Modelling Large-Scale Structure: Excursion Sets, Peaks and other creatures

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MPA - Garching

based on works with R. Sheth, A.Paranjape, T. Lazeyras, V. Desjacques
and scattered thoughts

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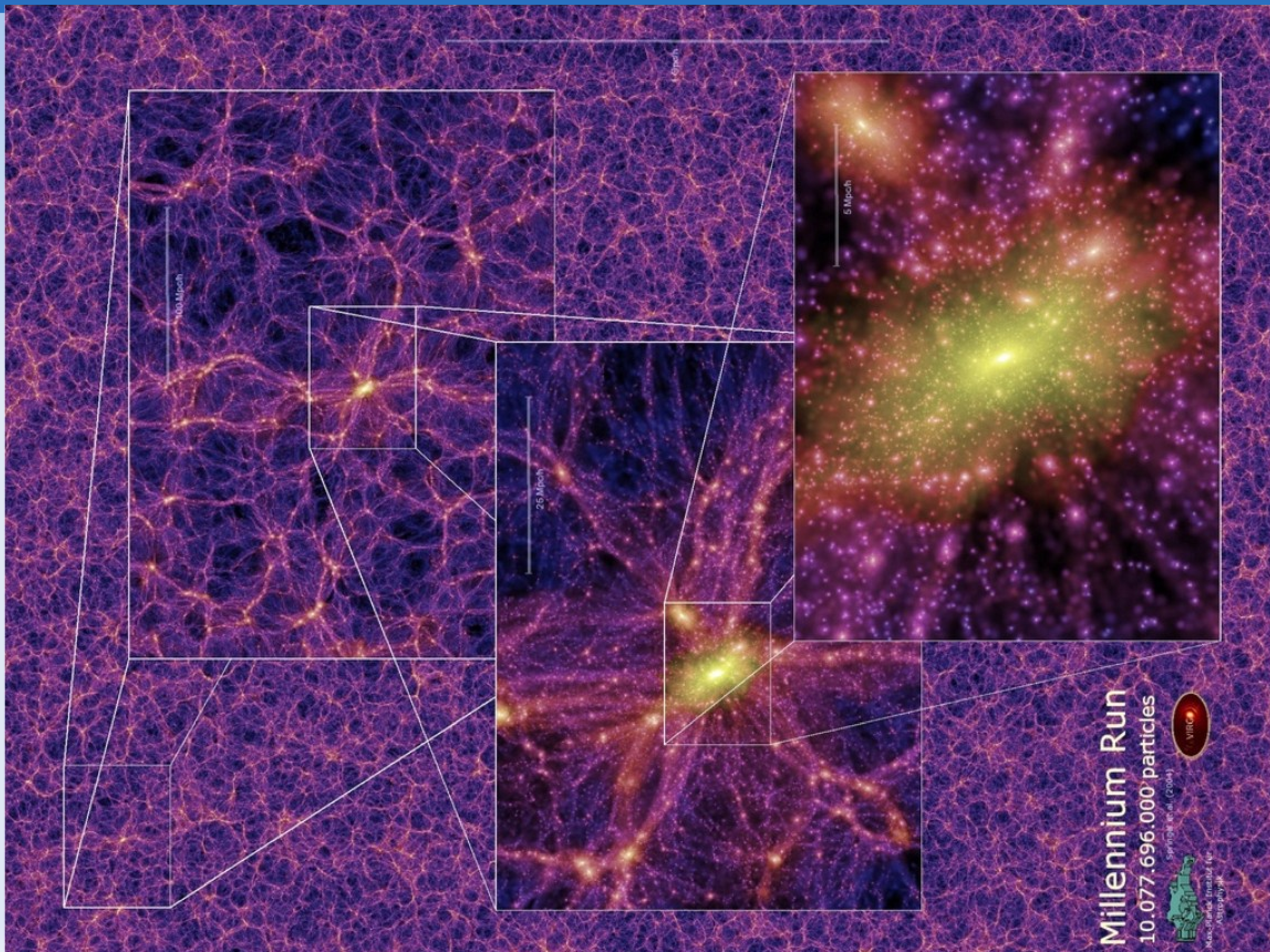
Why theory?

One might wonder why we put effort into approximate descriptions of cosmic structure formation given the tremendous recent and promised advances in computing power. Surely the not very distant future will bring computations of arbitrarily large simulation volumes with arbitrarily high resolution using arbitrarily adaptive hydrodynamical and N-body techniques. That will be so. But even so, we need a physical language to discuss the outcomes.

(Bond & Myers 96)

Really??



Really? Yes, for many reasons

- Understand N-body simulations
- Can't run a simulation for every choice of cosmological parameters!
- Explore non-standard cosmologies
- Huge degeneracy in parameter space: study deviation from universality
- Physically motivated fitting formulae (esp. for halo bias!)
- Improve data analysis

Which models?

- Excursion set theory.
  Halos are patches with high enough initial mean density to recollapse by today.
- Theory of peaks.
  Halos are peaks of the initial density field smoothed on their mass scale
- Peak-patch models. Excursion set peaks.
  Combination of the above, with more sophisticated models of collapse
- Models of halo motion. Patches have center of mass acceleration
- Models of bias. Should follow consistently

Sperical Collapse

- Spherical evolution sensitive only to the total mass M inside the shell, not to the inner density profile
- That is, only mean initial overdensity within R matters:

$$\delta_{R,in}(\mathbf{x}) \equiv \frac{1}{V_R}\int_{|\mathbf{y}|<R} \mathrm{d}^3y \, \frac{\rho_{in}(\mathbf{x}+\mathbf{y}) - \bar{\rho}_{in}}{\bar{\rho}_{in}}$$

- A shell of radius R containing $M = \bar{\rho}_{in}(4\pi/3)R^3$ collapses at $\mathbf{x}$ by z if

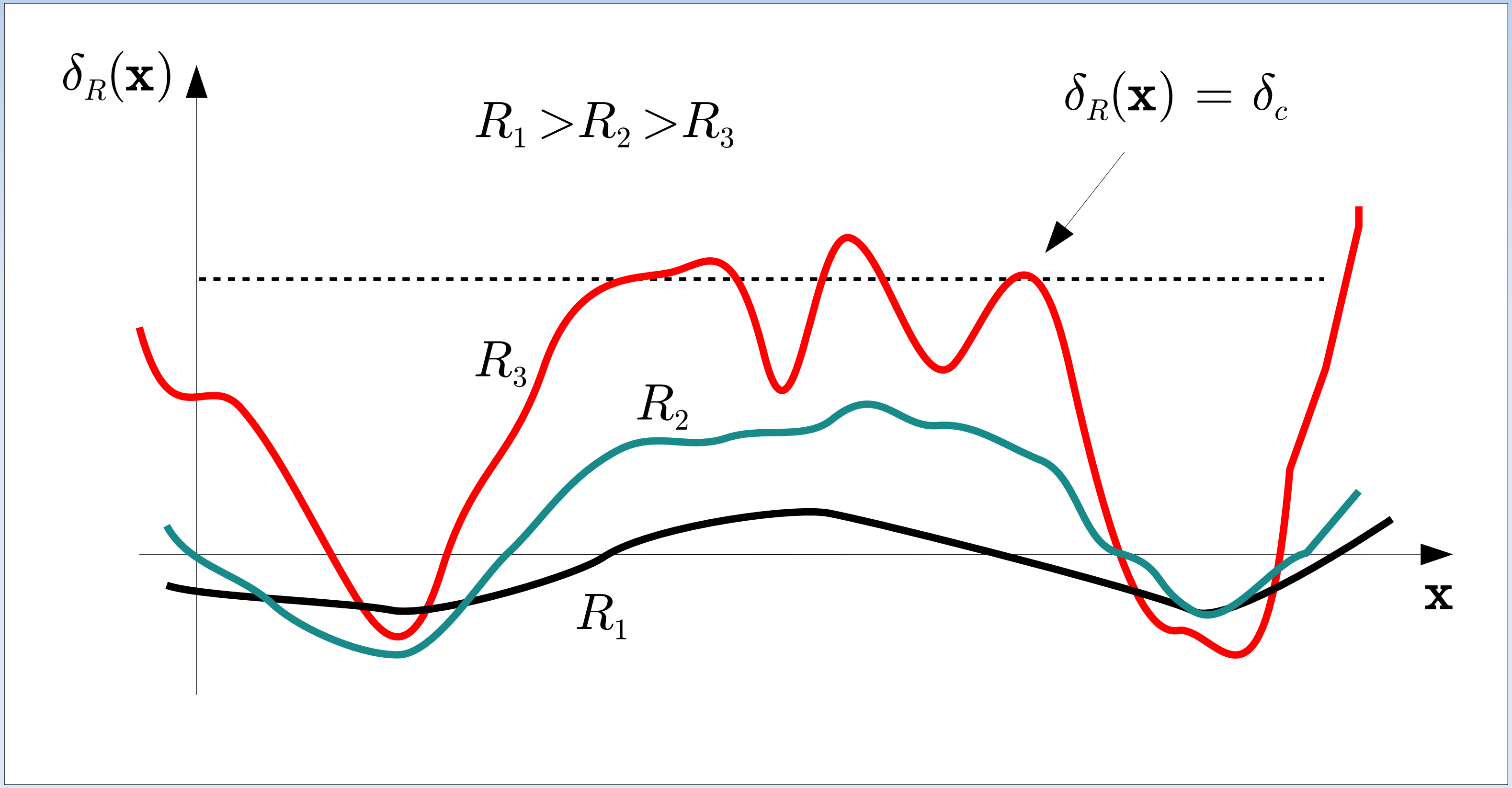
$$\delta_R(z,\mathbf{x}) \equiv D(z)\delta_{R,in}(\mathbf{x}) \gtrsim 1.69 \equiv \delta_c$$

- M sets the smoothing scale, the filter MUST be TopHat in real space
- In Fourier space:

$$\delta_R(\mathbf{x}) \equiv \int \frac{\mathrm{d}^3k}{(2\pi)^3} \, e^{i\mathbf{k}\cdot\mathbf{x}} \, \frac{3j_1(kR)}{kR} \, \delta_{in}(\mathbf{k}) \geq \frac{\delta_c(R)}{D(z)}$$

Finding proto-halos

- The INITIAL density field varies with position AND smoothing scale



- First crossing fixes R and $\mathbf{x}$: size and position of the proto-halo
- Mass conservation. Final mass is $M = \rho_{\mathrm{bg},in} 4\pi R^3/3$

