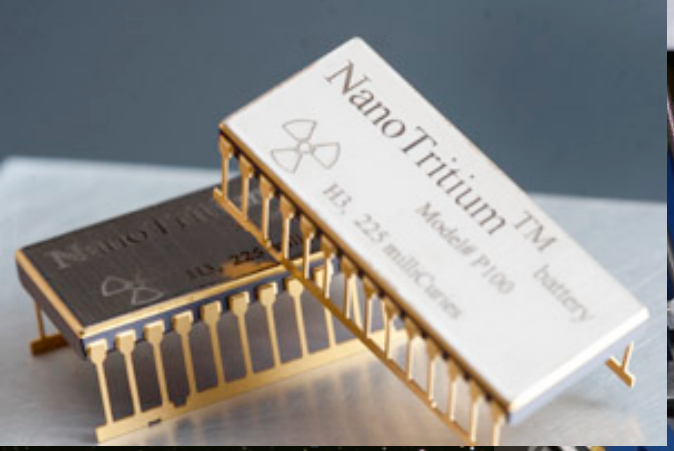
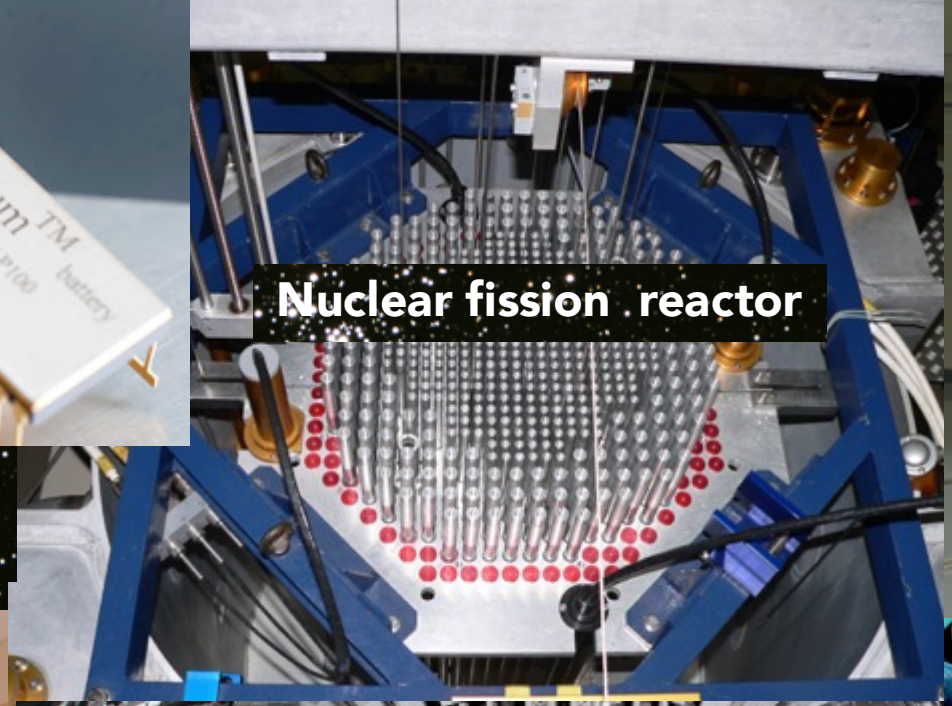
How to bend spacetime with electromagnets

André Füzfa
Namur Center for Complex Systems
and
Strategic Office for Bureaucratic Bullshit
University of Namur

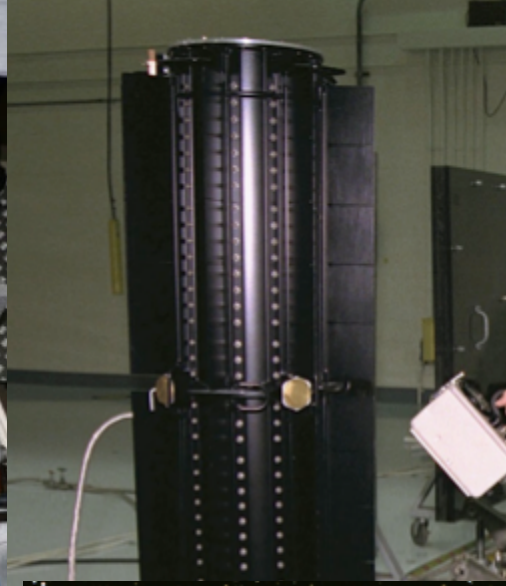
Physical Review D 93, 024014 (2016)
arXiv:1504.00333



