\mathcal{N} ovel \mathcal{A} spects of \mathcal{S} cattering \mathcal{E} quations

Zhengwen Liu

Center for Cosmology, Particle Physics and Phenomenology (CP3) Institut de Recherche en Mathématique et Physique (IRMP)



Louvain-la-Neuve, August 22, 2019





Backgrounds & Motivations

A mini introduction to the scattering equations

My PhD work

- Solving the scattering equations analytically
- Solving the scattering equations numerically

Conclusions & Outlook



- Ultimately, our (visible) universe is made of tiny particles, quarks and leptons
- Interactions between subatomic particles are described by quantum field theory (QFT)

Particles & Fields



 $\bigodot\ensuremath{\mathsf{A}}.\ensuremath{\mathsf{-P}}.$ Olivier

RMP UCLouvain

- The Standard Model of particle physics is a QFT
 - ▶ Quantum unification of electromagnetic, weak and strong forces
 - ► All matter is made up of quarks and leptons; the vector boson carries the interaction
 - ► The SM is now complete after the discovery of the Brout-Englert-Higgs boson

Scattering Amplitudes





©CMS.CERN

- Scattering of elementary particles is fundamental to our ability to unravel laws of nature
- Scattering amplitudes = probability amplitudes for the scattering of quantum particles
- They allow us to make predictions for physical observables in collider experiments
- Calculating scattering amplitudes efficiently is extremely important for collider physics
 - understand the properties of the BEH boson more precisely
 - ► search for new physics beyond the Standard Model

Scattering Amplitudes





- Amplitudes contain a remarkably rich mathematical structure
- A good understanding of the mathematical structures of amplitudes may lead to
 - ▶ new methods to perform computations
 - ► a deeper understanding of quantum field theory
- From both experimental (phenomenological) and theoretical sides, calculating amplitudes efficiently is important!

Feynman diagrams

Traditionally, we compute scattering amplitudes using Feynman diagrams perturbatively.

- A standard textbook method for calculating amplitudes
- A systematic procedure for calculations
 - ► Non-abelian gauge theory:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^{a} F^{a\mu\nu} \sim (\partial A)^{2} + (\partial A)A^{2} + A^{4}$$
• Feynman rules:

$$\mathcal{M}(gg \rightarrow gg) = \int_{R^{a}}^{A^{a}} \mathcal{M}_{R^{a}} \mathcal{M}_{R^{a}} + \int_{R^{a}}^{A^{a}} \mathcal{M}_{R^{a}} \mathcal{M}_{R^{a}} + \int_{R^{a}}^{A^{a}} \mathcal{M}_{R^{a}} \mathcal{M}_{R^{a}} + \dots$$
+ $\mathcal{M}(gg \rightarrow gg) = \int_{R^{a}}^{A^{a}} \mathcal{M}_{R^{a}} \mathcal{M}_{R^{a}} + \int_{R^{a}}^{A^{a}} \mathcal{M}_{R^{a}} \mathcal{M}_{R^{a}} + \dots$

Zhengwen Liu

Novel aspects of scattering equations

An Introduction to **QUANTUM**

Field

Theory

RMP UCLouvain

Feynman diagrams

Traditionally, we compute scattering amplitudes using Feynman diagrams perturbatively.

- A standard textbook method for calculating amplitudes
- A systematic procedure for calculations
 - ► Non-abelian gauge theory:

$$\mathcal{L} = -rac{1}{4}F^a_{\mu
u}F^{a\mu
u} \sim (\partial A)^2 + (\partial A)A^2 + A^4$$

► Feynman rules:

www.

► Feynman diagrams



Zhengwen Liu

Novel aspects of scattering equations

 $\sim f^{abe} f^{ecd} \eta^{\mu\nu} \eta^{\alpha\beta}$



RMP UCLouvain

Example: $gg \rightarrow ggg$



Example: $gg \rightarrow ggg$



Brute force calculation:

in and and and an example of an experiment of the set of and which constructions of the same ranks where the same where we have ready ranks the same ranks ranks are the ALC: 121-121

with reach with reach a real real real and real real reals reals reals real and reals reals reals reals reals reals reals reals An and the set of the set - Ալարելական ակելության, այնդարդայան, արառյունի արելույն, այնդարդելույն ելարդելույն եւ այնդանելու հետ անունեն ա ****************** ner ficht feite feite feit - fei feite f A REAL PROPERTY AND A REAL ADE: MAY HIS

```
en an a sea as a se an ab a se ab as a than a sector a sector ab a sector b
why any and all all a starting and an electric and all and all an electric and the start and all all all all all
ستها معاد المار معاد معاد المار المار المار المار المار المار المار معاد المار معاد المار المار المار المار الم
二、 "你们,你们,你们,你不是一些吗?"我是"我儿,我一家,你你,你你,你你,你你?"我说,你你,你你,你你?你你?你你?"你说:"你……"
dia intertal
```

