When the M meets the P : Gong session
Mal'tsev categories

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Theorem

Let G be a group, and let R be a reflexive relation compatible with the group structure (i.e. $R \leq G \times G$). Then R is an equivalence relation.

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Proof.

Assume that $(x, y), (y, z) \in R$. Then

$$(x, x)(x, y)^{-1}(y, y) = (xx^{-1}y, xy^{-1}y) = (y, x) \in R$$

and

$$(x, y)(y, y)^{-1}(y, z) = (xy^{-1}y, yy^{-1}z) = (x, z) \in R.$$

□

The proof only uses the fact that R is closed under $p(x, y, z) = xy^{-1}z$, and that

$$p(x, x, y) = y = p(y, x, x).$$

So the theorem applies to any algebraic structure equipped with such an operation.

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So the theorem applies to any algebraic structure equipped with such an operation.
The converse is true : an algebraic structure where any reflexive relation is an equivalence must have a ternary operation satisfying these conditions.
Such varieties are called *Mal'tsev varieties*.
Examples : groups, rings, modules, (Associative, Lie, Leibniz, Poisson) algebras, loops, Heyting and Boolean algebras...
Non-examples : sets, monoids

We can define identify a subobject $R \leq G \times G$ with the injective morphism $R \to G \times G$, and characterize equivalence relations in terms of morphisms.

Definition

A *Mal'tsev category* is a finitely complete category such that every reflexive relation is an equivalence relation.

Examples : any Mal'tsev variety, affine spaces, torsors, C^*-algebras, any additive category, (Hausdorff/compact) topological groups...
Non-examples : sets, topological spaces...

In a Mal'tsev category with appropriate notions of image and quotients:

1. the composition of two equivalence relations is an equivalence relation
2. this composition is commutative
3. every simplicial object has the Kan property
4. the lattice of equivalence relations on any object is modular
5. "Chinese Remainder Theorem" : for every equivalence relations R, S on X,
$$\frac{X}{R \cap S} \cong \frac{X}{R} \times_{X/(R \vee S)} \frac{X}{S}$$
6. there are notions of abelian objects, commutators, central extensions...

In fact 1,2,3 are equivalent to the Mal'tsev property for regular categories, and 5 is equivalent to the Mal'tsev property for exact categories.