Betavoltaic battery: weak force



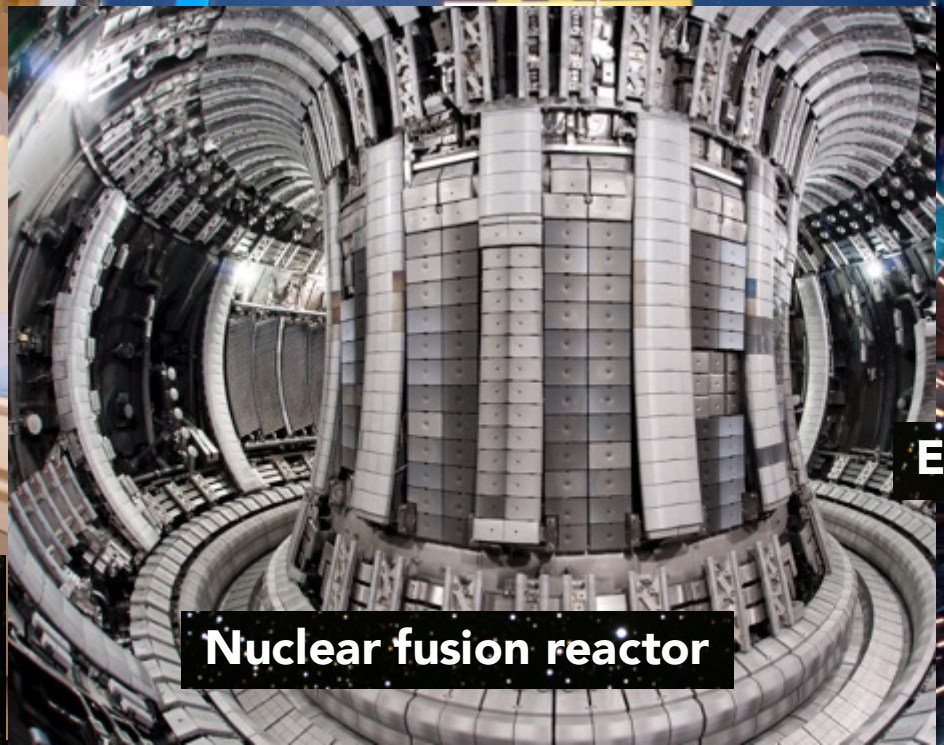
Nuclear fission reactor



Cassini's spacecraft radioisotope thermoelectric generator



MRI superconducting solenoid



Nuclear fusion reactor

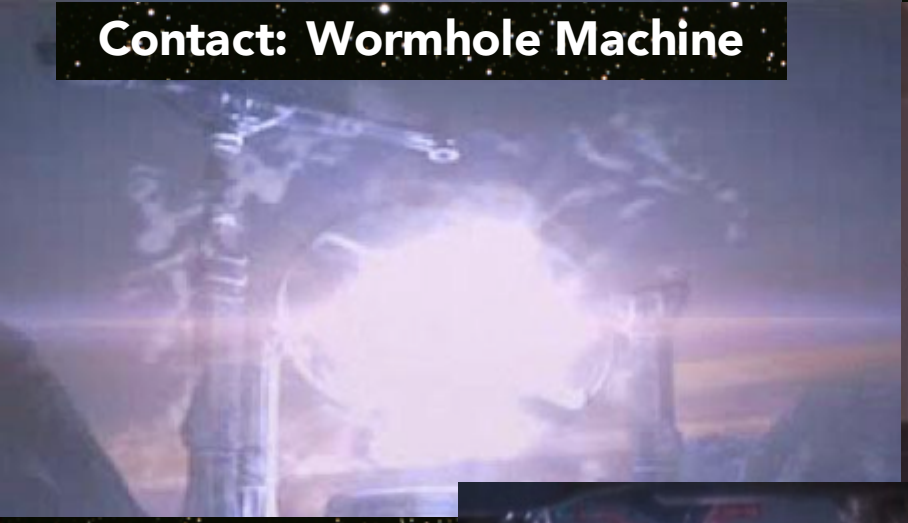


Electric arc welding




Gravitational Field Generators:
A technology from Sci-Fi

Contact: Wormhole Machine



Star Wars: Hyperdrive



Tomorrowland's portal

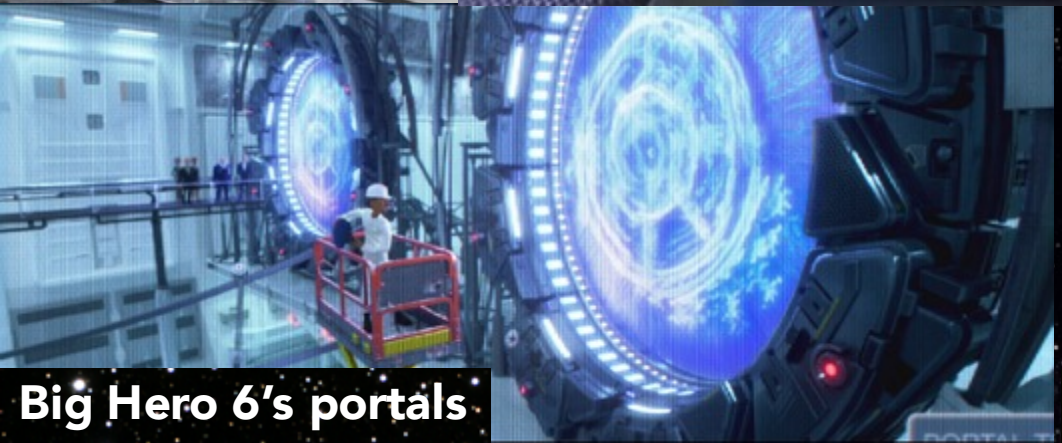




Back to the Future: Flux Capacitor



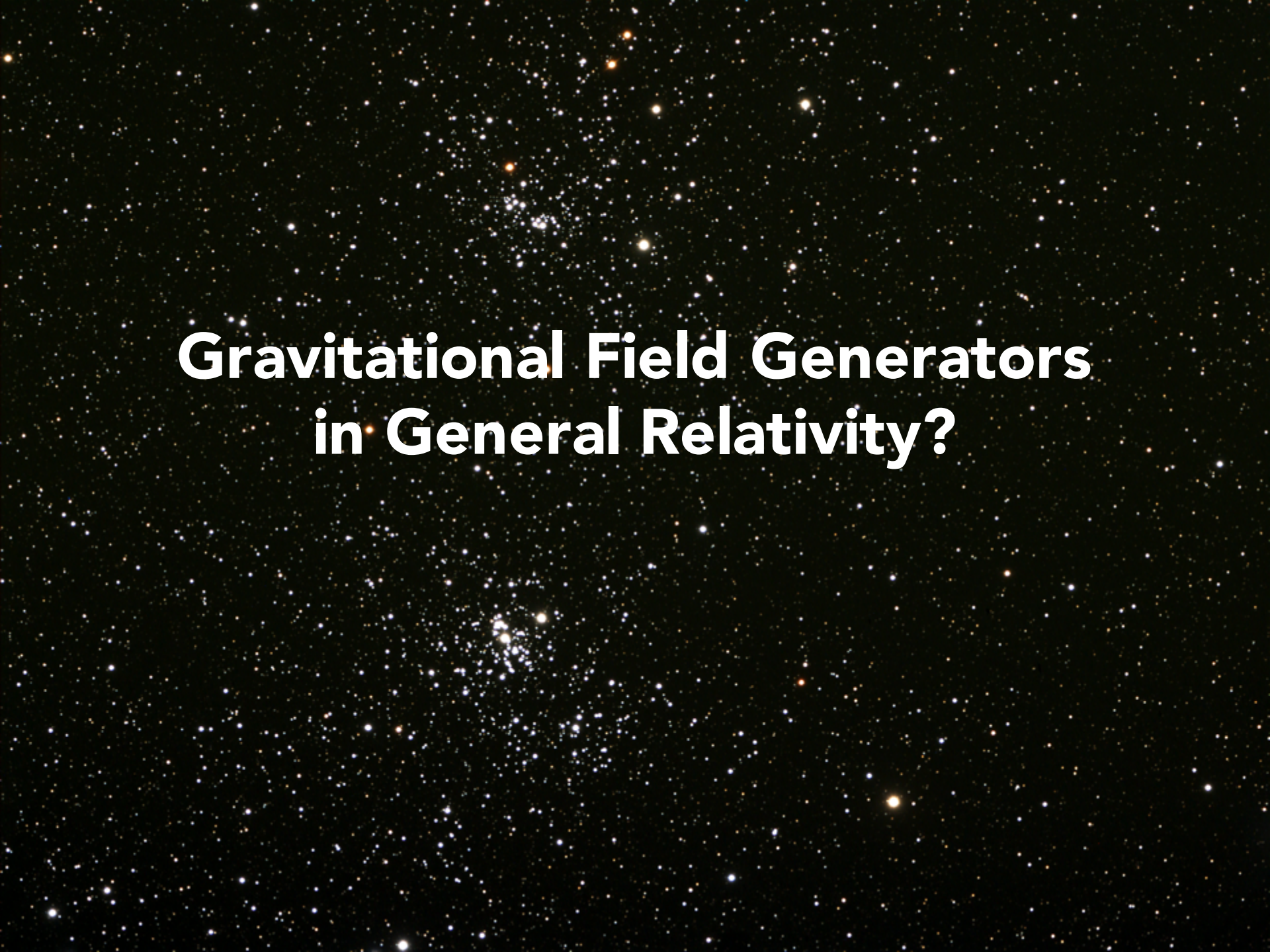
Star Gate SG1



Big Hero 6's portals



Star Trek: Warp drives

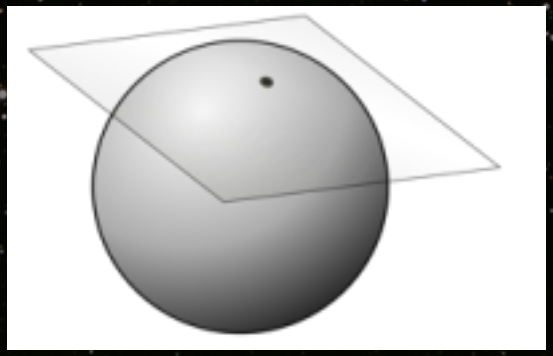

Gravitational Field Generators
in General Relativity?

Einstein's Equivalence Principle

★ Free fall and Earth curvature



Locally : you do not feel gravity



Locally : you do not feel curvature

★ Existence of a locally inertial frame: riemaniann geometry

→ metric fields are gravitational fields (Einstein 1908)

★ Universality of free fall :

→ all kinds of energies undergo gravitation

→ spacetime curvature is due to any kind of energy

★ Magnetic fields generators are also gravitational fields generators

Einstein-Maxwell (EM) System

★ Einstein's equations with matter and electromagnetism

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T^{(\mathrm{mat})}_{\mu\nu} + T^{(\mathrm{em})}_{\mu\nu}\right)$$

$$T^{(\mathrm{em})}_{\mu\nu} = -\frac{1}{\mu_0}\left(g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right)$$

Maxwell stress-energy tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Faraday tensor with A_μ electromagnetic vector potential

★ Pure Einstein-Maxwell system $T^{(\mathrm{mat})}_{\mu\nu} = 0$ **and** $T^{(\mathrm{em})} = 0 \Rightarrow R = 0$

$$R_{\mu\nu} = -\frac{8\pi G}{c^4} T^{(\mathrm{em})}_{\mu\nu}$$

Einstein equations

$$\nabla_\mu F^{\mu\nu} = \mu_0 J^\nu$$

Maxwell equations with J^ν current density

Some analytical results of EM system

★ Spacetime with uniform stationnary magnetic/electric fields:

→ Static axisymmetric cosmological solutions:

Levi-Civita,1917 ; Bertotti, 1959 ; Robinson, 1959 ; Bonnor, 1954 ; Melvin, 1964 ; Peblansky & Hacyan, 1979

→ Not asymptotically flat (divergence of metric fields at spatial infinity)

★ Electrically charged black hole:

→ Non-rotating: Reissner, 1916 and Nordström, 1918

→ Rotating: Kerr-Newman, 1965

→ Asymptotically flat