Excursion set theory

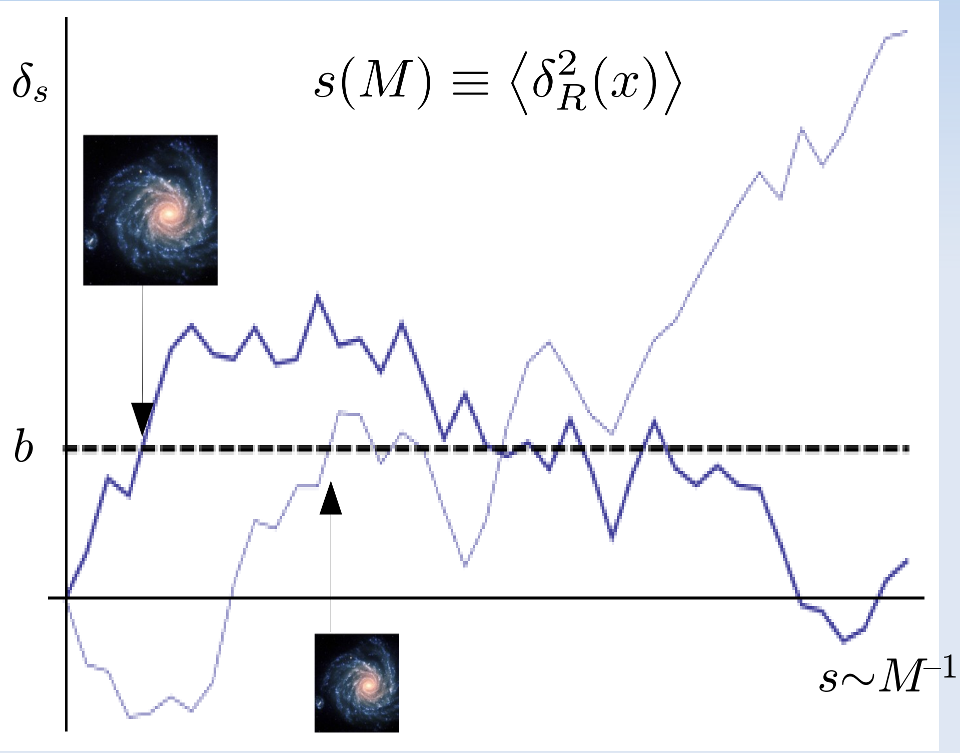
Halos as patches in the initial conditions that:

- are dense “enough” to have formed by today
- are not contained in larger patches of the same density (“no cloud-in-cloud”).
- “enough” is the initial overdensity of spherical collapse

Excursion set theory

- At each position $\mathbf{x}$, $\delta_R(\mathbf{x})$ follows a different random walk as R changes
- But the walks are not Markovian: steps correlate with each other
- True for any compact filter (include all Fourier modes)

FIRST PASSAGE of random walks w/ CORRELATED steps



- Abundance $n_h(M)$ $\longleftrightarrow$ first crossing probability $f(s)$ at scale $s(M)$
- But $f(s)$ is not known: need better maths

First crossing distribution

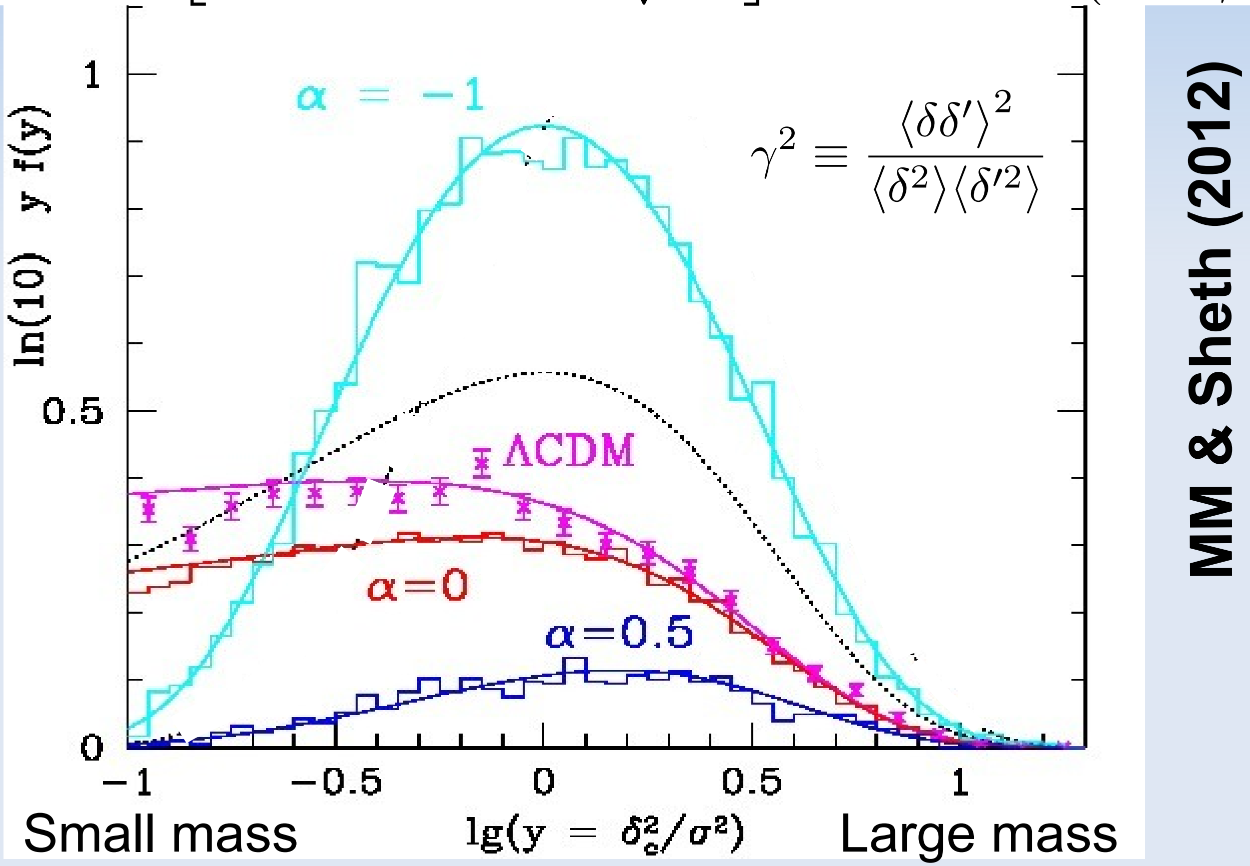
- Probability of ANY crossing at s **Press & Schechter (1974)**
- Want **FIRST** crossing to avoid cloud-in-cloud: $\delta_s > b(s)$ but $\delta_S < b(S)$ for $S < s$. Solved for Gaussian uncorrelated steps with constant/linear barrier
  Bond et al. (1991)
  Jedamzik (1995); Sheth(1998)
- May treat correlations as perturbations **Maggiore & Riotto (2010)**
  Corasaniti & Achitouv (2011)
- However: correlations make cloud-in-cloud less likely (less zig-zags)
  Paranjape, Lam & Sheth (2011)
- Can relax **FIRST** into **UPWARDS**: $\delta_s = b(s)$, $\delta'_s \equiv \mathrm{d}\delta/\mathrm{d}s \geq \mathrm{d}b/\mathrm{d}s$

$$\frac{M}{\bar\rho}\frac{\mathrm{d}n_h}{\mathrm{d}M} = \left|\frac{\mathrm{d}s}{\mathrm{d}M}\right| f_{\mathrm{up}}(s) = -\frac{\mathrm{d}s}{\mathrm{d}M}\int_{b'}^{\infty}\mathrm{d}\delta'_s(\delta'_s - b')\, p(\delta'_s, \delta = b)$$

MM & Sheth (2012)

Upcrossing distribution

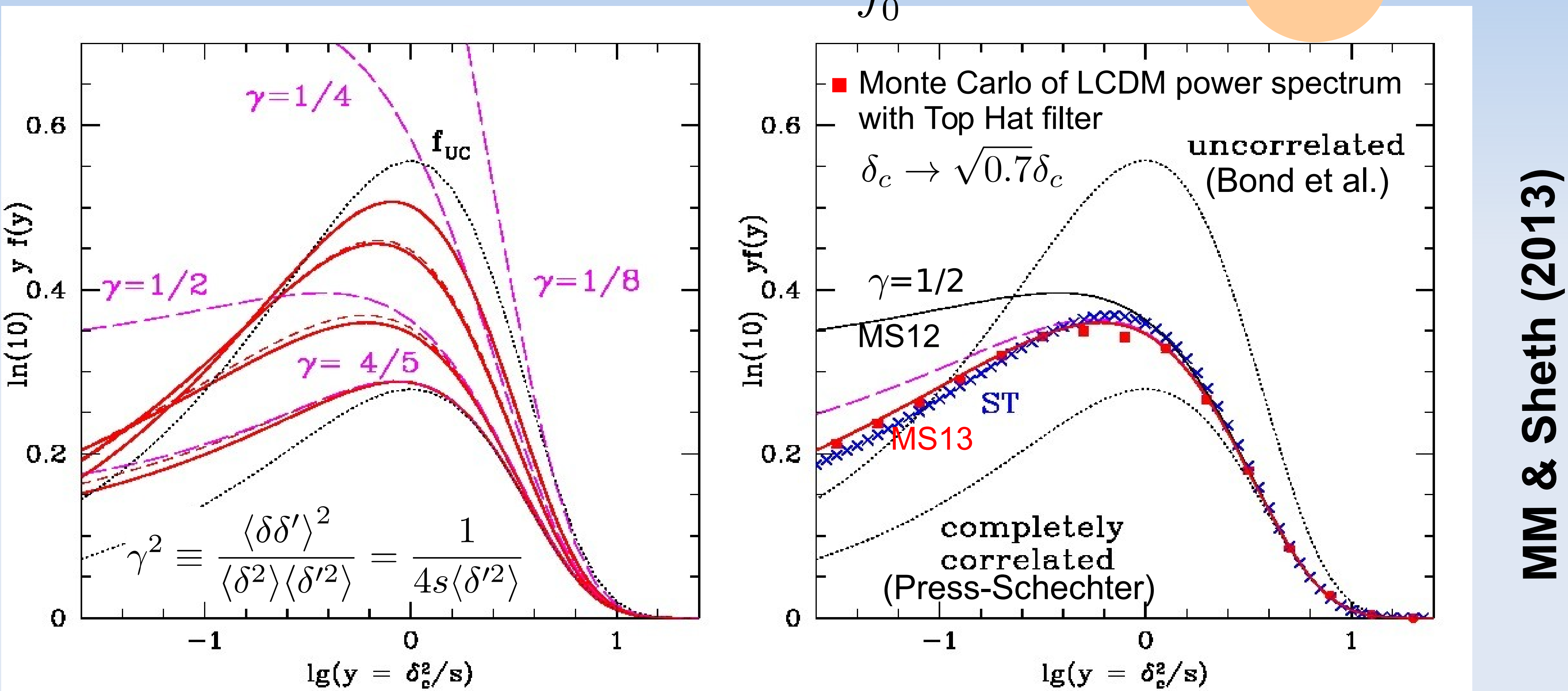
$$f_{\rm up}(s) = f_{\rm PS}(s)\left[\frac{1+{\rm erf}(X)}{2} + \frac{e^{-X/2}}{X\sqrt{2\pi}}\right], \qquad X^2 \equiv \frac{\gamma^2\delta_c^2}{(1-\gamma^2)s}$$



Check against exact first crossing of Monte Carlo walks (histograms) with various power spectra and generic barrier $b = \delta_c + \alpha s$. Dotted line is Bond et al.