الان المان المان الله الله الله الله المان ا والا الجافي اليافي المراف المراف المرافي المرافي المرافع المرافع المرافع المرافع المرافع المرافع المرافع المرافع جام المان المان المان الم الم الم المان جوز الوقر الوقر المرق الم المراجع المراجع المرق المرقع ا الاست المراجع المراجع المراجع والمراجع والم (ياريون من ال

يۇۋەر ئەرەپ مۇرىلىرىغى ئەرەپ بۇرۇر ئۇيۇر ئۇرۇ ئەرەپ يەرەپ ئەتلەر ئۇرۇر ئەرەپ بۇرۇر ئەرەپ ئەرەپ ئەرەپ ئۇرۇر ئەر¢ مۇسىن ئىش ئىلىرى بىر سەر ئىلىرى بىرى ئىلىرى بىرى ئىلىرى بىرى ئىلىرى بىرى ئىلىرى بىرى ئىلىرى بىرى بىرى بىرى بىر سان المان والمراجع - قر - مالد موجو البرام - الر- دوام موام - دوام - 4- - الله - الل تاريخ المراجع والمراجع and water water and a mile water water water to a figure and a star of a star water and a star water as a star water to الالار المالي المالية ال - در - ارتقار حالو - در - او - درای - درای - درای - درای - در - زوان حوال - درای - در - در او می مرای - در او -- ال - الله - الل 二素 化氯 化氯化氯化 有一句 化制制 化化 化化化二的 化物 化化 化化 化二角 人名 化化 化成化物 化化化物 化化化物 والمراجعة - to with spectrumly on - to to be state with the state with state with the - to to the spectrum with the - to the best with the المعادية المراقي الجرار المراجع المراقي الجراف المراقية المراجع ا ուղ աղելուցել անել անել 20,000 ուղել 10,000 ուղել 10,000 անել անել ուղել ուղել ուղել ուղել ուղել ուղել ուղել ուղե أبقدونوك والكار

ում (իլ դարդը, որել այս էսը, որել ու երել, որել ու երել, որել ու երել, որելիս երել ու երել երել երել այս երել (Միել երել) ու երել ու երել, որել, որել, ու երել, որել, ու երել, ու երել, որել, ու եղել, որել, որել, որ երել, որել, որել ու երել, ու երել, որել, որել երել, որել, որել, ու երել, որել, որել, որել, որել, որել, որել, որել, որել, որել, որել

[taken from Zvi Bern's talk]

```
(k_1 \cdot k_4)(\varepsilon_2 \cdot k_1)(\varepsilon_1 \cdot \varepsilon_3)(\varepsilon_4 \cdot \varepsilon_5)
```

Example: $gg \rightarrow ggg$



But the final result is extremely simple

Zhengwen Liu

Six gluons

C RMP UCLouvain

XXX XXX XXX XXX XXX XXX XXX XIFXXXIII XXXIII XXXXIII XXX IN WHERE IN EIFINE IN IN XXX XXX TXX SXF XXX XXX XXX KKN XIT XIF IT AN XXX XXX XXX XXX EFF EXEXXE BET EXE AFF IN XXX XXX XXX VEDXXXXXX 220 Feynman diagrams XYX XEX XXX

Six gluons

C RMP UCLouvain

XXX XXX XXX XXX XXX XXX XXX XIF XXX III IXX III IN WHERE IN EIFINE IN EVE KWW XXX XXX XXX XXX XXX KAK EEE EXE XXE EEXE DIG KIFIER XXXXXXER XXXXXXX VEDXXXXXX 220 Feynman diagrams ~ 100 pages of calculations XYX XEX XXX [Parke-Taylor 1985, Mangano-Parke-Xu 1987] Six gluons

C RMP UCLouvain

XXX XXX XXX XXX XXX XXX XXX XIF XXX III IXX III IN WHE REAL IN THE INF THE KWW XXX XXX XXX XXX XXX KAK ZEZ EXR XXE XRX XXX XXX KIFF FXX XXX XEF XXX XXX ISVAN NAN KAN VEDXXXXXX 220 Feynman diagrams ~ 100 pages of calculations [Parke-Taylor 1985, Mangano-Parke-Xu 1987]

The final result still can be one-line!