★ Infinite straight wire carrying steady current

→ Mukherjee, 1938 ; Witten, 1962

→ Not asymptotically flat

Some analytical results of EM system

★ Bonnor's theorem (1953):

→ Duality correspondance between electrostatic and magnetostatic solutions outside of the sources

→ Proof of existence of the solution for current loops and partial asymptotics for the magnetic dipole

Asymptotically flat solutions in axisymmetric spacetime?

★ Current loop in general relativity (Bonnor, 1960):

→ field equations and particular analytical solution

★ Infinitely long solenoid in weak field regime (Ivanov, 1994)

Solutions of full EM for current loops and solenoids carrying arbitrarily large currents?

Field equations

★ Gauge choice for metric (Weyl gauge) and vector potential:

$$ds^2 = c^2 e^{\rho(r,z)} dt^2 - e^{\lambda(r,z)} \left(dr^2 + dz^2\right) - e^{-\rho(r,z)} r^2 d\varphi^2$$

$$A = \frac{a(r,z)}{r} d\varphi$$

★ Einstein's equations:

$$\nabla^2_{(r,z)} \rho = \frac{8\pi G}{c^4 \mu_0} \frac{e^\rho}{r^2} \left((\partial_r a)^2 + (\partial_z a)^2 \right)$$

← Magnetic energy density

$$\nabla^2_{(r,z)} \lambda + (\partial_z \rho)^2 = \frac{8\pi G}{c^4 \mu_0} \frac{e^\rho}{r^2} \left((\partial_r a)^2 - (\partial_z a)^2 \right)$$

★ Maxwell equation:

~ Magnetism on curved space

$$\nabla^2_{(r,z)} a - \frac{2}{r} \partial_r a = - \left(\partial_r a \partial_r \rho + \partial_z a \partial_z \rho \right) - r \mu_0 J$$

$$\nabla^2_{(r,z)} = \partial_r^2 + \partial_r / r + \partial_z^2$$

← Laplacian operator on flat space!

Maxwell equation on curved space

$$\nabla^2_{(r,z)} a - \frac{2}{r}\partial_r a = -\left(\partial_r a \partial_r \rho + \partial_z a \partial_z \rho\right) - r\mu_0 J$$

Vector potential decomposition: $a = a_{\mathrm{nr}} + a_{\mathrm{rel}}$

★ Non-relativistic Maxwell equation : flat background ρ=λ=0

$$\nabla^2_{(r,z)} a_{\mathrm{nr}} - \frac{2}{r}\partial_r a_{\mathrm{nr}} = -r\mu_0 J$$

★ Sources for a_{nr}:

→ **Current loop**: infinitely thin ring (J~δ(z).δ(r-l))

→ **Solenoid**: infinitely thin sheet (J~δ(r-l) for z in [-L/2;L/2])

→ Classical solutions in terms of complete elliptic functions of first, second (and third) kind

→ can also be obtained through Biot-Savart law

→Avoids dealing with point-like sources

Dimensionless field equations

★ Current loop of length l : u=r/l, v=z/l, a′= a /(μ_0 I l)

★ Solenoid of length L and radius l: u=r/l, v=z/L, a′= a /(μ_0 I nL l)

