Some instances of broken symmetry in categorical algebra

Tim Van der Linden

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Understanding group cohomology via categorical algebra

In my work I develop and apply categorical algebra
in its interactions with homology theory.

- Concrete aim: understanding (co)homology of groups
- Which aspects of group cohomology are typical for groups, and which function for more general reasons, so that a categorical argument suffices to understand and apply these in other settings?
- Conversely, what do the needs of homological algebra tell us about categories of non-abelian algebraic structures?

Today's subject: I would like to

- sketch how our work environment arises out of a broken symmetry
- give an idea of how the broken symmetry between homology and cohomology “may be fixed”

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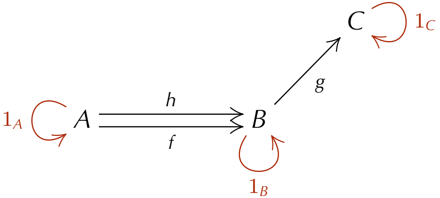
[Eilenberg-MacLane, 1945]



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vertices called **objects** and edges called **morphisms** or **arrows**, having an associative **composition** of arrows, and loops which act as **identities**.

Examples

- an ordered set $(S, \leqslant)$, where $x \leqslant y$ determines an arrow $x \to y$

  here any two parallel arrows are equal

- **concrete** categories: structured sets & structure-preserving functions

  Set (sets & functions); *Top* (topological spaces & continuous maps); *Ab* and *Gp* ((abelian) groups & homomorphisms); $Vect_{\mathbb{K}}$ (vector spaces & linear maps, $\mathbb{K}$ is a field); $Lie_{\mathbb{K}}$ ($\mathbb{K}$-Lie algebras & Lie algebra morphisms); Mod_R (R-modules and linear maps, R is a ring)

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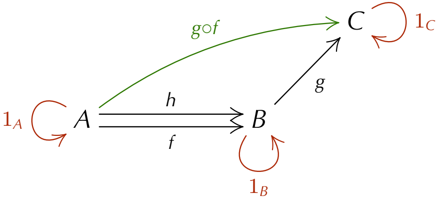

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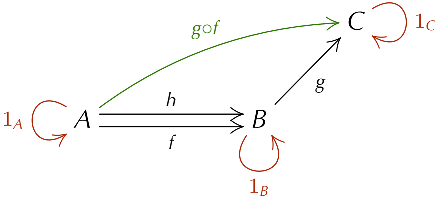

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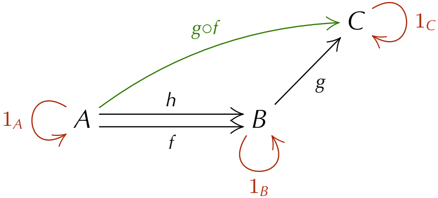

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Some universal properties/constructions

Category theory deals with objects "from the outside" via their interactions, through universal properties and constructions.

For instance, a **terminal object** is an object 1 such that for every object X there exists a unique arrow $X \to 1$.

- In an ordered set $(S, \leqslant)$ viewed as a category, a terminal object is the same thing as a maximum.
- In *Set*, an object is terminal if and only if it is a singleton set.
- In *Top*, *Ab*, *Gp*, $\mathit{Vect}_{\mathbb{K}}$, Mod_R and $\mathit{Lie}_{\mathbb{K}}$, an object is terminal if and only if its underlying set is a singleton.

This is the simplest example of a universal property, but it is relevant in what follows.

Another example is that of a **monomorphism** $m \colon M \to A$: for every pair of parallel arrows $f, g \colon X \to M$, if $m \circ f = m \circ g$, then $f = g$.

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A **monoid** is a set M equipped with an associative multiplication $\cdot$ which admits a unit e_M. (So, a monoid is a one-object category.)
A **group** is a monoid whose elements are invertible.

In the category *Mon* of monoids and monoid homomorphisms, the groups may be characterised as follows: B is a group iff for all

$$0 \longrightarrow X \overset{k}{\rightarrowtail} A \underset{f}{\overset{s}{\leftrightarrows}} B, \qquad k = \ker(f), \qquad f \circ s = 1_B$$

k and s are **jointly strongly epimorphic**, which means that $k(X)$ and $s(B)$ together generate A, or more precisely that k and s do not both factor though a monomorphism $m\colon M \to A$, unless m is an isomorphism.