Einstein gravity

• Einstein's theory provides an elegant geometric description of the fundamental interaction of gravitation.

- QFT: Einstein gravity is equivalent to the QFT of a massless, self-interacting, spin-2 particle (graviton).
- Feynman rules:



RMP 📕 UCLouvain



• It is unimaginable to calculate graviton amplitudes via Feynman diagrams.

Einstein gravity

• Einstein's theory provides an elegant geometric description of the fundamental interaction of gravitation.

- QFT: Einstein gravity is equivalent to the QFT of a massless, self-interacting, spin-2 particle (graviton).
- Feynman rules:



RMP UCLouvain



• It is unimaginable to calculate graviton amplitudes via Feynman diagrams. But the final results can be still extraordinarily simple! [Kawai-Lewellen-Tye 1986]

$$\mathcal{M}_3^{(GR)} \sim \mathcal{A}_3^{(YM)} \times \mathcal{A}_3^{(YM)}$$
, $\mathcal{M}_4^{(GR)} \sim \mathcal{A}_4^{(YM)} \times \mathcal{A}_4^{(YM)}$

• Gravity as a double copy of Yang-Mills

$$\mathcal{M}_n^{(GR)} \sim \mathcal{A}_n^{(YM)} \times \mathcal{A}_n^{(YM)}$$

Zhengwen Liu

Reformulate the S-matrix

- The Feynman diagram method is often impractical (for high-multiplicity processes)!
 - ► The number of diagrams grows rapidly with the number of external legs, e.g.,

gg ightarrow ng	2	3	4	5	6	7	8
♯ of diagrams	4	25	220	2 485	34 300	559 405	10 525 900

- Each diagram has many terms
- ► Individual Feynman diagrams is unphysical
- ► The result of adding the contributions of many diagrams can be extraordinarily simple!
- Substantial simplifications and hidden structures of amplitudes are invisible in FDs
- It has motivated theorists to look for a better way to calculate amplitudes, many novel techniques have been developed, such as
 - Unitarity methods
 - ► On-shell recursions
 - Scattering equations

Zhengwen Liu

Novel aspects of scattering equations

IRMP UCLouvain



Introduction to Scattering Equations

Scattering equations





$$f_a(z,k) = \sum_{b \neq a} \frac{k_a \cdot k_b}{z_a - z_b} = 0, \quad a = 1, \dots, n$$

- This system has a redundancy, only (n-3) out of n equations are independent.
- The total number of independent solutions is (n-3)!.

IRMP UCLouvain

Cachazo-He-Yuan formalism

Based on the scattering equations, Cachazo, He and Yuan proposed a compact formula for tree-level scattering amplitudes in 2013 [Cachazo-He-Yuan 2013]



C RMP UCLouvain

- \bullet The contour ${\mathcal C}$ is entirely determined by the zeros of the scattering equations
- The scattering equations, $f_a = 0$, are universal for all theories.
- The integrand \mathcal{I}_n encodes dynamics of the specific theory.
- This formula is valid for many massless QFTs, such as Yang-Mills, Einstein gravity.

Advantages

C RMP UCLouvain

• Summing many many Feynman diagrams is equal to a one-line formula! E.g.,

$$\mathcal{A}_{8}(gg \rightarrow ggggggg) = \bigwedge_{k=1}^{n} \bigwedge_{$$

In scattering equations: summing $\sim 35 K$ diagrams is equal to summing ~ 100 residues.

• In 4D, the scattering eqs can be naturally decomposed into "smaller parts" (helicity sectors).

$$\mathcal{A}_n(g^+g^+ \to g^+ \cdots g^+) = rac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

• An elegant description for the mysterious relation between different theories, e.g.

$$\mathsf{Gravity} = \mathsf{Yang-Mills}^2: \quad \mathcal{I}_n^{\mathsf{Gravity}} \, \sim \, \mathcal{I}_n^{\mathsf{YM}} \times \mathcal{I}_n^{\mathsf{YM}}$$

• Powerful to reveal mathematical structures behind amplitudes, e.g. soft and Regge limits.