★ Set of 3 coupled dimensionless non-linear elliptic PDEs

$$\nabla^2\rho = \mathcal{C}_I \frac{L^2}{l^2}\frac{e^\rho}{u^2}\left(\left(\partial_u(a_{\rm nr}+a_{\rm rel})\right)^2 + \frac{l^2}{L^2}\left(\partial_v(a_{\rm nr}+a_{\rm rel})\right)^2 \right)$$

$$\nabla^2\lambda + \frac{l^2}{L^2}\left(\partial_v\rho\right)^2 = \mathcal{C}_I \frac{L^2}{l^2}\frac{e^\rho}{u^2}\left(\left(\partial_u(a_{\rm nr}+a_{\rm rel})\right)^2 - \frac{l^2}{L^2}\left(\partial_v(a_{\rm nr}+a_{\rm rel})\right)^2 \right)$$

$$\nabla^2 a_{\rm rel} - \frac{2}{u}\partial_u a_{\rm rel} = -\left(\partial_u(a_{\rm nr}+a_{\rm rel})\partial_u\rho + \frac{l^2}{L^2}\partial_v(a_{\rm nr}+a_{\rm rel})\partial_v\rho \right)$$

★ Magneto-gravitational coupling: $\boxed{\mathcal{C}_I = \frac{8\pi G}{c^4}\mu_0 I_{\rm tot}^2 = 8\pi\left(\frac{I_{\rm tot}}{I_{\rm Pl}}\right)^2}$

★ The total current matters!

Planck current: $I_{\rm Pl} = \dfrac{c^2}{\sqrt{G\mu_0}} = 9.8169\times 10^{24}\,A$

The gravitational fields we can hope to generate are extremely tiny !

$$\mathcal{C}_I \sim \left(\frac{I_{\rm tot}}{10^{25}\, A} \right)^2$$

Compactness of massive objects: $s = \dfrac{GM}{c^2 R}$

Boundary conditions

★ Regularity conditions at r=0: vanishing radial derivatives on axis

★ Asymptotic flatness: gravitational fields around a magnetic dipole:

$$\rho \sim \frac{\mathcal{C}_{\mathcal{I}}}{32}\frac{L^4}{l^4}\frac{v^2}{d^6}$$

With d the Euclidean distance: $$d = \left(u^2 + \frac{L^2}{l^2}v^2\right)^{1/2}$$

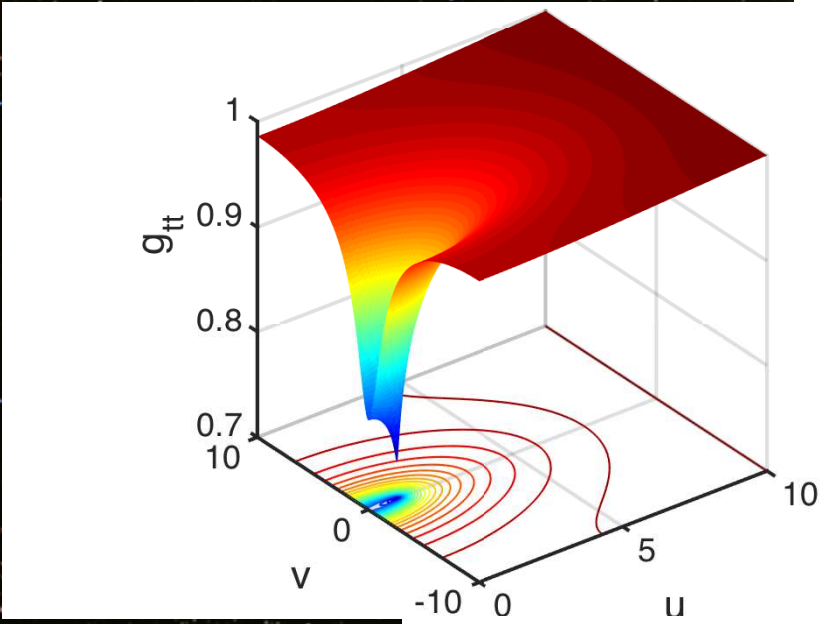
$$\lambda \sim \frac{\mathcal{C}_{\mathcal{I}}}{16}\frac{L^2}{l^2}\left[2\frac{u^2}{d^6} - \frac{L^2}{l^2}\frac{v^2}{2d^6} - \frac{9u^4}{4d^8}\right]$$

