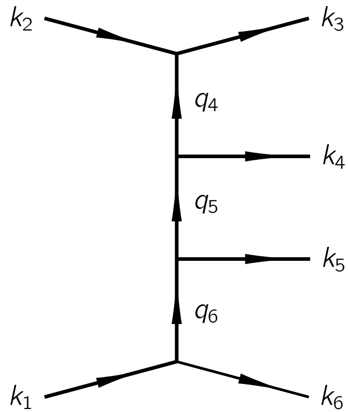




  - Large forward energies s
  - Fixed momentum transfer $s \gg |t|$

- Multi-Regge Kinematics:





  - Generalizes to $2 \to n{-}2$ scattering $(n > 4)$
  - Large rapidity separations between the final-state particles

$$y_3 \gg y_4 \gg \cdots \gg y_n \qquad s \sim e^{y_3 - y_n}$$

  - No hierarchy in transverse directions

Scattering equations in MRK

- In MRK: $y_3 \gg y_4 \gg \cdots \gg y_n$, we proposed the following conjecture [Duhr & ZL, JHEP 1901 146]

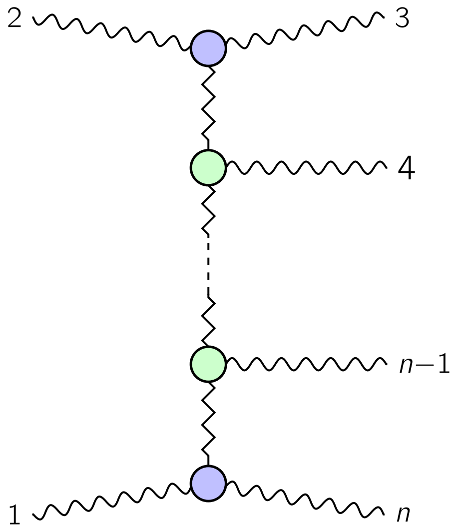
$$\text{“}z_3 \gg z_4 \gg \cdots \gg z_n\text{''}$$

- We have performed a large number of detailed numerical analyses
- Consequence of conjecture:
  - ▶ We can simplify the scattering equations vastly
  - ▶ We obtained the exact solution for any multiplicity n (and for any helicity sector)!

$$z_a = \frac{k_a^+}{k_a^\perp} \times \begin{cases} \left(\prod_{l \in \overline{\mathfrak{N}}_{<a}} \frac{q_l^\perp}{q_{l+1}^\perp} \right)^* \left(\prod_{l \in \overline{\mathfrak{N}}_{>a}} \frac{q_l^\perp}{q_{l+1}^\perp} \right), & a \in \mathfrak{P} \\ \frac{k_a^\perp}{q_{a+1}^\perp} \left(\frac{q_a^\perp}{k_a^\perp} \right)^* \left(\prod_{l \in \overline{\mathfrak{N}}_{<a}} \frac{q_l^\perp}{q_{l+1}^\perp} \right)^* \left(\prod_{l \in \overline{\mathfrak{N}}_{>a}} \frac{q_l^\perp}{q_{l+1}^\perp} \right), & a \in \overline{\mathfrak{N}} \end{cases}$$

- It is very rare that one can analytically solve the scattering eqs for arbitrary multiplicities!

Scattering amplitudes in MRK



• In MRK, scattering amplitudes in both Yang-Mills theory and Einstein gravity have an amazingly simple structure.

• Using the unique solution in MRK, we obtain the expected factorized form of amplitudes.

[Duhr & ZL, JHEP 1901 146; ZL, JHEP 1902 112]

• Our conjecture implies the expected results; conversely, this gives a strong support to the validity of our conjecture!

Scattering equations in MRK

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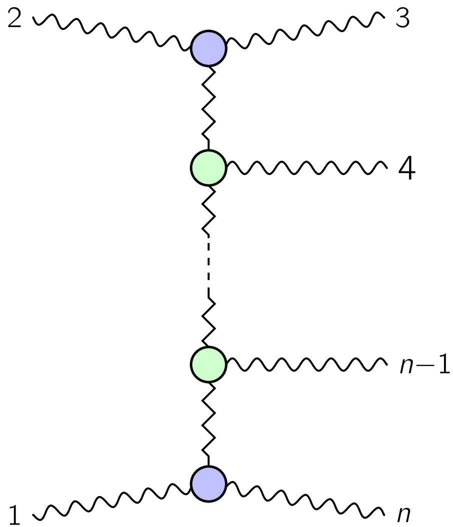