Solution by back substitution

- An even better approx: $p(\delta_s \geq b) = \int_0^s \mathrm{d}S f(S) p(\delta_s \geq b|\mathrm{up}, S)$



- Upcrossing captures $f(s)$ for all $P(k)$, filters and barriers. Yet, the mass function works only if $\delta_c \rightarrow .84\ \delta_c$. There is a flaw in the ansatz!

Peak Theory (BBKS 86)

Halos as local maxima of the initial density field

Peak Theory (BBKS 86)

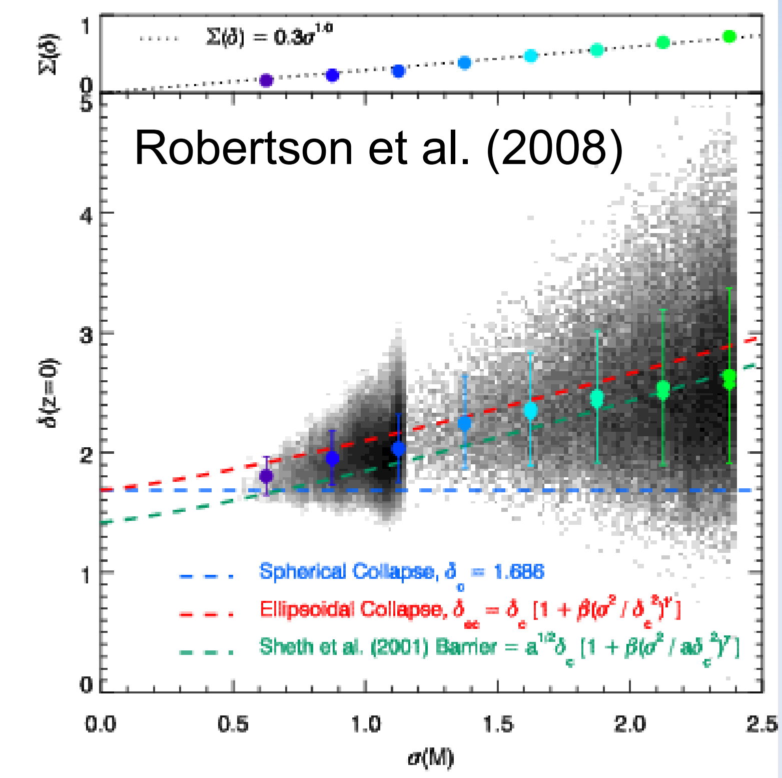
- Halos do not form at random locations, but near peaks of $\delta_R(\mathbf{r})$
- Critical height: $\nu \equiv \delta_R/\sigma_0 = \nu_c$
- Maxima: $\eta \equiv \nabla\delta_R/\sigma_1 = 0$, $\zeta \equiv -\nabla\nabla\delta_R/\sigma_2$ has eigenv. $\zeta_1 > \zeta_2 > \zeta_3 > 0$
- Volume element: $\mathrm{d}^3\eta = (\sigma_2/\sigma_1)^3 \, |\det(\zeta)| \, \mathrm{d}^3x$

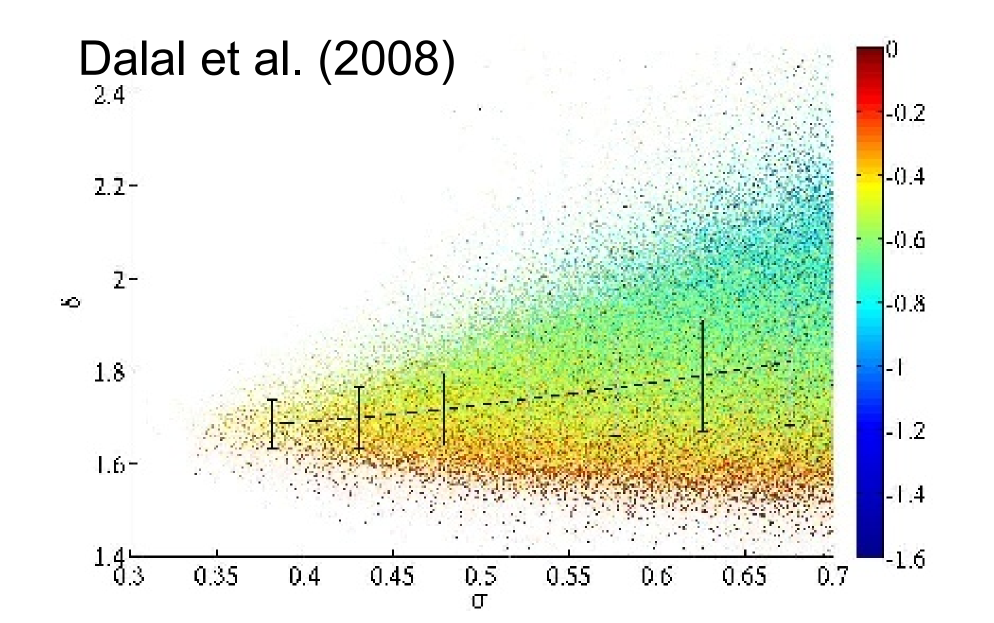
$$\frac{\mathrm{d}n_{\rm pk}}{\mathrm{d}\nu} = \frac{\sigma_2^3}{\sigma_1^3} \langle |\det(\zeta)| \theta(\zeta_3) \delta_{\rm D}(\nu - \nu_c) \delta_{\rm D}(\eta) \rangle$$

- Exact peak number density, but not precise for halo mass function
- Halos are not just critically high peaks either!
- Also, $\sigma_2 = \infty$ for Top-Hat filter. Need Gaussian smoothing

The critical density

- At small mass, barrier b becomes “stochastic” (other variables play a role, e.g. shear, shape, velocity dispersion) and scale-dependent





- Not just height of b : different types prefer different $\mathrm{d}b/\mathrm{d}s$

- At same mass, they select different $\mathrm{d}\delta/\mathrm{d}s$ (need a model!)

Peaks vs Excursion Sets

- Key ingredient for peaks: Jacobian $\det(\zeta)$ of the 3D mapping $\boldsymbol{\eta} \to \mathbf{r}$
- Excursion sets rely on the 1D mapping $\delta \to \sigma(M)$:

$$\delta_{\rm D}(\sigma - \sigma_c) = \frac{{\rm d}(\delta - b)}{{\rm d}\sigma}\delta_{\rm D}(\delta - b)$$

- The two can be combined: 4D mapping $(\delta,\boldsymbol{\eta}) \to (\sigma,\mathbf{r})$ with

$$\delta_{\rm D}^{(3)}(\mathbf{r} - \mathbf{r}_0)\delta_{\rm D}(\sigma - \sigma_c) = \frac{\sigma_2^3}{\sigma_1^3}|\det(\zeta)|\delta_{\rm D}^{(3)}(\eta)\frac{{\rm d}(\delta - b)}{{\rm d}\sigma}\delta_{\rm D}(\delta - b)$$

- Conventionally one uses normalized slope $x = \gamma\delta'$.
  For a moving barrier: $x \to x - \gamma\beta$, with $\beta \equiv {\rm d}b/{\rm d}\sigma$

Excursion Set Peaks

- Assume Gaussian filter, linear barrier $b = \delta_c + \beta\sigma_0$

$$f_{\rm ESP}(\sigma;\beta) = \frac{\nu_c p(\nu_c+\beta)}{\sigma}\int_{\gamma\beta}^{\infty}{\rm d}x\,\frac{x-\gamma\beta}{\gamma\nu_c}F(x)\,p(x|\nu_c)$$

Paranjape, Sheth (2012)

- The initial density from N-body simulations has a roughly log-normal scatter around a mean that is compatible with ellipsoidal collapse

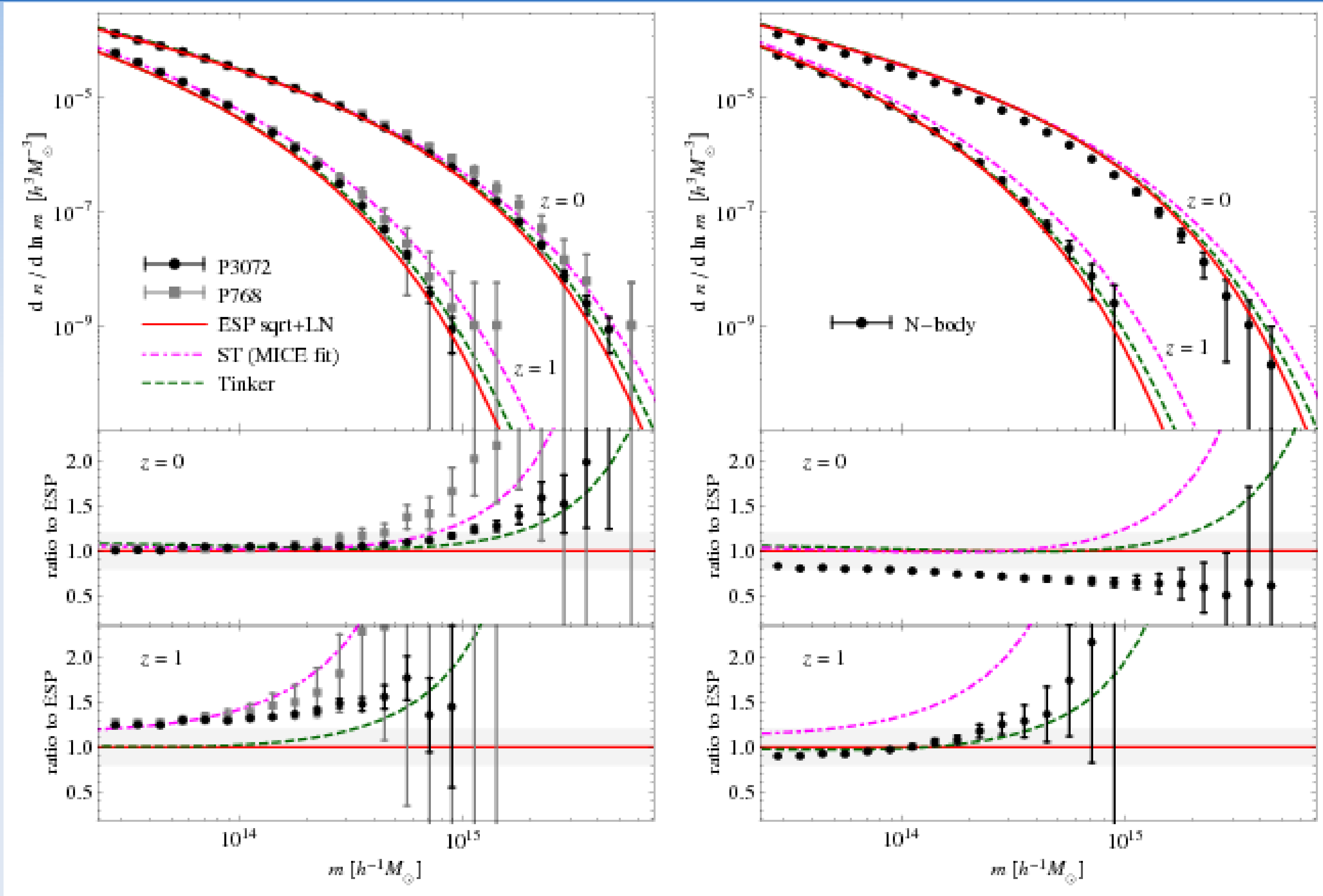
Robertson et al. (2008)