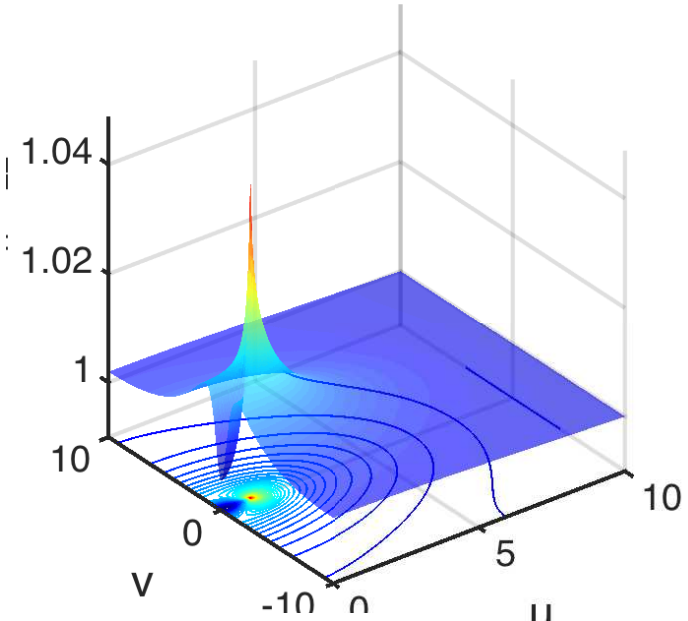
$$a_{\mathrm{rel}} \sim 0$$

★ These gravitational fields are not monopolar

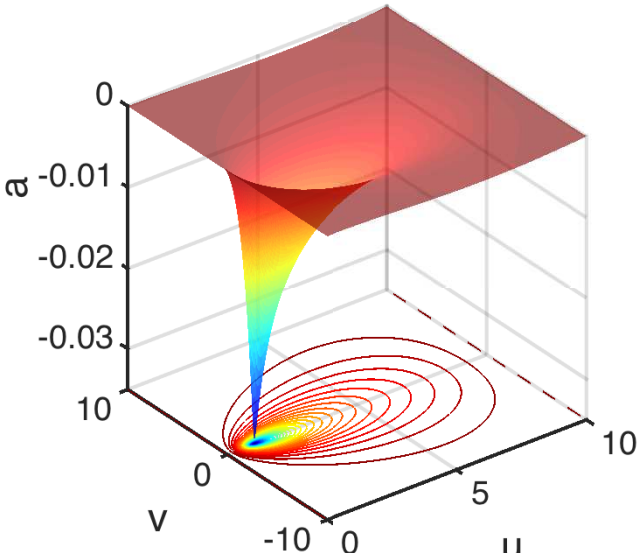
Numerical Results for the current loop

Metric components





Relativistic part of Magnetic component

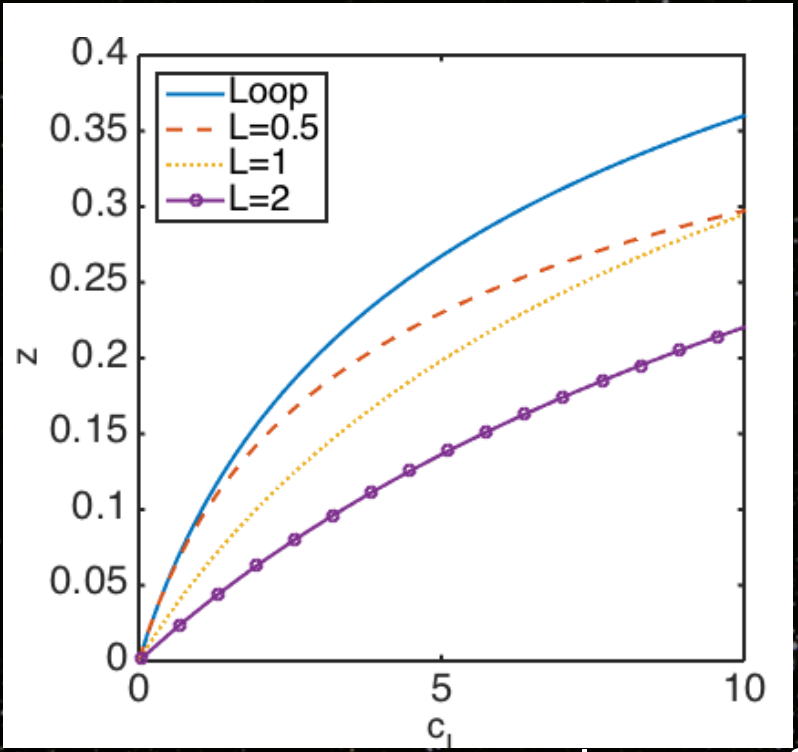


Gravitational redshift from coils

★ Metric components are not observables

→ Move to gauge invariant quantities

★ Einstein effect: gravitational redshift is tiny!



$$z = 1 - e^{-\rho(0,0)/2} \sim c_I$$

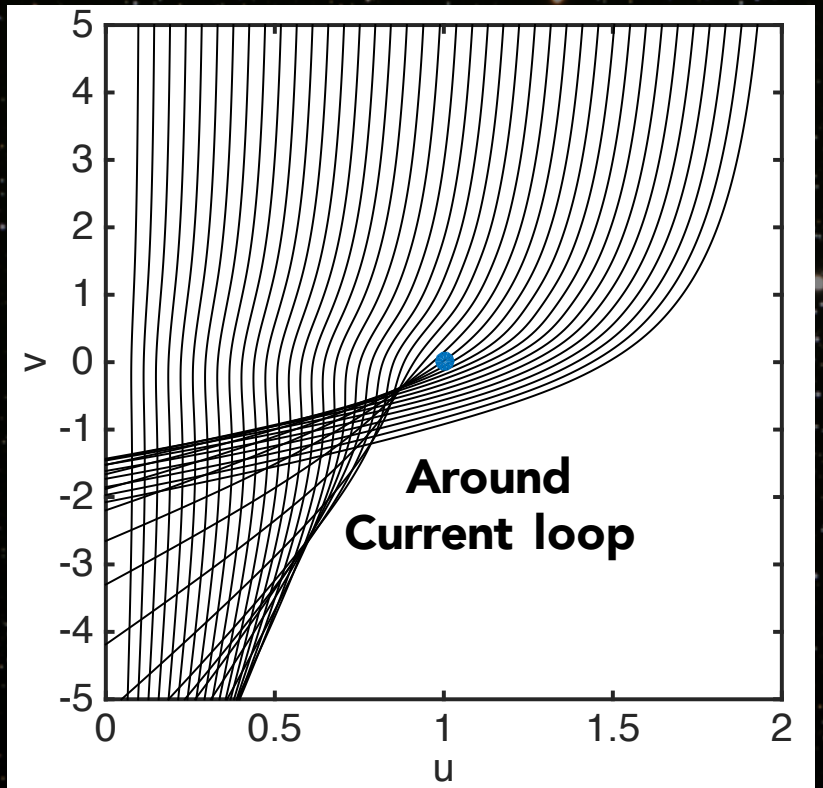
$$c_I \sim \left(\frac{I_{\rm tot}}{10^{25}\, A} \right)^2$$

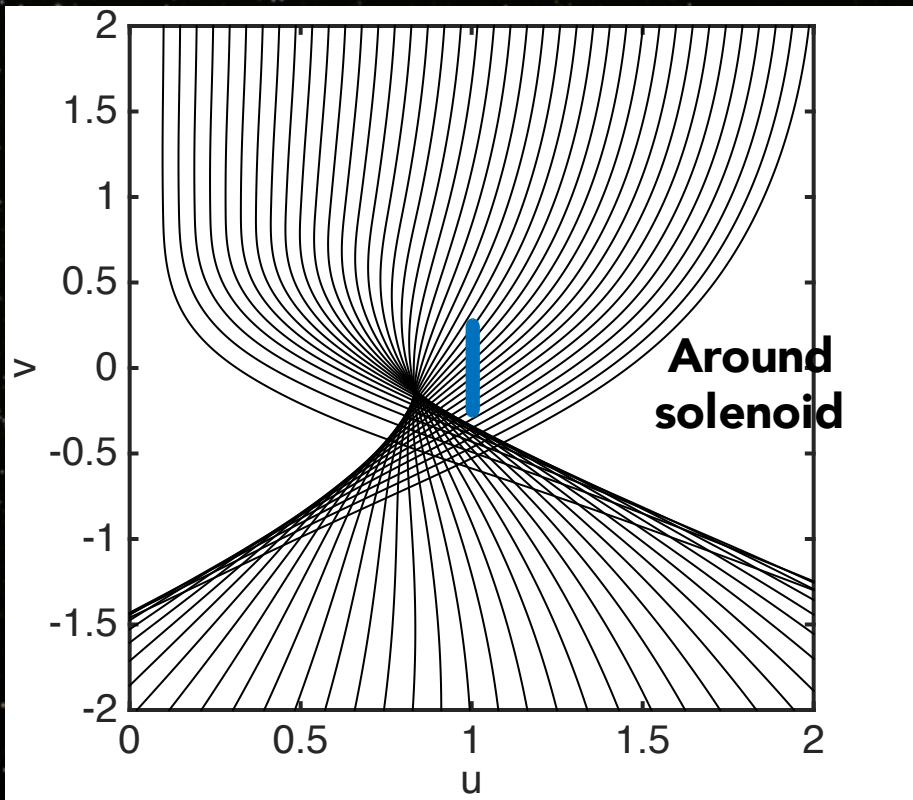
★ No spacetime singularities for finite current?