Conjecture → Exact Solutions → Correct Physical Results

Scattering equations in MRK

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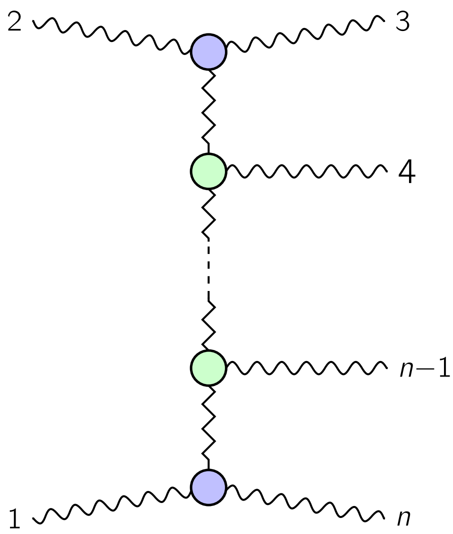
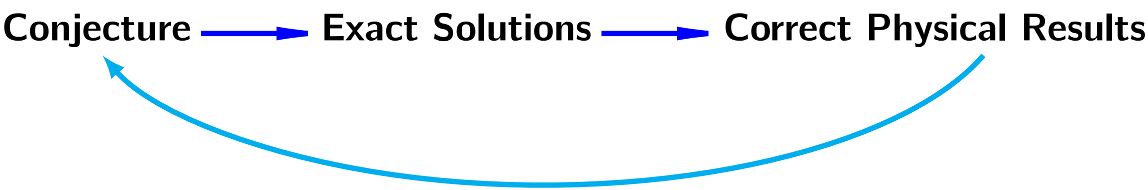




Numerically solving scattering eqs

- Amplitude = summing over all $(n-3)!$ solutions of the scattering equations

$$\mathcal{A}_n = \sum_{\text{all solutions}} F_n(z,k)$$

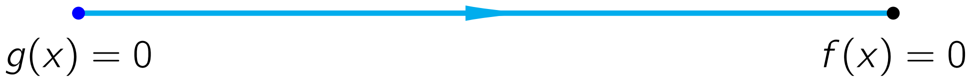
$$\sum_{b\neq a} \frac{k_a \cdot k_b}{z_a - z_b} = 0$$

- The CHY formula shows the great potential to find more efficient ways to evaluate amplitudes, in particular for phenomenologically-relevant processes
  - ▶ An essential step is to solve the scattering equations
  - ▶ In general, it is very difficult to obtain all solutions valid for a high multiplicity

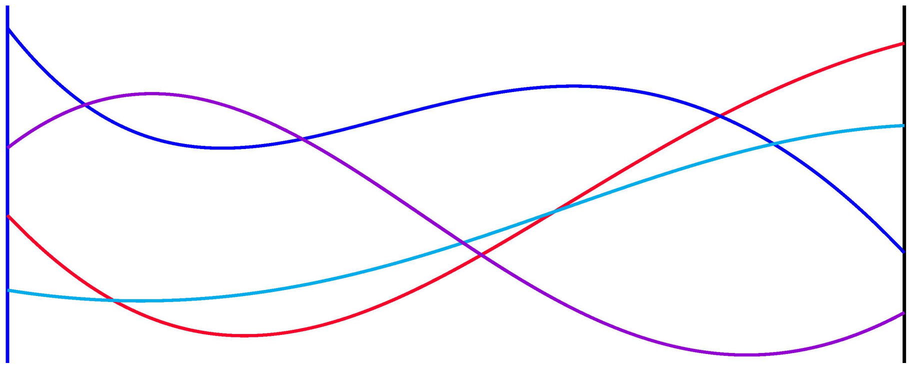
- I will introduce a novel method to solve the scattering equations based on the numerical algebraic geometry.