Zhengwen Liu





My PhD thesis



gravitational moduli space nonlinear sigma model Galileon theory tree-level amplitudes scalar operators high energy limit super Yang-Mills theory gluon massless Witten tree-level Lipatov bootstrap, form factors soft limit algebra **SCALTERING OPERATIONS** Dirac-Born-Infeld soft photon off-shell tree amplitudes twistor string maximally supersymmetric theories twistor string maximally supersymmetric Einstein gravity impact factor scattering amplitudes graviton homotopy massless particles Riemann four dimensions algebraic geometry

My PhD thesis



gravitational moduli space nonlinear sigma model Galileon theory tree-level amplitudes scalar operators high energy limit super Yang-Mills theory gluon massless Witten tree-level Lipatov bootstrap. form factors soft limit algebra **SCALTERING EQUATIONS** Dirac-Born-Infeld soft photon off-shell tree amplitudes multiplets twistor string maximally supersymmetric theories twistor string maximally supersymmetric Einstein gravity impact factor scattering amplitudes graviton homotopy massless particles Riemann four dimensions algebraic geometry

My PhD work:

- New formulas for tree amplitudes in EFTs
- Extend the scattering equation to form factors
- Solving the scattering equations in Regge kinematics analytically
- Solving the scattering equations by homotopy continuation numerically

Zhengwen Liu

Solving the scattering equations *C* RMP UCLouvain

• A tree amplitude = a sum over the (n-3)! independent solutions of the scattering equations.

$$f_a(z, k) = \sum_{b \neq a} \frac{k_a \cdot k_b}{z_a - z_b} = 0, \quad a = 1, \dots, n$$

- It is easily understandable that solving scattering equations is crucially important!
- However, solving scattering equations is very challenging due to their complexity.
 - When n > 6, solving the scattering equations analytically is impossible!

n	4	5	6	7	8	9	10	11	12
(<i>n</i> -3)!	1	2	6	24	120	720	5040	40 320	362 880

- Only several very special solutions are previously known!
- My work has made it possible to solve the scattering equations
 - ▶ in a special kinematic regime for any multiplicity *n* analytically
 - ► for a high multiplicity via numerical algebraic geometry

• Regge limit:



- ► Large forward energies *s*
- Fixed momentum transfer $s \gg |t|$
- Multi-Regge Kinematics:



- Generalizes to $2 \rightarrow n-2$ scattering (n > 4)
- ► Large rapidity separations between the final-state particles
 - $y_3 \gg y_4 \gg \cdots \gg y_n$

$$s \sim e^{y_3-y_n}$$





► No hierarchy in transverse directions

Zhengwen Liu

Novel aspects of scattering equations

17/25

Scattering equations in MRK *C* RMP UCLouvain

• In MRK: $y_3 \gg y_4 \gg \cdots \gg y_n$, we proposed the following conjecture [Duhr & ZL, JHEP 1901 146]

"
$$z_3 \gg z_4 \gg \cdots \gg z_n$$
"

- We have performed a large number of detailed numerical analyses
- Consequence of conjecture:
 - ► We can simplify the scattering equations vastly
 - ▶ We obtained the exact solution for any multiplicity *n* (and for any helicity sector)!

$$z_{a} = \frac{k_{a}^{+}}{k_{a}^{\perp}} \times \begin{cases} \left(\prod_{l \in \overline{\mathfrak{N}}_{< a}} \frac{q_{l}^{\perp}}{q_{l+1}^{\perp}}\right)^{*} \left(\prod_{l \in \overline{\mathfrak{N}}_{> a}} \frac{q_{l}^{\perp}}{q_{l+1}^{\perp}}\right), & a \in \mathfrak{P} \\ \frac{k_{a}^{\perp}}{q_{a+1}^{\perp}} \left(\frac{q_{a}^{\perp}}{k_{a}^{\perp}}\right)^{*} \left(\prod_{l \in \overline{\mathfrak{N}}_{< a}} \frac{q_{l}^{\perp}}{q_{l+1}^{\perp}}\right)^{*} \left(\prod_{l \in \overline{\mathfrak{N}}_{> a}} \frac{q_{l}^{\perp}}{q_{l+1}^{\perp}}\right), & a \in \overline{\mathfrak{N}} \end{cases}$$

• It is very rare that one can analytically solve the scattering eqs for arbitrary multiplicities!

Zhengwen Liu

Scattering amplitudes in MRK *C*

• In MRK, scattering amplitudes in both Yang-Mills theory and Einstein gravity have an amazingly simple structure.

• Using the unique solution in MRK, we obtain the expected factorized form of amplitudes.

[Duhr & ZL, JHEP 1901 146; ZL, JHEP 1902 112]

• Our conjecture implies the expected results; conversely, this gives a strong support to the validity of our conjecture!