- Basic idea: $a = sf(a) \cdot \big(s(f(a)^{-1}) \cdot a\big)$
- All concepts here are categorical:
  they make sense outside the context of groups and monoids.
- Recent work [Montoli-Rodelo-VdL, 2017] [García, 2017].
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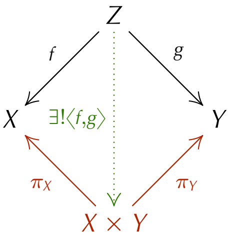
Another example is that of a **product** $(X \times Y, \pi_X, \pi_Y)$ of objects X and Y, which is such that any pair of arrows (f, g) as in



factors uniquely through the pair (π_X, π_Y).

In other words, it is terminal amongst pairs of arrows $X \leftarrow Z \rightarrow Y$.

- In an ordered set $(S, \leqslant)$ viewed as a category, the product of two elements x and y is $x \wedge y = \min\{x, y\}$.
  (Indeed, $z \leqslant x \wedge y$ iff $z \leqslant x$ and $z \leqslant y$. $x \wedge y$ is the largest such z.)
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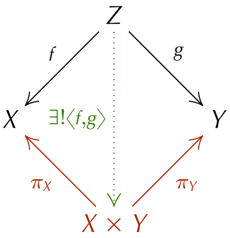

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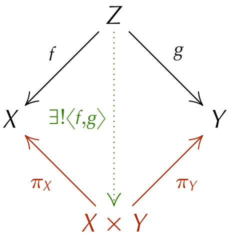

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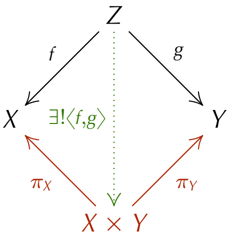

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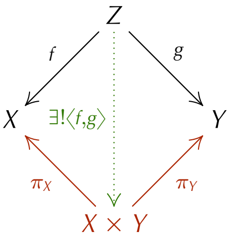

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Duality: the basic type of symmetry in category theory

If the arrows in a category $\mathbb{X}$ are reversed
then we find a new, "**opposite**" category $\mathbb{X}^{\text{op}}$.

- Sometimes this opposite is known: algebraic geometers understand that $(\text{affine schemes})^{\text{op}} \simeq (\text{commutative rings})$, for instance.

Any categorical concept has a **dual**,
which is this concept, considered in the opposite category.

- The dual of a terminal object 1 is an **initial** object 0: $\forall X\ \exists!(X \to 1)$
  For instance, $\varnothing$ in *Set*, the one-element algebra in *Gp*, *Ab*, $Vect_{\mathbb{K}}$, etc.
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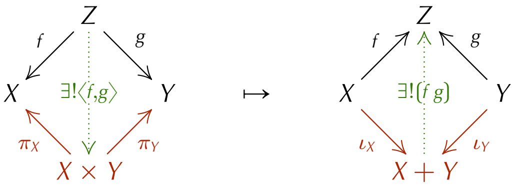

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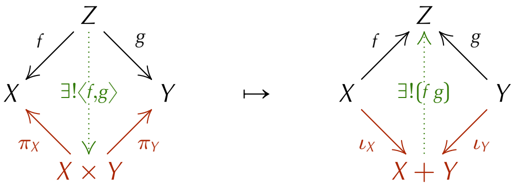

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Certain concepts are invariant under duality

A **zero object** is an object which is both initial and terminal: $0 = 1$.
The concept of a zero object is **self-dual**, so *invariant under duality*: in the opposite category, it will still be zero.

A **pointed category** is a category with a zero object.

- *Mon*, *Ab*, *Gp*, $\mathit{Vect}_{\mathbb{K}}$, Mod_R, $\mathit{Lie}_{\mathbb{K}}$ are pointed, while *Set* and *Top* are not.
- If $(S, \leqslant)$ has a zero object, then S is a singleton.

A **biproduct** is a diagram

$$X \underset{\pi_X}{\overset{\iota_X}{\rightleftarrows}} X \oplus Y \underset{\pi_Y}{\overset{\iota_Y}{\leftrightarrows}} Y,$$

where $(X \oplus Y, \pi_X, \pi_Y)$ is a product and $(X \oplus Y, \iota_X, \iota_Y)$ is a coproduct.
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Abelian and non-abelian categories, first symmetry break

An **abelian group** a commutative group: $x \cdot y = y \cdot x$.

Ab is an **abelian category**: it is

- finitely (co)complete: universal constructions exist, and
- pointed, and such that
- every monomorphism is a kernel, every epimorphism is a cokernel.

This axiom set is self-dual. Abelian categories have biproducts.

- Framework for homological algebra, algebraic geometry etc. [Buchsbaum, 1955; Grothendieck, 1957; Yoneda, 1960; Freyd, 1964]
- Examples: Mod_R (*Ab* and $Vect_{\mathbb{K}}$), sheaves of abelian groups.