★ Measure of the depth of gravitational potential well: too weak to be detected!

Deviation of light by electromagnets

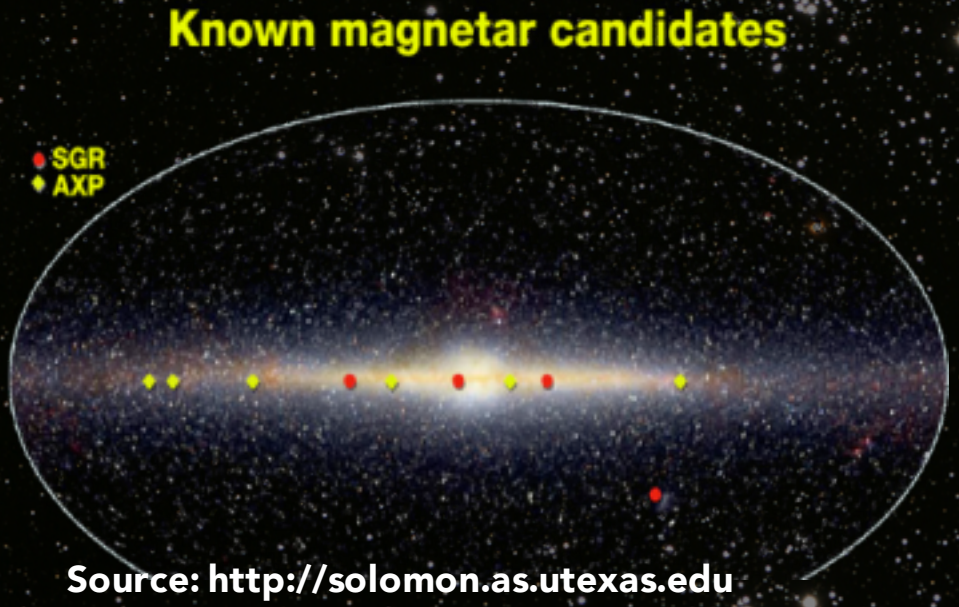
- ★ Classical (non quantum) effect from general relativity (curved spacetime)
- ★ Amplification for detection: observables cumulative along the path of light (phase shift, polarization, etc.):
- ★ Integration of null geodesics for a numerical metric



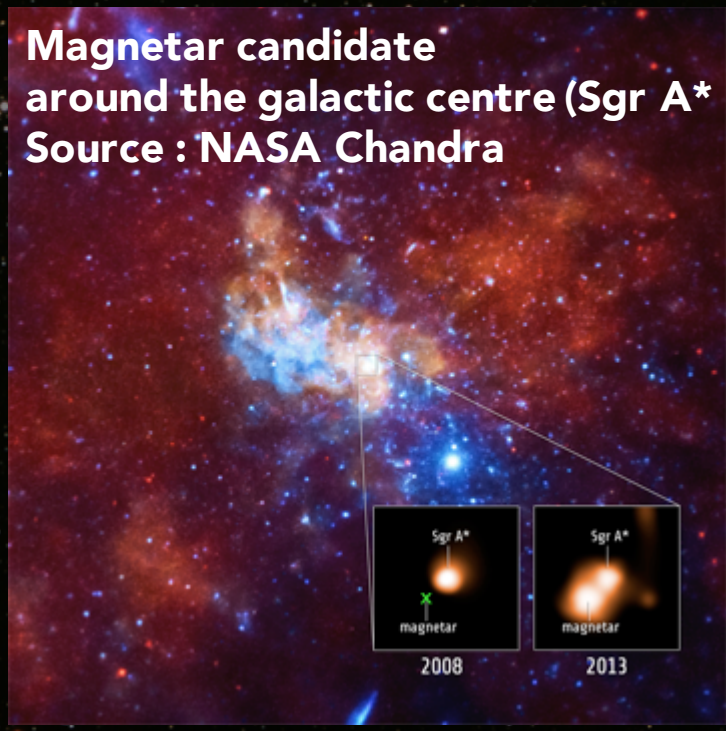


Application to astrophysics : magnetars

★ Neutron stars with ultra-strong magnetic fields (B~10^8-10^{11} T)



Source: http://solomon.as.utexas.edu



Magnetar candidate
around the galactic centre (Sgr A*
Source : NASA Chandra

★ Additional spacetime curvature due to magnetic energy will contribute to gravitational lensing and quantum effects on polarization

Generation and detection of artificial gravitational fields

★ Phase shift of light from gravitational field

$$\Delta\Phi = \frac{1}{2}\int_\gamma h_{\mu\nu} K^{(0)\nu} dx^\mu = \frac{\pi}{\Lambda}\int_0^{\mathcal{L}} (\rho(0,z) - \lambda(0,z))\, dz$$

→ $h_{\mu\nu}$: perturbed metric ; K^ν: unperturbed wave vector ; Λ: wavelength

★ Cumulative along the trajectory : signal amplification in cavities

★ Setup similar to ground-based interferometric detectors of GW (LIGO, VIRGO, GEO600, etc.)