- Can choose β stochastic and log-normal

$$\frac{{\rm d}n_{\rm ESP}}{{\rm d}M} = \frac{{\rm d}\sigma}{{\rm d}M}\frac{1}{V_*}\int {\rm d}\beta\, f_{\rm ESP}(\sigma;\beta)\,p(\beta)$$

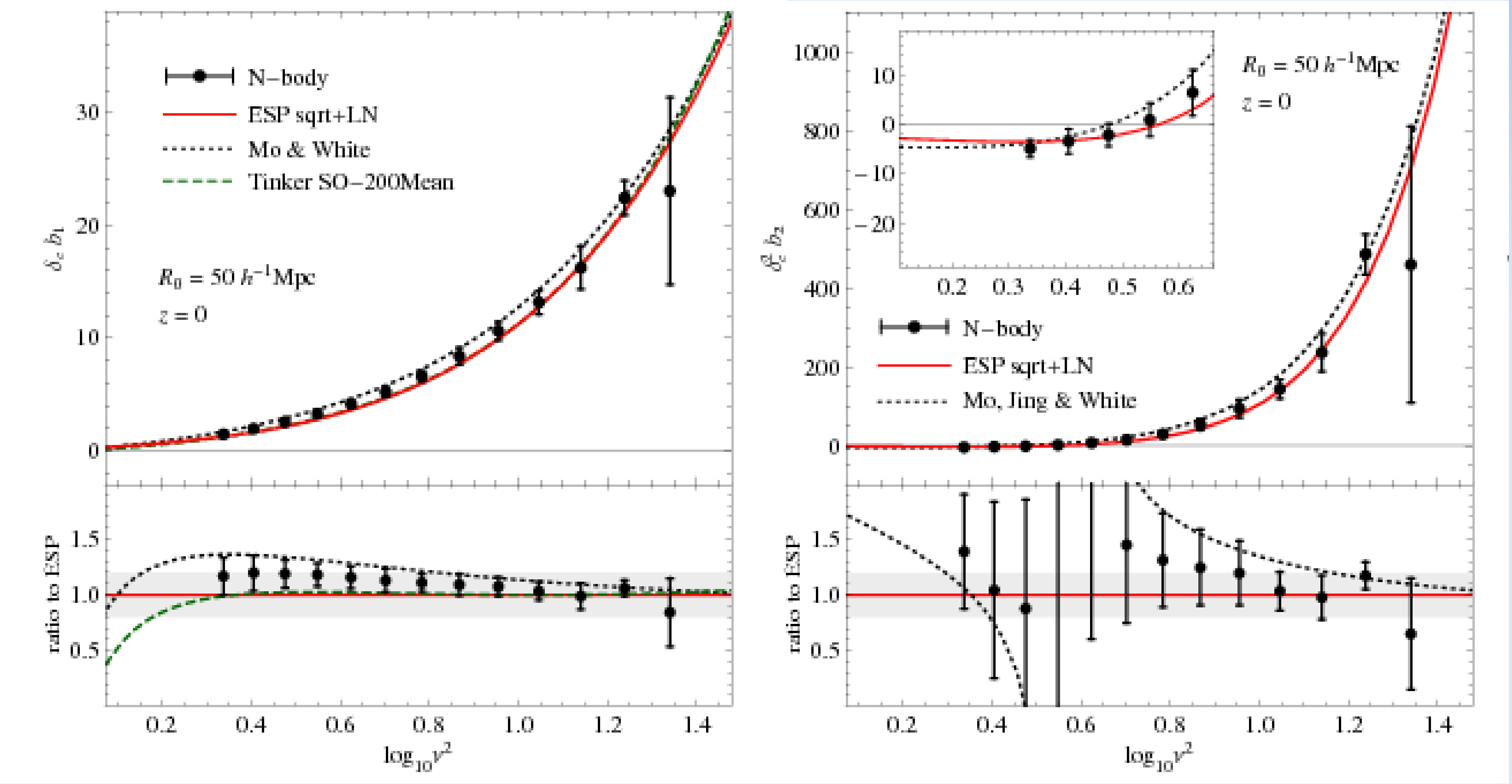
Paranjape, Sheth & Desjacques (2012)

Excursion Set Peaks



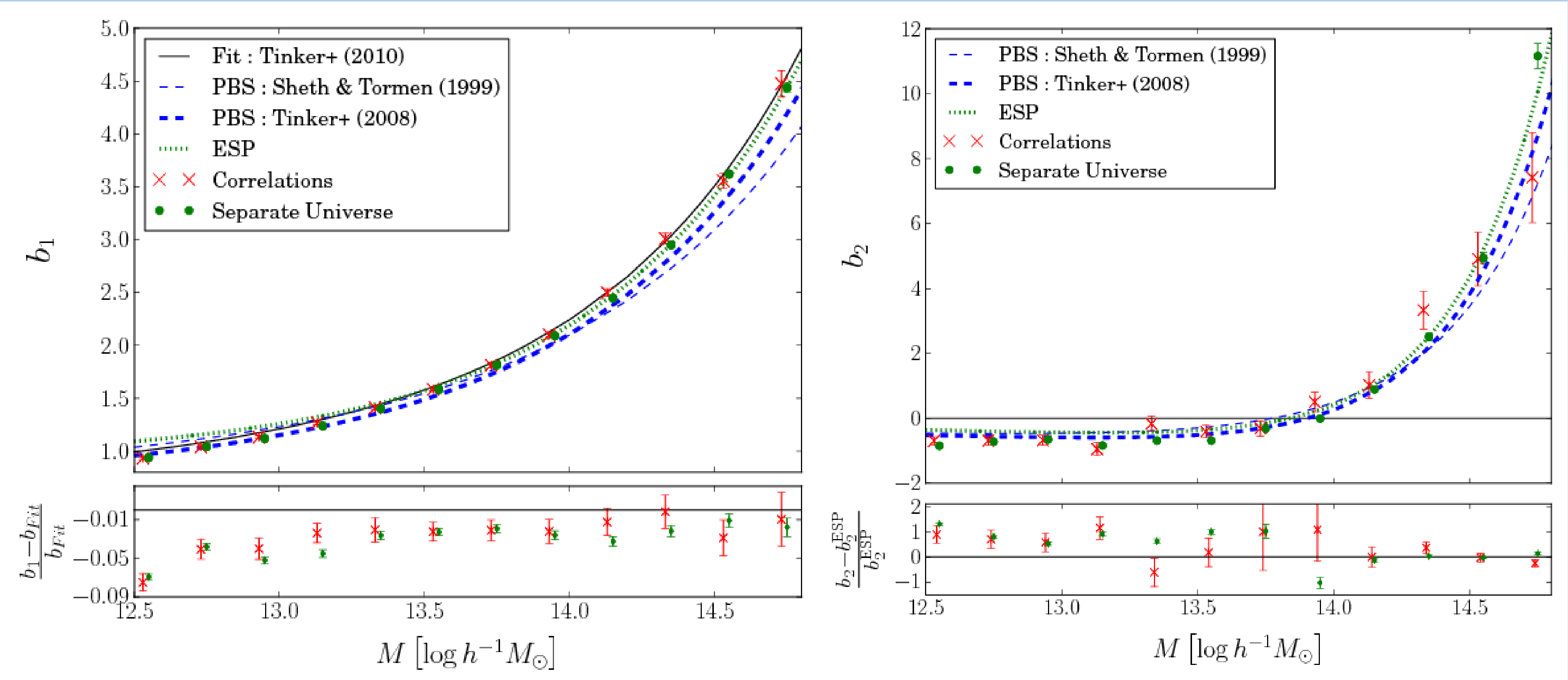
Paranjape et al (2013)

Excursion Set Peaks



Paranjape et al (2013)