UCLouvain

Scattering equations in MRK

• In MRK, scattering amplitudes in both Yang-Mills theory and Einstein gravity have an amazingly simple structure.

• Using the unique solution in MRK, we obtain the expected factorized form of amplitudes.

[Duhr & ZL, JHEP 1901 146; ZL, JHEP 1902 112]

• Our conjecture implies the expected results; conversely, this gives a strong support to the validity of our conjecture!



IRMP 📕 UCLouvain

Conjecture — Exact Solutions — Correct Physical Results

Scattering equations in MRK

• In MRK, scattering amplitudes in both Yang-Mills theory and Einstein gravity have an amazingly simple structure.

• Using the unique solution in MRK, we obtain the expected factorized form of amplitudes.

[Duhr & ZL, JHEP 1901 146; ZL, JHEP 1902 112]

• Our conjecture implies the expected results; conversely, this gives a strong support to the validity of our conjecture!



RMP UCLouvain



Numerically solving scattering eqs RMP UCLouvain

• Amplitude = summing over all (n-3)! solutions of the scattering equations

$$\mathcal{A}_n = \sum_{\text{all solutions}} F_n(z,k)$$

$$\sum_{b\neq a}\frac{k_a\cdot k_b}{z_a-z_b}=0$$

- The CHY formula shows the great potential to find more efficient ways to evaluate amplitudes, in particular for phenomenologically-relevant processes
 - ► An essential step is to solve the scattering equations
 - ▶ In general, it is very difficult to obtain all solutions valid for a high multiplicity
- I will introduce a novel method to solve the scattering equations based on the numerical algebraic geometry.

Homotopy continuation

• Fundamental idea: to establish a path between hard problems and easy problems

$$g(x) = 0 \qquad \qquad f(x) = 0$$

• The we may have a link between their solutions



- Technically, tracking the solutions via integrating ordinary differential equations
- The homotopy method has been well-studied in mathematics community.

Zhengwen Liu

IRMP UCLouvain

Solving scattering equations

- This technique can be straightforwardly applied to the (polynomial) scattering equations; many packages are available.
- But we can do better!

[ZL & Zhao, JHEP 1902 071]

C RMP UCLouvain

▶ Introduce a physical path in the space of kinematical invariants $s_{ij} \rightarrow s_{ij}(t)$



Solving scattering equations efficiently P UCLouvain

- This technique can be straightforwardly applied to the (polynomial) scattering equations; many packages are available.
- But we can do better!

[ZL & Zhao, JHEP 1902 071]

- ▶ Introduce a physical path in the space of kinematical invariants $s_{ij} \rightarrow s_{ij}(t)$
- ▶ It naturally induces a continuous deformation of scattering equations



► Design an ingenious algorithm based on the physical properties of scattering equations

• We can easily generate all solutions of the scattering equations with a high accuracy.

Zhengwen Liu

Solving scattering equations efficiently P UCLouvain

• Our method has been implemented into a C++ program [https://github.com/zxrlha/sehomo]

n	#(n)	ta	\overline{t}_{n} (ms)
	#(")	υ	
5	2	1.3 ms	0.7
6	6	5.0 ms	0.8
7	24	35 ms	1.5
8	120	0.22 s	1.8
9	720	1.3s	1.8
10	5040	13 s	2.5
11	40 320	2.3 min	3.2
12	362 880	30 min	4.9
13	3 628 800	5.6 h	5.5

- All solutions have been checked within an accuracy of 10^{-16}
- Totally independent to compute each solution $(\mathcal{O}(ms)) \Longrightarrow$ parallel computing
- Pave a way towards a more efficient method of generating amplitudes via scattering eqs

Zhengwen Liu



Conclusions & Outlook

Conclusion

- The scattering equations provide a new way to make predictions
- Scattering amplitude = summing over the solutions of the scattering equations
- During my PhD, I have explored various aspects of the scattering equations and obtained many new results.
 - ► We initiated the study of the scattering equations in Regge kinematics
 - ► We developed an efficient method to solve the scattering equations

These results have not only led to a deeper understanding of the mathematical structures underlying the scattering equations, but also broadened the scope for their applications.

IRMP UCLouvain





- Regge behavior for more theories via scattering equations
- A rigorous mathematical proof of the conjecture on the scattering equations in MRK
- Scattering equations for more quantities in QFT
- A more efficient method of calculating amplitudes via homotopy continuation
- Applications in phenomenology (new representation for the SM + efficient algorithms)
- How to understand the scattering equation formalism from the first principle, in particular from the viewpoint of QFT?



Thanks for your attention!