

We introduce the fraction of the neutrino energy transferred to the hadron

$$\gamma = \frac{2\vec{p} \cdot \vec{q}}{2\vec{p} \cdot \vec{p}_1}$$

In particular, in the hadron rest frame,
 $\vec{P} = (P^0, \vec{0}) \Rightarrow \gamma = \frac{P^0}{P_1^0}$,
 which justifies the definition.

We calculate, with $\vec{p}_2 = \chi \vec{P}$

$$\gamma = \frac{2\vec{p}_2 \cdot \vec{q}}{2\vec{p}_2 \cdot \vec{p}_1} = \frac{2\vec{p}_2 \cdot \vec{p}_1 - 2\vec{p}_2 \cdot \vec{p}_3}{\Delta} = 1 + \frac{u}{\Delta}$$

$$\Leftrightarrow u = \Delta(\gamma - 1)$$

$$\begin{aligned} \gamma &= 1 + \frac{1}{\Delta} (-2)\vec{p}_2 \cdot \vec{p}_3 = 1 - 2\frac{1}{\Delta} (E_2 E_3 - |\vec{p}_2||\vec{p}_3|\cos\theta) \\ &= 1 - \frac{2}{\Delta} E_2 E_3 (1 - \cos\theta) \\ &= 1 - \frac{1}{\Delta} E_{\text{CM}} E_3 (1 - \cos\theta) \end{aligned}$$

Due to $\delta(E_3 - \frac{E_{\text{CM}}}{2})$, we have

$$\begin{aligned} \gamma &= 1 - \frac{1}{\Delta} E_{\text{CM}} \frac{E_{\text{CM}}}{2} (1 - \cos\theta) \\ &= 1 - \frac{1}{2} (1 - \cos\theta) \\ &= \frac{1}{2} (1 + \cos\theta) \end{aligned}$$

$$\Rightarrow d\gamma = \frac{1}{2} d\cos\theta$$

$$\cos\theta = 1 \Rightarrow \gamma = 1$$

$$\cos\theta = -1 \Rightarrow \gamma = 0$$

Thus

$$\sigma = \frac{G_F^2 |L_{NC}^u|^2 \Delta}{2\pi} 2 \int_0^1 d\gamma \int_0^\infty dE_3 \delta(E_3 - \frac{E_{\text{CM}}}{2})$$

$$= \frac{G_F^2 |L_{NC}^u|^2 \Delta}{\pi} \int_0^1 d\gamma$$

$$\Rightarrow \frac{d\sigma}{d\gamma} = \frac{G_F^2 |L_{NC}^u|^2 \Delta}{\pi} = \frac{G_F^2 \Delta}{\pi} \left(\frac{1}{2} - \frac{2}{3} \sin^2 \theta_w \right)^2$$

Similarly

$$\frac{d\sigma}{dy}(v_\mu d_L \rightarrow v_\mu d_L) = \frac{G_F^2 \Lambda}{\pi} \left(\frac{1}{2} - \frac{1}{3} \sin^2 \theta_w \right)^2$$

For the right-handed part, the only difference is the sign in the ϵ -tensor, so

$$\begin{aligned} & \frac{1}{2} \frac{1}{N_c} \sum_{\text{spin}} \sum_{\text{color}} |M(v_\mu d_R \rightarrow v_\mu u_R)|^2 \\ &= 16 G_F^2 |R_{NC}^u|^2 \left[2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3) \right. \\ & \quad \left. - 2(\delta_\omega^\rho \delta_\tau^\sigma - \delta_\tau^\rho \delta_\omega^\sigma) p_{3\rho} p_{4\sigma} p_4^\omega p_2^\tau \right] \\ &= \dots \\ &= 64 G_F^2 |R_{NC}^u|^2 (p_1 \cdot p_4)(p_2 \cdot p_3) \\ &= 16 G_F^2 |R_{NC}^u|^2 u^2 \\ &= 16 G_F^2 |R_{NC}^u|^2 \Lambda^2 (1-y)^2 \end{aligned}$$

Following exactly the same lines as for the left-handed part, we get

$$\sigma(v_\mu u_R \rightarrow v_\mu u_R) = \frac{G_F^2 |R_{NC}^u|^2 \Lambda}{\pi} \int_0^1 dy (1-y)^2$$

$$\Rightarrow \frac{d\sigma}{dy}(v_\mu u_R \rightarrow v_\mu u_R) = \frac{G_F^2 \Lambda}{\pi} (1-y)^2 \frac{4}{9} \sin^4 \theta_w$$

$$\frac{d\sigma}{dy}(v_\mu d_R \rightarrow v_\mu d_R) = \frac{G_F^2 \Lambda}{\pi} (1-y)^2 \frac{1}{9} \sin^4 \theta_w$$

The charged currents are similar

$$\frac{d\sigma}{dy} (\nu_\mu d_L \rightarrow \mu^- u_L) = \frac{G_F^2 \Delta}{\pi}$$

$$\frac{d\sigma}{dy} (\nu_\mu d_R \rightarrow \mu^- u_R) = 0$$

Furthermore:

$$\overline{L_{NC}^2} = \frac{1}{2} (L_{NC}^{u^2} + L_{NC}^{d^2})$$

$$= \frac{1}{2} \left[\left(\frac{1}{2} - \frac{2}{3} \sin^2 \theta_w \right)^2 + \left(\frac{1}{2} - \frac{1}{3} \sin^2 \theta_w \right)^2 \right]$$

$$= \frac{1}{2} \left(\frac{1}{4} - \frac{2}{3} \sin^2 \theta_w + \frac{4}{9} \sin^4 \theta_w + \frac{1}{4} - \frac{1}{3} \sin^2 \theta_w \right)$$

$$= \frac{1}{4} - \frac{1}{2} \sin^2 \theta_w + \frac{5}{18} \sin^4 \theta_w + \frac{1}{9} \sin^4 \theta_w$$

$$\overline{R_{NC}^2} = \frac{1}{2} \left(\frac{4}{9} \sin^4 \theta_w + \frac{1}{9} \sin^4 \theta_w \right)$$

$$= \frac{5}{18} \sin^4 \theta_w$$

$$\overline{L_{CC}^2} = \frac{1}{2}$$

$$\overline{R_{CC}^2} = 0$$

The integrated cross-sections are

$$\sigma_L = \frac{G_F^2 \Delta}{\pi} \overline{Q_W^2} \int_0^1 dy = \frac{G_F^2 \Delta}{\pi}$$

$$\sigma_R = \frac{G_F^2 \Delta}{\pi} \overline{Q_W^2} \int_0^1 dy (1-y)^2$$

$$= \frac{G_F^2 \Delta}{\pi} \overline{Q_W^2} \left(1 - 1 + \frac{1}{3} \right)$$

$$= \frac{G_F^2 \Delta}{3\pi} \overline{Q_W^2}$$

Thus

$$\sigma_{NC} = \frac{GFD^2}{\pi} \left(\frac{1}{4} - \frac{1}{2} \sin^2 \theta_w + \frac{5}{18} \sin^4 \theta_w + \frac{1}{3} \frac{5}{18} \sin^4 \theta_w \right)$$

$$\sigma_{CC} = \frac{GFD^2}{2\pi}$$

$$\Rightarrow R = \frac{1}{2} - \sin^2 \theta_w + \frac{4}{3} \frac{5}{9} \sin^4 \theta_w$$

$$\boxed{R = \frac{1}{2} - \sin^2 \theta_w + \frac{20}{27} \sin^4 \theta_w}$$