Excursion Set Peaks



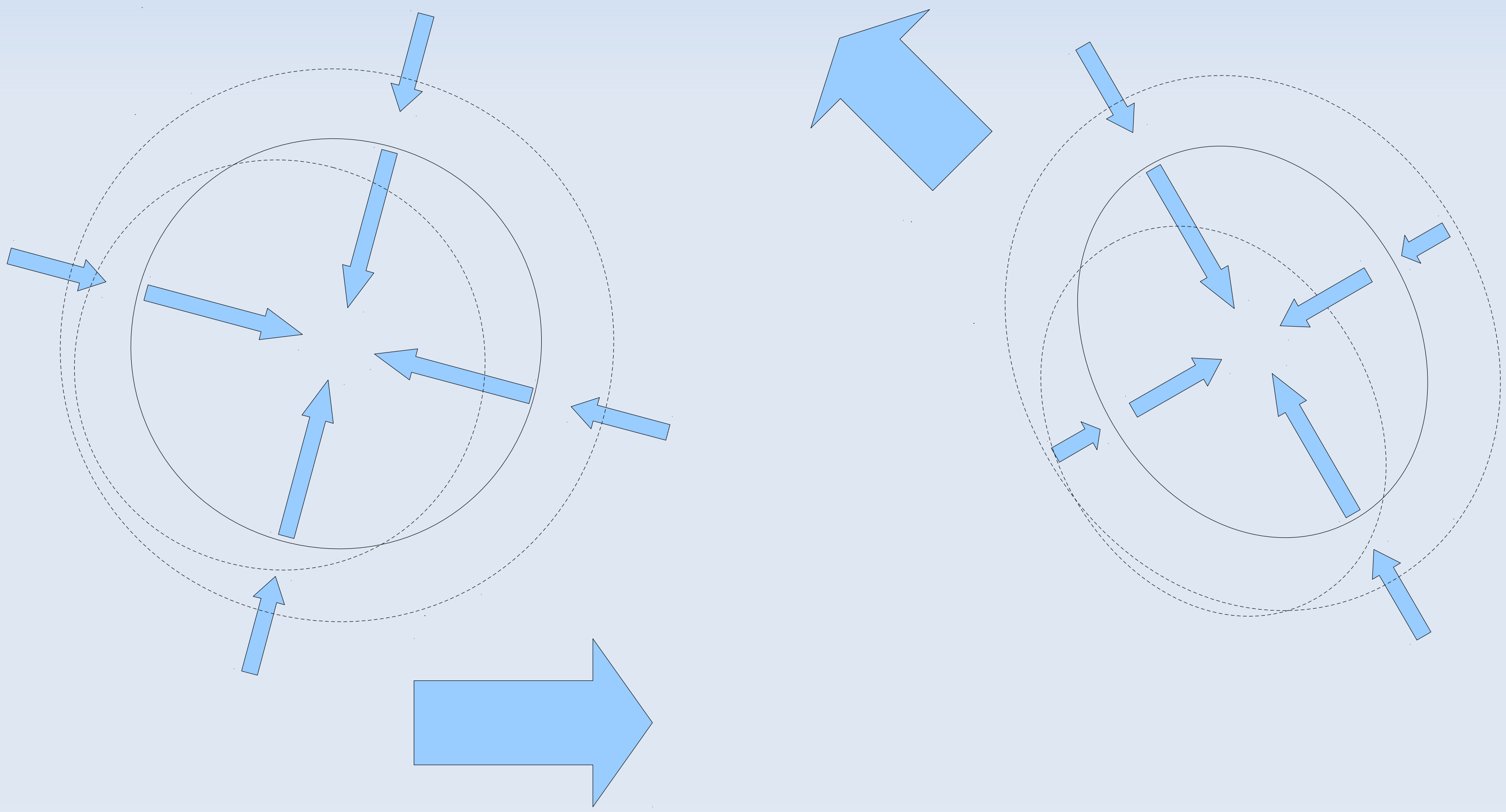
Lazeyras and Schmidt (2015)

ESP: the good, the bad, the ugly

- The good.
  It works well for mass function and bias.
  Successful prediction of bias is non-trivial
- The bad.
  At smaller mass, not all halos are peaks.
  Only upcrossing. First crossing would lower the low-mass tale
  The scatter of the threshold agrees with N-body only up to 30%.
  No information on the initial ellipticity of protohalos
- The ugly.
  Mixed Gaussian and Top Hat filtering with ad-hoc matching
  What is β? Shear, ellipticity? Connection with the physics blurred by degeneracies. No obvious way to improve

What's next?

Halos as centers of convergence of the velocity field



What's next?

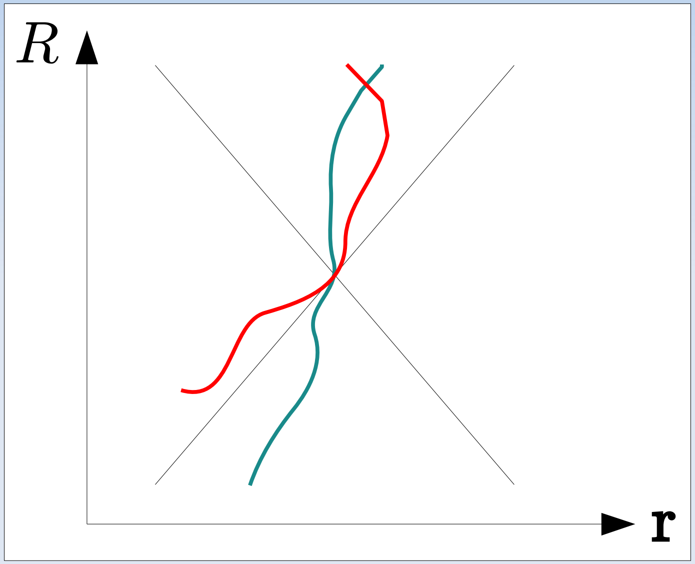
- Halos as convergence points of the acceleration field
- Identified by spheres with null dipole moment D_i. That is, set the origin of the coordinates on the center of mass.
- Replace $\nabla_i\delta = 0$ with $D_i = 0$, $\zeta_{ij} = -\nabla_i\nabla_j\delta$ with $-\nabla_i D_j$
- 4D density: $$\frac{\sigma_2^3}{\sigma_1^3}\left|\det\begin{bmatrix}\partial\delta/\partial R & \nabla_j\delta \\ \partial D_i/\partial R & \nabla_j D_i\end{bmatrix}\right|\delta_{\rm D}^{(3)}(\mathbf{D})\delta_{\rm D}(\delta - b)$$
- For TH filter: $$\zeta_{ij} \equiv -\nabla_i D_j = \int \frac{{\rm d}^3k}{(2\pi)^3}\frac{k_i k_j}{k^2}\delta(\mathbf{k})\frac{\partial W_{\rm TH}(kR)}{\partial R}$$
- Describes change of δ as any axis shrinks. Triaxial excursion sets!
- Infall from any direction must decrease with distance: pos def ζ_{ij} , like for peaks

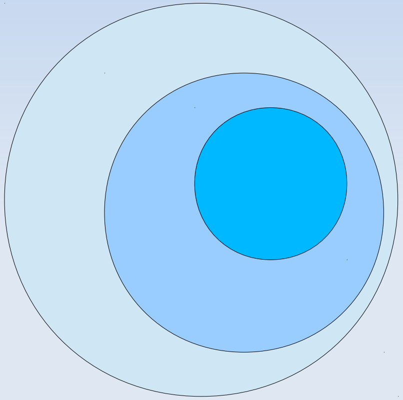
M.Musso (in prep)

What's next?

- Non-concentric ES trajectories following the $D_i = 0$ condition

$$\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}R}\right| < 1$$





- Upcrossing ruled by convective derivative along the trajectory

$$\frac{\mathrm{d}\delta}{\mathrm{d}R} = \frac{\partial\delta}{\partial R} + \frac{\mathrm{d}r_i}{\mathrm{d}R}\nabla_i\delta \qquad 0 = \frac{\partial D_i}{\partial R} + \frac{\mathrm{d}r_j}{\mathrm{d}R}\nabla_j D_i$$

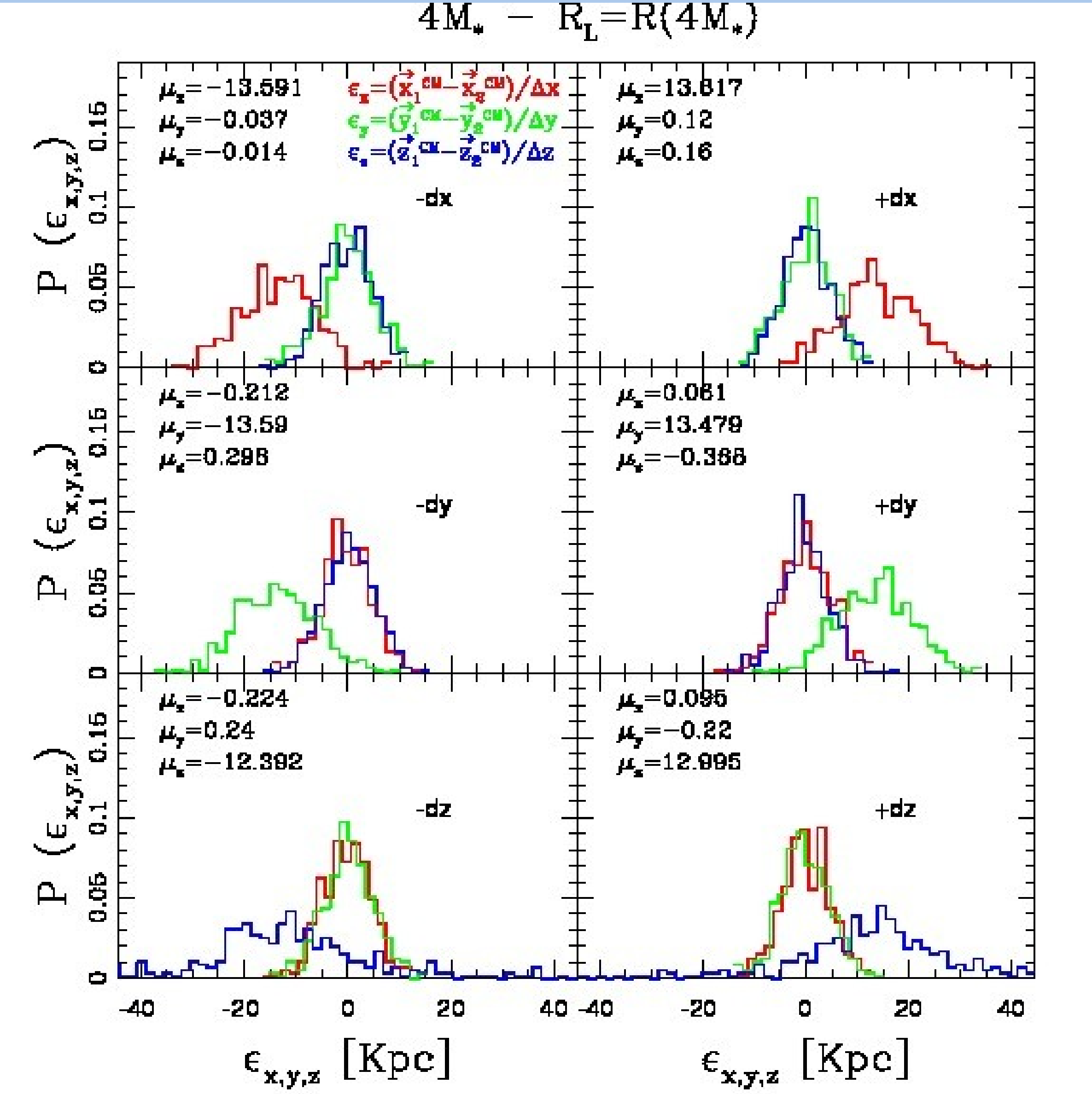
- Limit on $\mathrm{d}r_i/\mathrm{d}R \propto \nabla_i\delta$. Add nonlocal info on nearby halos!