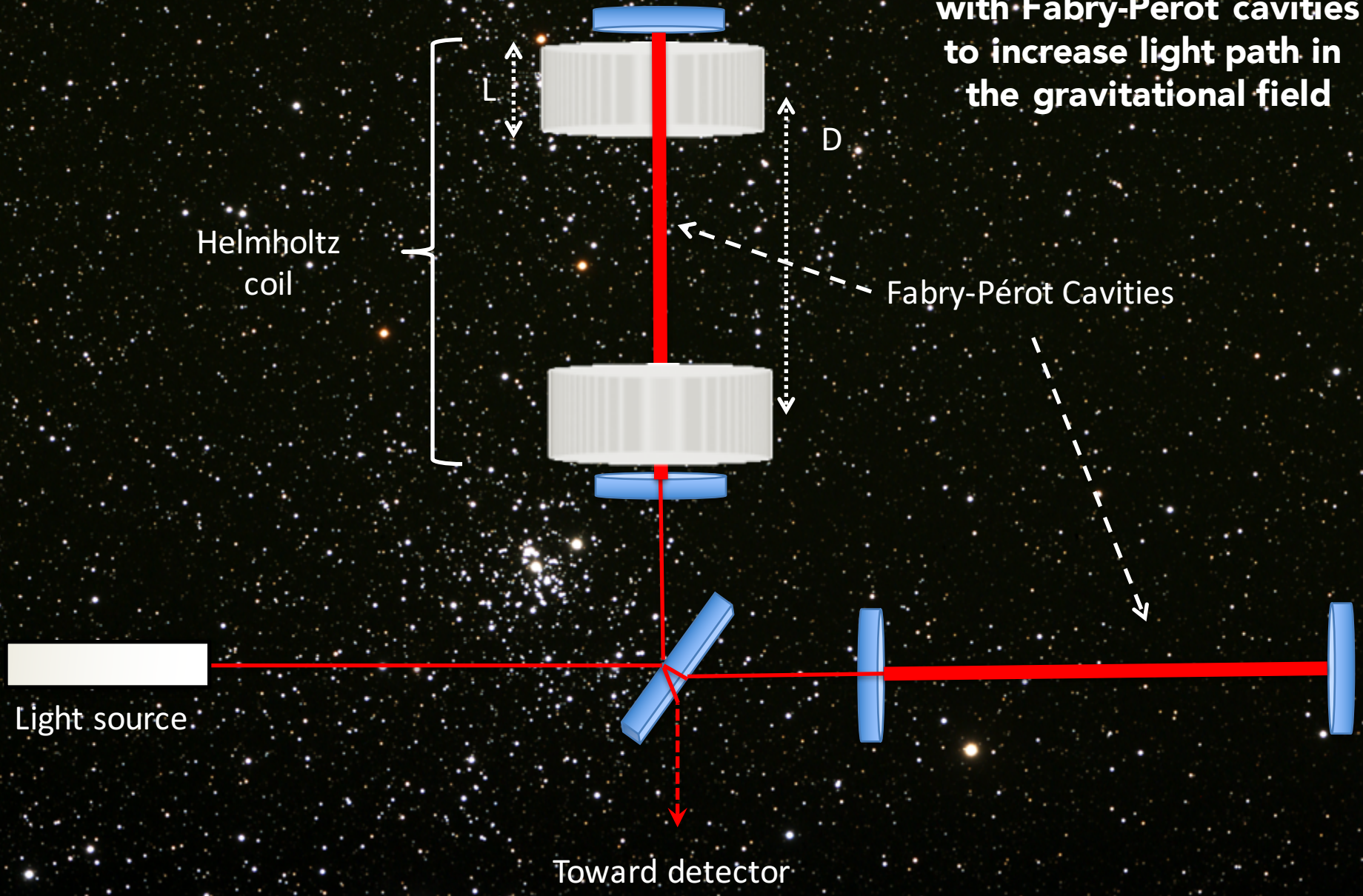
★ Use of Helmholtz coils:

$$\nabla^2 (\rho - \lambda) = \frac{2\mathcal{C}_I}{u^2} \left(\partial_v a_{\mathrm{nr}}\right)^2 \sim |B_r|^2$$

★ Compromise between amplitude generated and detection threshold

A proposed setup:

Michelson interferometer with Fabry-Pérot cavities to increase light path in the gravitational field



Laboratory application : testing both Einstein and Maxwell theories

★ Equivalence principle tests with binding energies of composite systems :

→ universality of free fall, atomic clocks, stellar physics, quasars spectra, BBN, CMB, solar system (PPNs), binary systems with pulsars, pulsar timing, etc.

★ No specific tests exclusively on gravitation + electromagnetism

→ Intense magnetic fields : purely electromagnetic and at high energies

★ Gravitation-Electromagnetism unification:

→ Kaluza-Klein (1920s): compactification of an extra-dimension

$$\bar{g}^{(5)}_{AB} = \begin{pmatrix} g^{(4)}_{\mu\nu} + \phi^2 A_\mu A_\nu & \phi^2 A_\mu \\ \phi^2 A_\mu & \phi^2 \end{pmatrix}$$

Φ^2: Variation of both the gravitational and the gauge couplings

→ Low-energy limits of the (many) compactified string theories

★ Nonlinear electromagnetism (e.g. Born-Infeld)

Some perspectives

★ Theoretical side:

→ Prospects from alternative theories of gravitation

★ Experimental side:

→ feasibility study with light interferometry

→ Other possible detection channels

- Polarization
- bunch of charged particles and synchrotron radiation
- Aharonov-Bohm effect?
- Atom/neutron interferometry?

★ Possible testbed for high-precision gravitational wave/field detectors

★ Gersenshtein-Zeldovich effect:

→ resonant gravitational wave generation from electromagnetic wave in strong magnetic field

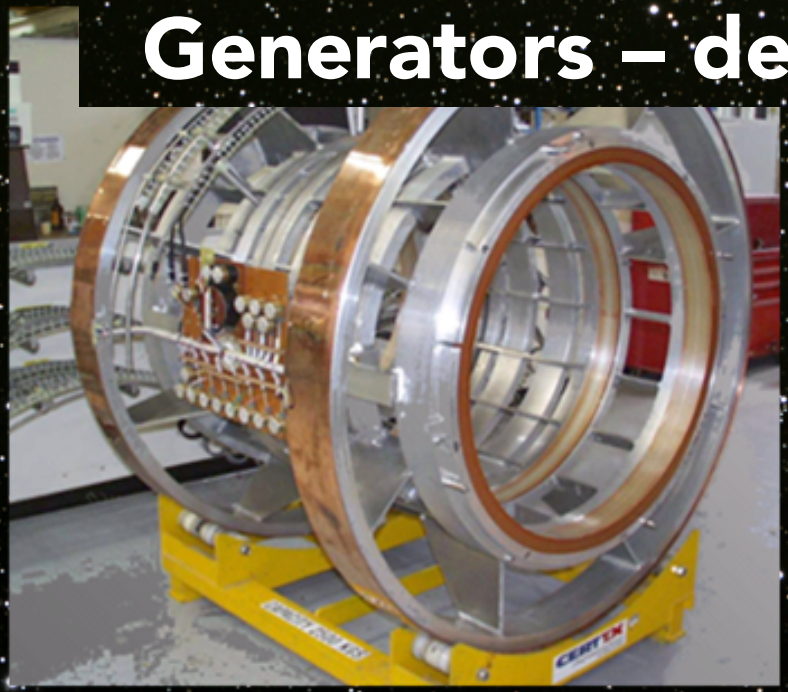
→ technological use : challenge of generators/detectors development