Removing commutativity breaks the categorical symmetry:
no longer self-dual, the situation becomes radically different.

- Free products of groups are non-cartesian $\Rightarrow$ no biproducts; furthermore, non-normal subgroups exist.

This is where our work starts:

- to extend the framework to include non-abelian categories such as *Gp*, $Lie_{\mathbb{K}}$, $Alg_{\mathbb{K}}$, *XMod*, *Loop*, $HopfAlg_{\mathbb{K},coc}$, C^*-*Alg*; and
- to develop a unified homology theory.

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Abelian and non-abelian categories, first symmetry break

An **abelian group** a commutative group: $x \cdot y = y \cdot x$.

Ab is an **abelian category**: it is

- finitely (co)complete: universal constructions exist, and
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This axiom set is self-dual. Abelian categories have biproducts.

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Aim: extend basic group (co)homology to "all those" categories.

- When is a variety of algebras "sufficiently close" to *Gp*?
- How to capture homological properties of *Gp* categorically?

Answer: [Janelidze-Márki-Tholen, 2002; Borceux-Bourn, 2004]

A variety of algebras is **semi-abelian** iff it is pointed and **protomodular**: for all

$$0 \longrightarrow X \overset{k}{\rightarrowtriangle} A \underset{f}{\overset{s}{\leftrightarrows}} B, \qquad k = \ker(f), \qquad f \circ s = 1_B$$

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- This is the condition that distinguishes groups amongst monoids.
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Homology and cohomology, second symmetry break

Theorem [Hopf, 1942; Brown-Ellis, 1988; Donadze-Inassaridze-Porter, 2005] [Everaert-Gran-VdL, 2008]

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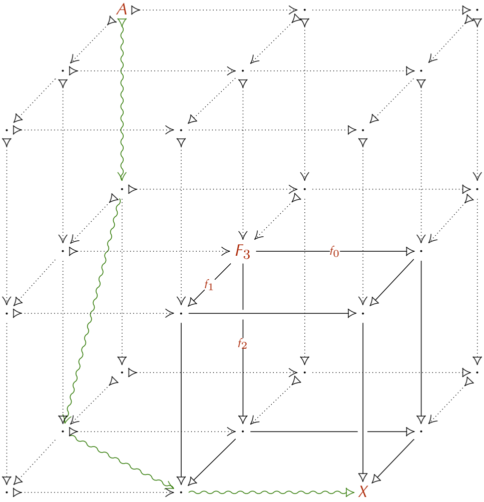
Cohomology: abelian vs. semi-abelian



Theorem [Yoneda, 1960] [Rodelo-VdL, 2016]

If X is an object, and A an abelian object, in $\mathbb{X}$ that satisfies (SH), then

$$H^{n+1}(X, A) \cong CExt^n(X, A).$$

The dual space

A simple example of duality is the *dual vector space* construction:

$$\begin{aligned}(-)^*\colon \mathit{Vect}_{\mathbb{K}} &\to \mathit{Vect}_{\mathbb{K}}^{\mathrm{op}}\colon\\ V &\mapsto V^* = \mathit{Hom}(V,\mathbb{K})\\ (f\colon V\to W) &\mapsto (f^* = (-)\circ f\colon W^*\to V^*)\end{aligned}$$

If V is finite-dimensional then $V^{**} \cong V$, but in general not.

The relationship between homology and cohomology of groups (with trivial coefficients) may be simplified by viewing it this way:

Theorem [Peschke-VdL, 2016]

Let G be a group and $n \geqslant 1$. Then for $ab\colon Gp \to Ab\colon X \mapsto X/[X,X]$,

$$H_{n+1}(G, ab) \cong \mathit{Hom}(H^{n+1}(G,-), 1_{Ab}).$$

- So here 1_{Ab} acts as some kind of a dualising object.
- This is a consequence of a *non-additive derived Yoneda lemma*.

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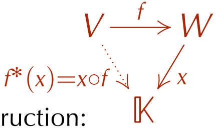

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- In the present case, it allowed us to simplify aspects of a classical theory—group (co)homology— from a new perspective and find new results.
- Since we eliminate those arguments that depend on $\mathbb{X} = Gp$, such results are automatically true for many algebraic categories $\mathbb{X}$.
- We single out conditions on $\mathbb{X}$ that “bring it closer” to Gp or $Lie_{\mathbb{K}}$.
- My personal project of “understanding cohomology of groups” is, in a first approach, almost complete.
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