M.Musso (in prep)

What's next?

- The center of mass of a sphere of Lagrangian radius near the center of mass of the protohalo moves in the direction opposite to the displacement
- $D_i = 0$ at the center of mass of the protohalo
- $\nabla_i D_j$ is indeed neg. definite

P.Gasparotto and MM (in prep.)



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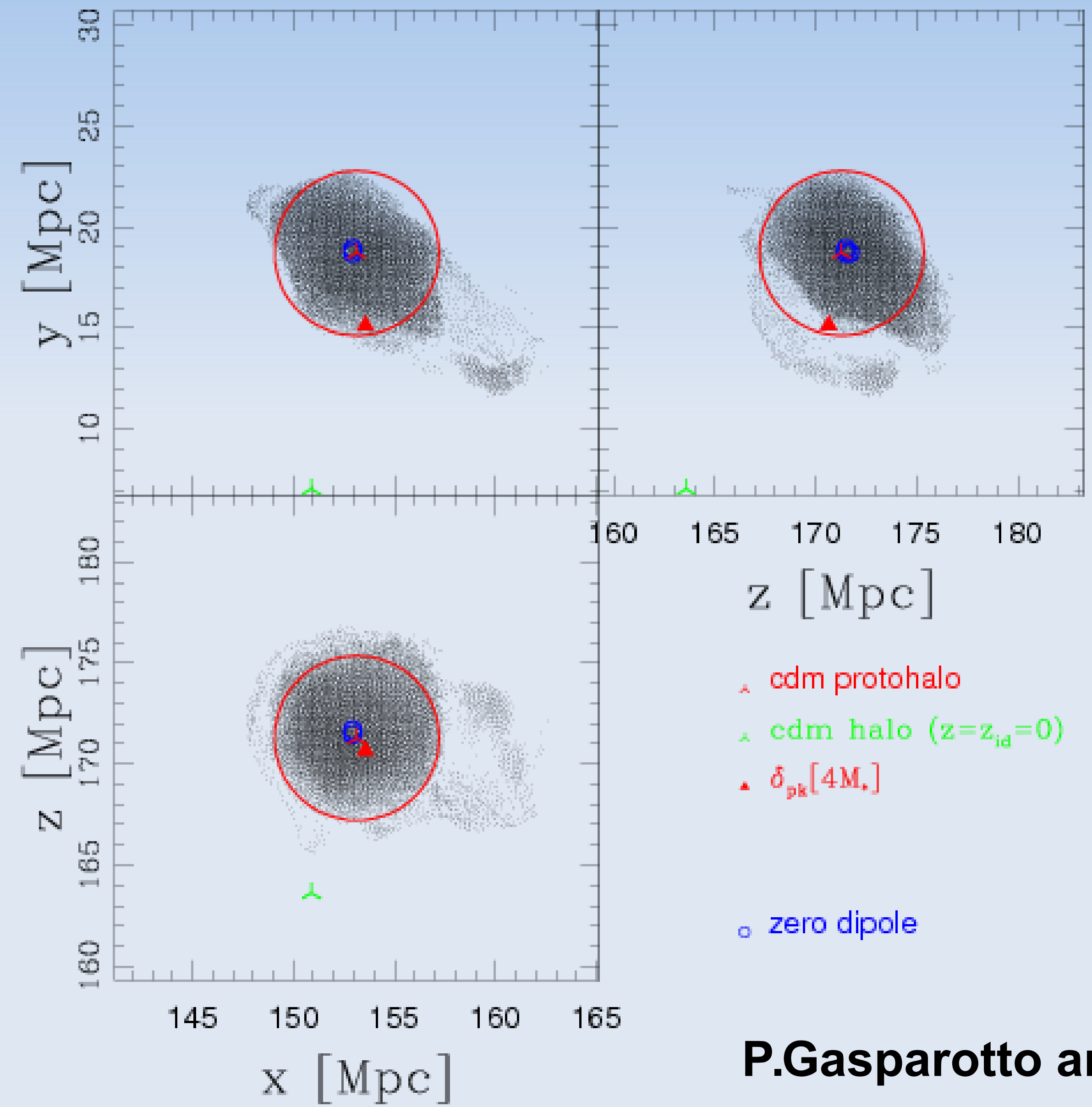

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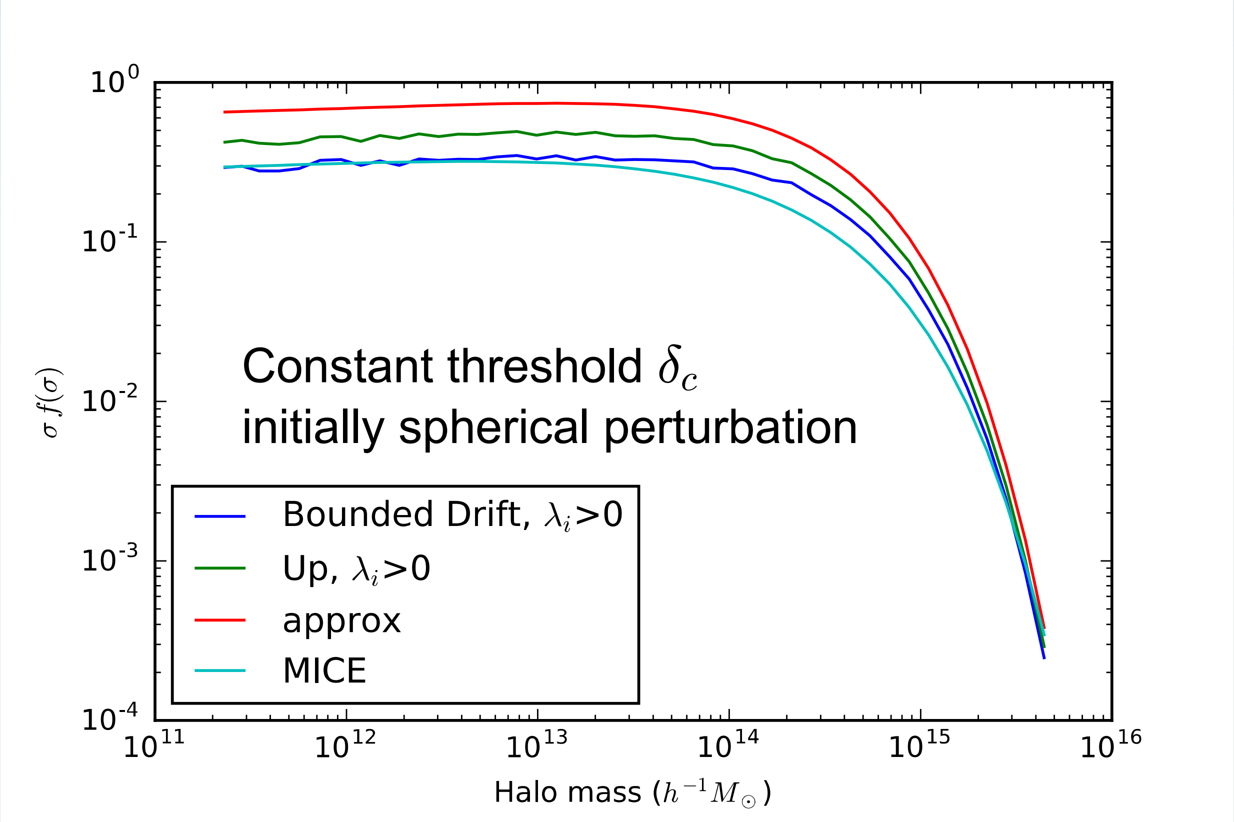
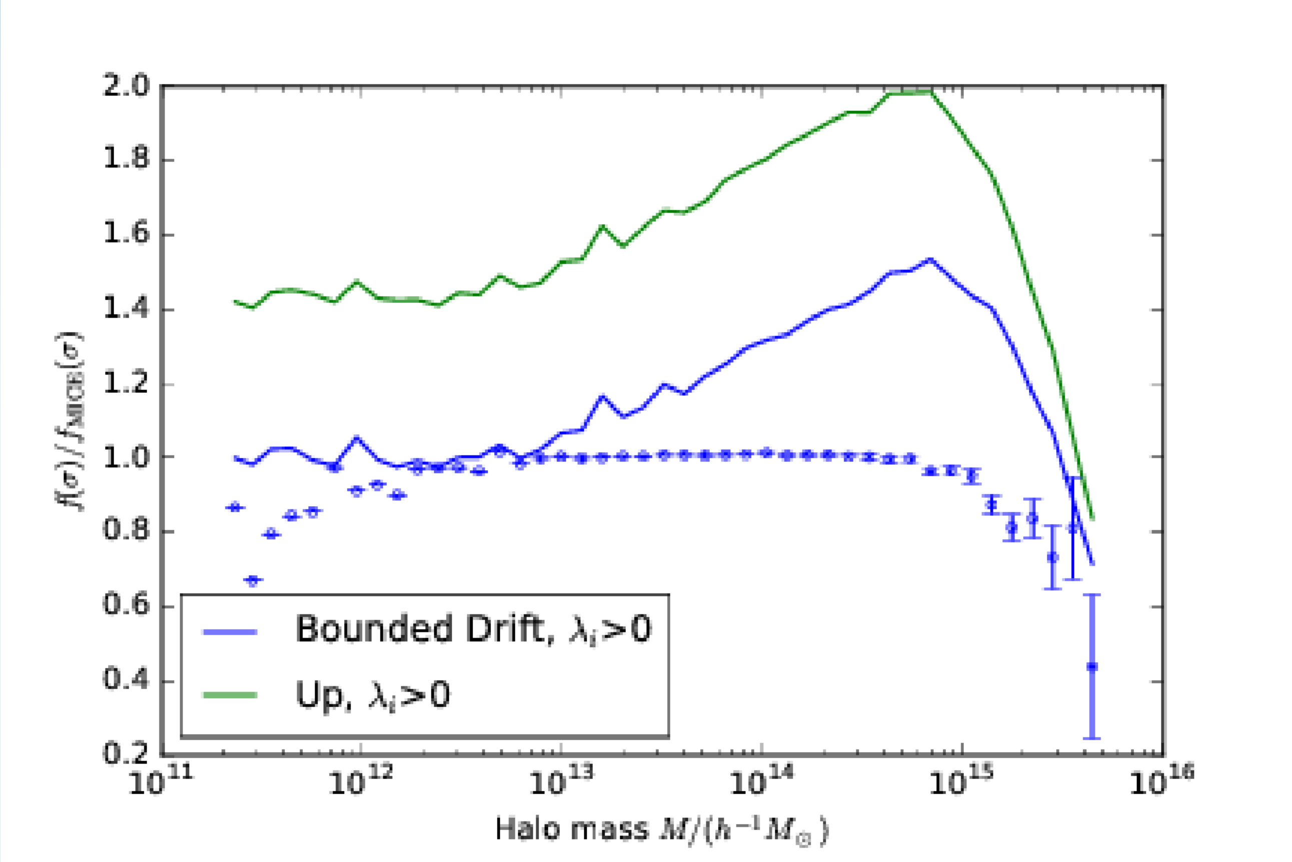


- Discrepancy will be reduced by introducing initial ellipticity as a response to shear. Hopefully by the right amount...