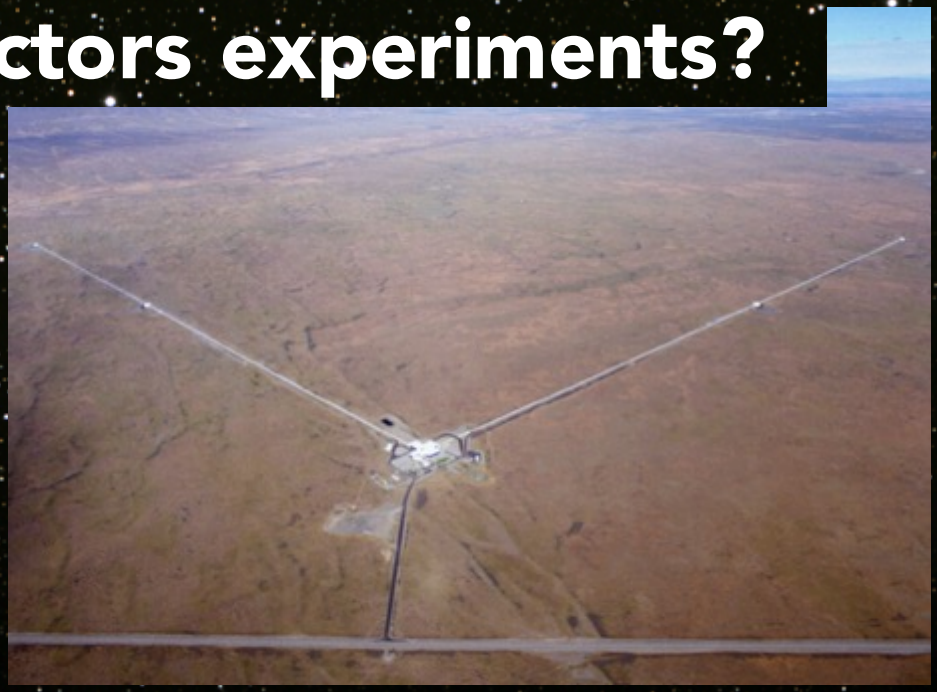




New labs for gravitational physics?

Generators – detectors experiments?









Numerical method (1/2)

★ Relaxation method: $\nabla^2 \rho^{(n+1)} = S_1 \left[\rho^{(n)}, a_{\rm rel}^{(n)} \right]$, etc.

→ 3 PDEs for 3 unknowns: ρ,λ,$a_{\rm rel}$

→ Initial guess (n=0): non-relativistic solution

→ at each step n, solve a set of linear inhomogeneous elliptic PDEs

→ stop when relative updates below tolerance threshold

★ Fourier decomposition with cosines (regularity conditions):

$$f(u,v) = \sum_{k=0}^{N} \hat{f}_k(u) \cos\left(\frac{k\pi}{V} v\right) \; ; \; (u,v) \in [0,U] \times [-V,+V]$$

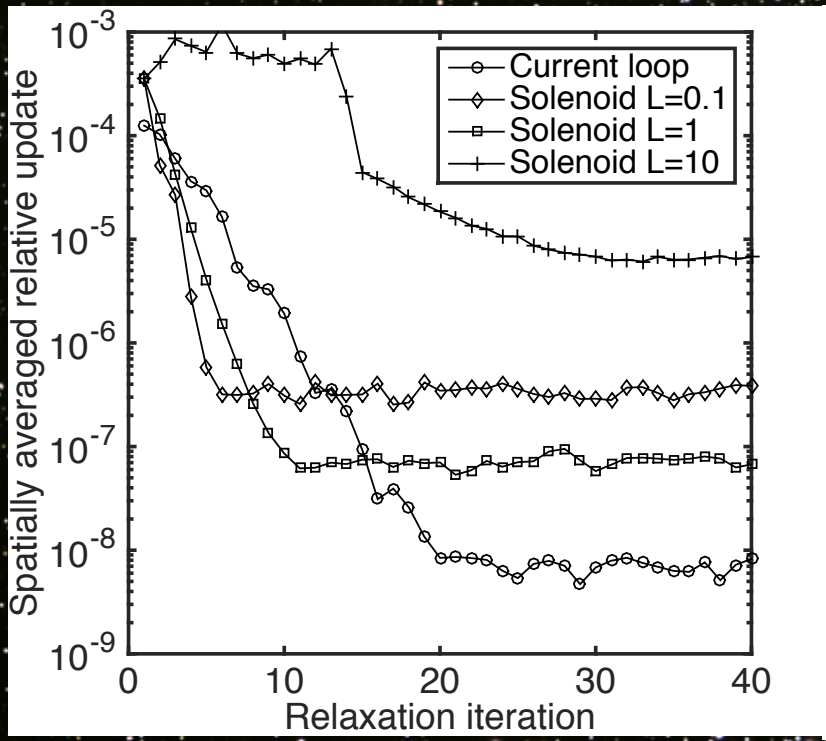
$$\hat{f}_k(u) = \frac{1}{V} \int_{-V}^{V} f(u,v) \cos\left(\frac{k\pi}{V} v\right) dv$$

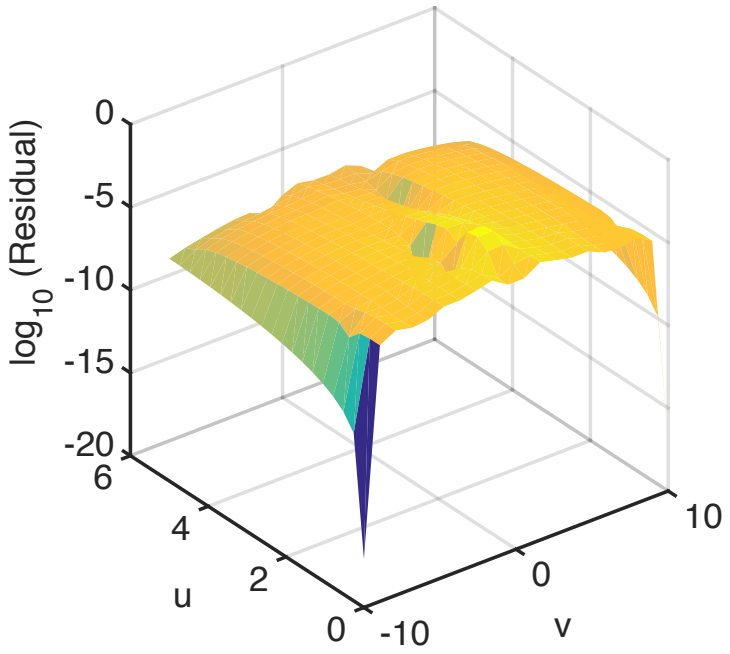
★ Boundary value problem for each Fourier mode at each step n:

$$\frac{d^2 \hat{\rho}_k}{du^2} + \frac{1}{u} \frac{d\hat{\rho}_k}{du} - \frac{l^2}{L^2} \left(\frac{k\pi}{V}\right)^2 \hat{\rho}_k = \hat{S}_{k,1}(u)$$

Numerical method (2/2)

Convergence of the relaxation algorithm





Residual of off-diagonal Einstein equation

Current loops and solenoids in GR

- ★ Metric in Weyl gauge
- ★ Field equations
- ★ Sources
- ★ Dimensionless equations and magneto-gravitational coupling
- ★ Boundary conditions
- ★ Numerical method and convergence
- ★ Example of Numerical Results
- ★ Gauge invariant characterization: null geodesics
- ★ Application:
  - → generation and detection of artificial gravitational fields
  - → Alternative tests of general relativity and electromagnetism on curved spacetime