Homotopy continuation

- Fundamental idea: to establish a path between hard problems and easy problems



- The we may have a link between their solutions



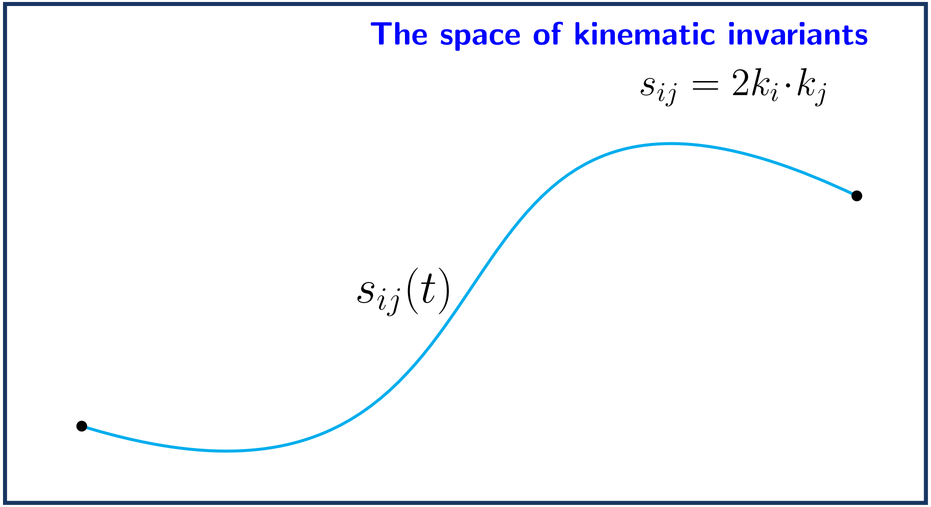
- Technically, tracking the solutions via integrating *ordinary differential equations*
- The homotopy method has been well-studied in mathematics community.

Solving scattering equations

• This technique can be straightforwardly applied to the (polynomial) scattering equations; many packages are available.

• But we can do better! [ZL & Zhao, JHEP 1902 071]

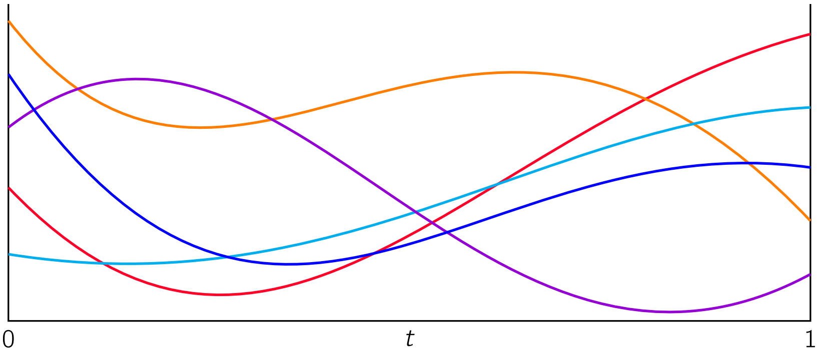
▶ Introduce a physical path in the space of kinematical invariants $s_{ij} \to s_{ij}(t)$



Solving scattering equations efficiently

- This technique can be straightforwardly applied to the (polynomial) scattering equations; many packages are available.

- But we can do better! [ZL & Zhao, JHEP 1902 071]
  - ▶ Introduce a physical path in the space of kinematical invariants $s_{ij} \to s_{ij}(t)$
  - ▶ It naturally induces a continuous deformation of scattering equations



  - ▶ Design an ingenious algorithm based on the physical properties of scattering equations

- We can easily generate all solutions of the scattering equations with a high accuracy.

Solving scattering equations efficiently

- Our method has been implemented into a C++ program [https://github.com/zxrlha/sehomo]

n	$\sharp(n)$	t_n	$\bar{t}_n$ (ms)
5	2	1.3 ms	0.7
6	6	5.0 ms	0.8
7	24	35 ms	1.5
8	120	0.22 s	1.8
9	720	1.3 s	1.8
10	5040	13 s	2.5
11	40 320	2.3 min	3.2
12	362 880	30 min	4.9
13	3 628 800	5.6 h	5.5

$$\sharp(n) = (n-3)!$$

$$\bar{t}_n \equiv \frac{t_n}{(n-3)!}$$

Language: C++

- All solutions have been checked within an accuracy of 10^{-16}
- Totally independent to compute each solution ($\mathcal{O}(\text{ms})$) $\Longrightarrow$ parallel computing
- Pave a way towards a more efficient method of generating amplitudes via scattering eqs

Conclusions & Outlook

Conclusion

• **The scattering equations provide a new way to make predictions**

• **Scattering amplitude = summing over the solutions of the scattering equations**

• **During my PhD, I have explored various aspects of the scattering equations and obtained many new results.**

- We initiated the study of the scattering equations in Regge kinematics
- We developed an efficient method to solve the scattering equations

These results have not only led to a deeper understanding of the mathematical structures underlying the scattering equations, but also broadened the scope for their applications.

Outlook

- Regge behavior for more theories via scattering equations

- A rigorous mathematical proof of the conjecture on the scattering equations in MRK

- Scattering equations for more quantities in QFT

- A more efficient method of calculating amplitudes via homotopy continuation

- Applications in phenomenology (new representation for the SM + efficient algorithms)

- How to understand the scattering equation formalism from the first principle, in particular from the viewpoint of QFT?

Thanks for your attention!