M.Musso (in prep)

What's next?



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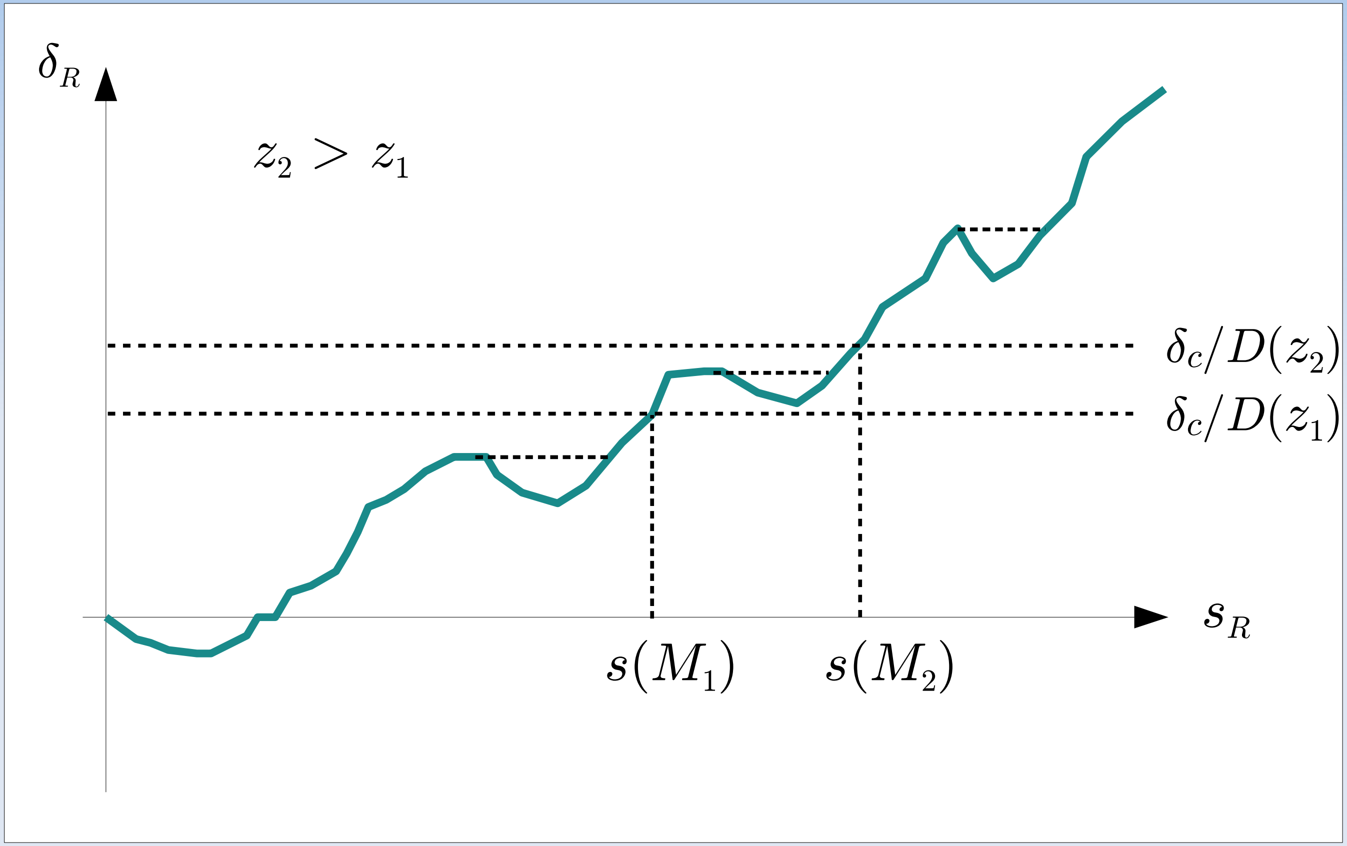
M.Musso (in prep)

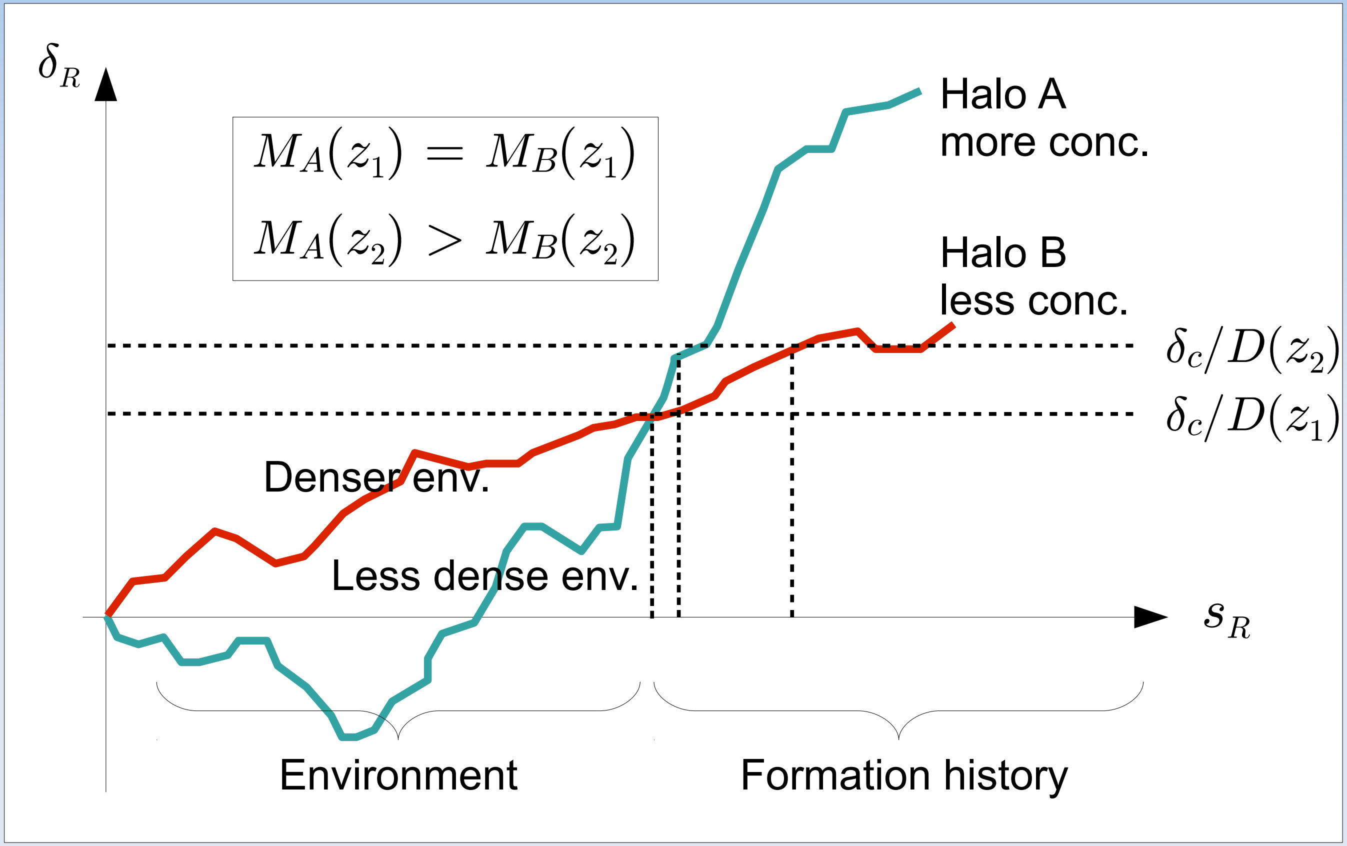
Formation history



- As threshold drops with z, first crossing moves/jumps to larger M
- Continuous growth of M is accretion, finite jumps are mergers. Can get whole formation history $M(z)$ from the initial density profile

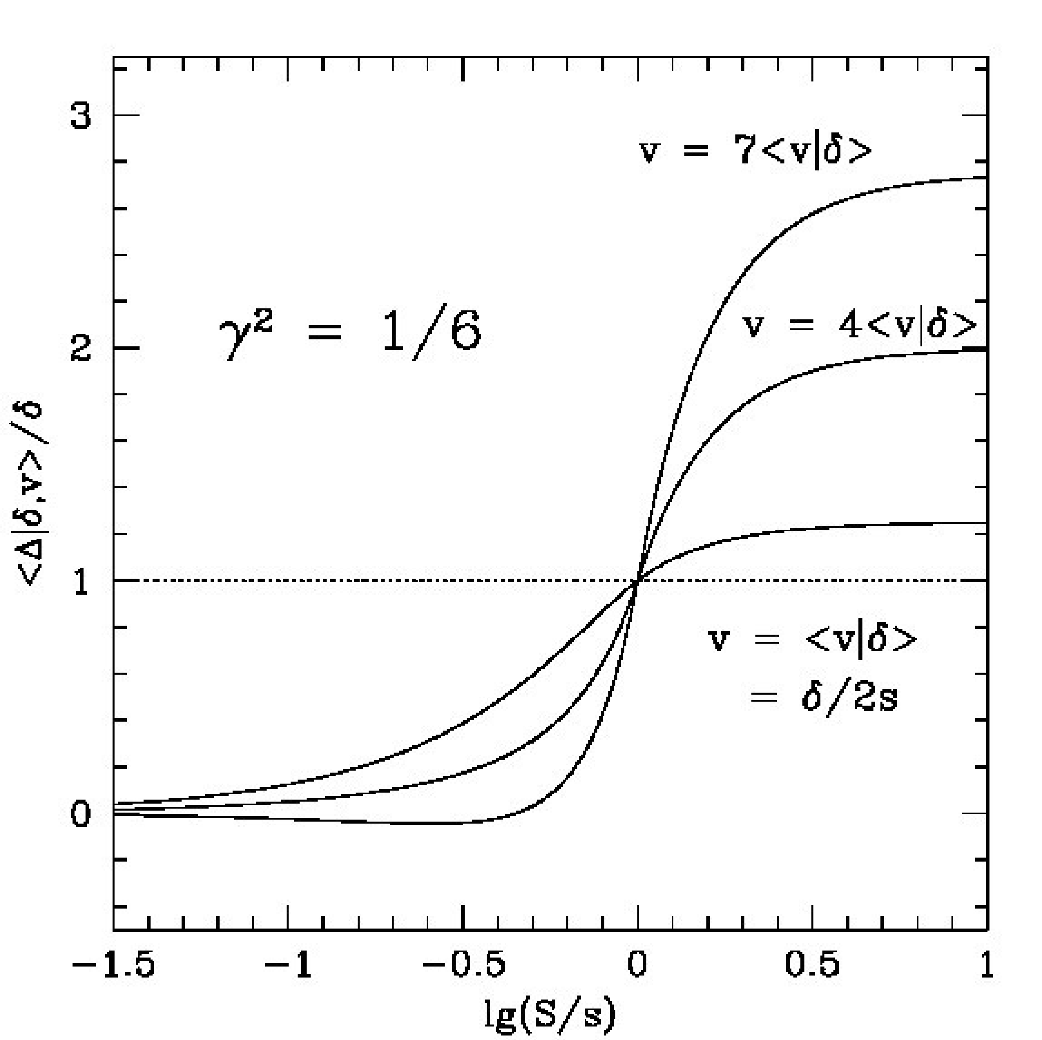
Lacey and Cole (1993)

Assembly bias



- Same mass at z_1, but A has more mass than B at $z_2 > z_1$ as it crossed the higher threshold earlier: more concentrated, slower accretion
- But sharp turns are unlikely: B prefers denser large-scale environment than A (not so for uncorrelated steps). Assembly bias!

Assembly bias



Conditional mean value of $\delta(S)$ for given $\delta(s)$ and $\mathrm{d}\delta/\mathrm{d}s$. A steeper slope favors lower density environment, higher concentration and earlier formation time.
Only at fixed mass AND slope, mass accretion history and large-scale environment become uncorrelated

MM & Sheth (2014)
Dalal et al. (2008)

"Markov velocity"

- For Top Hat filter, δ is not Markovian

$$p(\delta(s)|\delta(S),\mathrm{history}) \neq p(\delta(s)|\delta(S))$$

- But, for LCDM spectrum, the slope $\mathrm{d}\delta/\mathrm{d}R$ nearly is:

$$\frac{\mathrm{d}^2\delta}{\mathrm{d}R^2} - \frac{1}{R}\frac{\mathrm{d}\delta}{\mathrm{d}R} = \eta \qquad \langle\eta(s)\eta(S)\rangle \sim \delta_{\mathrm{D}}(s-S)$$

  Same divergence as the variance of the Top Hat peak curvature!
  Decay of space correlations with distance

- Once δ and $\mathrm{d}\delta/\mathrm{d}R$ are given, further memory is erased:

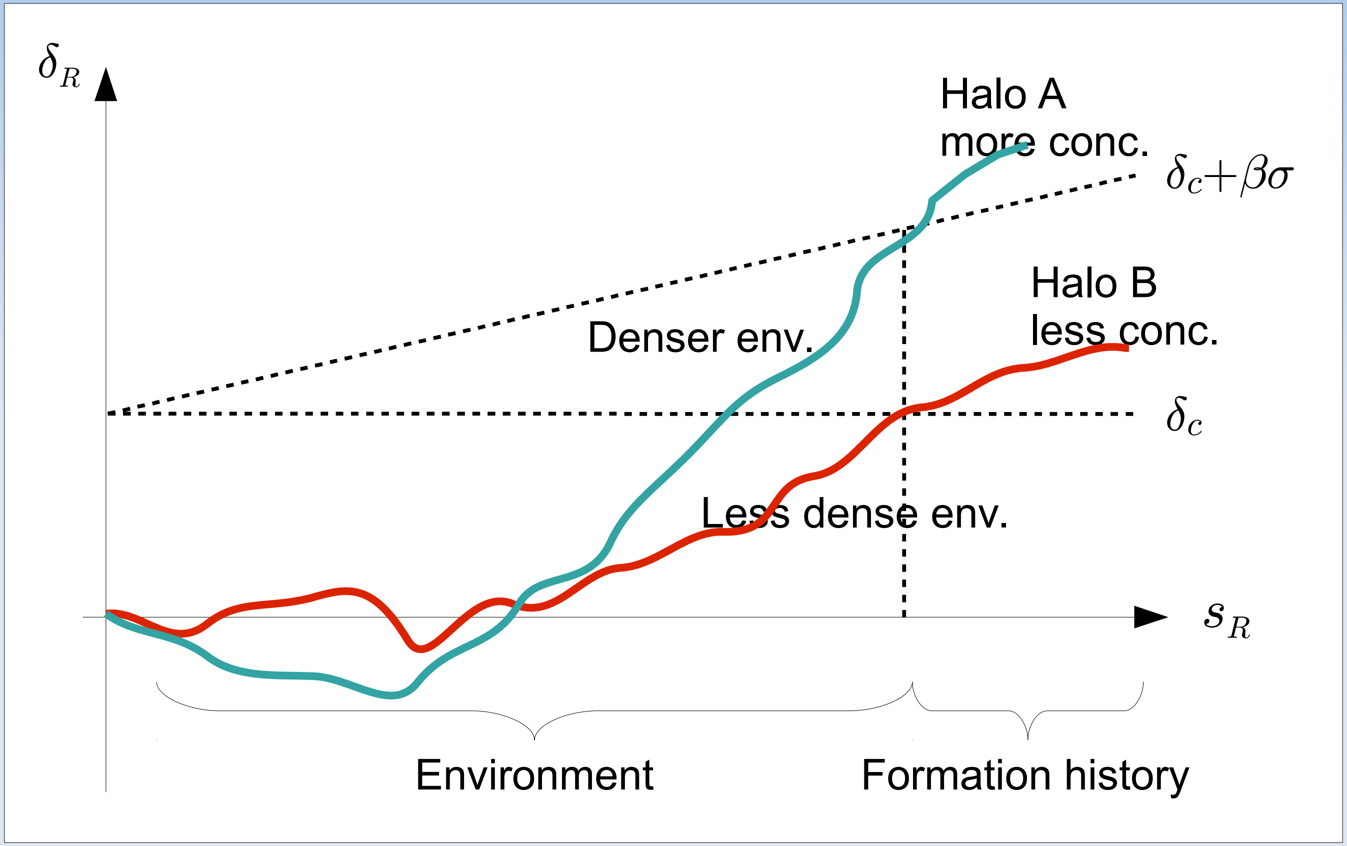
$$n_h(m|\delta(S),\delta(S_1),\delta(S_2),\dots) \simeq n_h(m|\delta(S),\mathrm{d}\delta/\mathrm{d}S)$$

  At fixed slope, formation history or clustering properties do not depend on the environment!

- Easy merger trees

MM & Sheth (2014)

More assembly bias



- At smaller mass, scatter in b becomes significant. Larger β selects steeper slopes (low accretion, high c) but also denser environment
- Reversed assembly bias because of β. But what is β ??

Conclusions

- Approximate analytical models still capture key statistical properties. The “cloud-in-cloud” problem is solved in full generality (upcrossing)
- Combining ES and peaks, the agreement with N-body is very good at large masses
- Low-mass behavior likely improved by dynamical models based on the convergence of mass flows (e.g. null dipoles)
- Need to include initial ellipticity as a response to shear
- Correlation between formation history and environment gives simple models of assembly bias.
- Can predict merger trees or reionization bubbles

Thanks!!

Large scale bias

- Parametrize the dependence of halo field on large-scale matter field:

$$n_h(\mathbf{x}) = \sum_k \frac{c_k}{k!} H_k(\delta/\sigma)$$

- Expansion in orthogonal (Hermite) polynomials, not Taylor series
- The coefficients are:

$$c_k = \langle n_h(\mathbf{x}) H_k(\delta/\sigma) \rangle = \sigma^k \left\langle \frac{\partial n_h}{\partial \delta} \right\rangle$$

- Mathematically:

$$\frac{c_k}{\sigma^k} = (-1)^k \frac{\partial^k \langle n_h \rangle}{\partial^k \langle \delta \rangle}\bigg|_{\langle \delta \rangle = 0} = \frac{\sigma_0^k \langle n_h(\mathbf{x}) H_k(\delta_0/\sigma_0) \rangle}{\langle \delta \delta_0 \rangle^k}$$

- Two different strategies to measure them in simulations: “separate Universe”, or halo by halo

Large scale bias

- Realistic models involve additional variables, which need not be scalars. For instance, ESP has $\eta_i \equiv \nabla_i\delta$ and $\zeta_{ij} \equiv -\nabla_i\nabla_j\delta$
- Expansion in derivatives of the field, inducing scale dependent bias
- Only rotational invariant combinations η^2, $\mathrm{tr}(\zeta)$, $\mathrm{tr}(\zeta^2)$, $\det(\zeta)$ are relevant. But they are no longer Gaussian variables
- Need to find the appropriate orthogonal basis to expand n_h. This is a suitable, non-trivial combination of Hermite, Laguerre and Legendre

$$H_{ij}(\nu,(\zeta))L_k^{(1/2)}(3\eta^2/2)F_{lm}(\mathrm{tr}(\bar{\zeta}^2),\mathrm{tr}(\bar{\zeta}^3))$$

Lazeyras, MM & Desjacques (2015)

- Can now compute all the scale dependent coefficients, and measure them by cross-correlating halos with these orthogonal polynomials!

A self-consistent model?

- Accretion of shells on their center of mass governed by gravitational potential. Do a simple multipole expansion:

$$\phi_{\rm int}(\mathbf{r}) = -\frac{GM}{r} - \frac{G r_i r_j \bar{Q}_{ij}}{r^5} + \dots \qquad [\mathbf{r} \equiv \mathbf{x} - \mathbf{x}_{CM}]$$

  First term gives spherical collapse.

- Center-of-mass motion and tidal torque from

$$\phi_{\rm ext}(\mathbf{r}) = \phi_0 + P_i r_i + \frac{1}{2}\bar{\lambda}_{ij} r_i r_j + \dots$$

  May be treated with your favorite PT (except during mergers)

- Traceless quadrupole Q_{ij} and shear λ_{ij} give ellipsoidal collapse
- Evolution described by the virial equations. Are higher orders small?