Divergent series and physical observables: An introduction to Sudakov resummation

Centre for Cosmology, Particle Physics and Phenomenology
Université Catholique de Louvain
November 24$^{\text{th}}$, 2011

Giovanni Ridolfi
Università di Genova and INFN Genova, Italy

Plan of the talk:

1. **A review of Sudakov resummation**
2. **Ambiguities in resummed results**
3. **Standard solution**
4. **A different approach**

Results obtained in collaboration with R. Abbate, M. Bonvini, S. Forte, J. Rojo, M. Ubiali.

1. Sudakov resummation: a short review*

Physical cross sections are always inclusive over arbitrarily soft particles in the final state, because of finite detector resolution.

Inclusiveness plays a crucial role in QCD: infrared divergences from virtual corrections are cancelled by radiation of undetected real gluons.

The finite left-over of these cancellations give large contributions if the tagged final state is forced to take most of the available energy.

*S. Catani, hep-ph/9610413

Schematically: $(1-z)\sqrt{s}$ **total energy carried by unobserved radiation. Virtual and real emission corrections:**

$$\frac{dw_{\rm virtual}}{dz} = -2C\,\alpha_{\rm S}\,\delta(1-z)\int_0^{1-\epsilon}\frac{dy}{1-y}\,\log\frac{1}{1-y}$$

$$\frac{dw_{\rm real}}{dz} = +2C\,\alpha_{\rm S}\,\frac{1}{1-z}\,\log\frac{1}{1-z}\,\theta(1-\epsilon-z)$$

because of the bremsstrahlung spectrum $d\omega/\omega$ **and the spectrum for collinear emission,** dk_T^2/k_T^2**.**

The sum of the two contributions is finite as the infrared cut-off ϵ **goes to 0:**

$$\frac{dw(z)}{dz} = \frac{dw_{\rm virtual}}{dz} + \frac{dw_{\rm real}}{dz} = 2\,C\,\alpha_{\rm S}\,\left[\frac{1}{1-z}\,\log\frac{1}{1-z}\right]_+$$

where, by definition,

$$\int_0^1 dz\,[D(z)]_+ f(z) = \int_0^1 dz\,D(z)\,[f(z)-f(1)]$$

- **Virtual contribution concentrated at $z = 1$**
- **Real emission contribution spread over the interval $x < z < 1$, where x is the fraction of total energy carried by the observed final state.**

Hence, the contribution of soft emission to the cross section is proportional to

$$\int_x^1 dz \, \frac{dw}{dz} = -C \, \alpha_{\mathrm{S}} \, \log^2(1 - x),$$

a finite left-over of the cancellation of infrared divergences.

As $x \to 1$ in the final state, the phase space for real emission is suppressed, and the finite left-over becomes large.

At order n, at most two powers of $\log(1-x)$ for each power of $\alpha_{\rm S}$ appear in the perturbative coefficients:

$$C_n\,\alpha_{\rm S}^n = \alpha_{\rm S}^n \sum_{m=1}^{2n} c_{nm}\,\log^m(1-x) + \text{non singular terms}$$

The perturbative expansion becomes unreliable; logarithmically enhanced contributions must be resummed to all orders.

Logarithmic contributions are expected to be relevant when $x \to 1$.

Side remark: **resummation may have an impact even at smaller values of x, depending on the shape of parton densities.**

See e.g.

[M. Bonvini, S. Forte, GR, NPB874 (2011) 93]

[T. Becher, M. Neubert and G. Xu, JHEP 0807 (2008) 030]

Examples:

1. **lepton-nucleon scattering in the quasi-elastic limit:**

$$x \to x_{\mathrm{Bj}} = \frac{Q^2}{2p \cdot q}, \qquad x \to 1$$

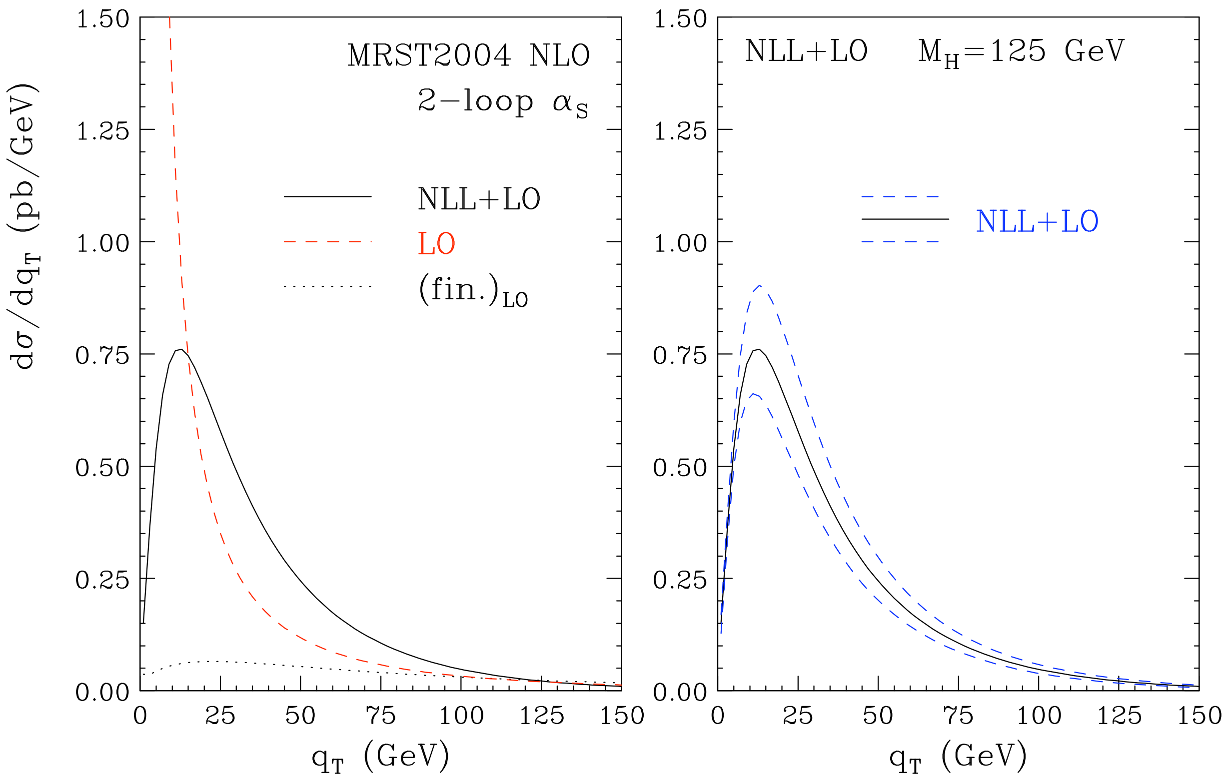
2. **production of heavy systems (Drell-Yan pairs, Higgs) close to threshold:**

$$x \to \tau = \frac{Q^2}{s}, \qquad s \gtrsim Q^2$$

3. **transverse momentum spectrum in the small-q_T region:**

$$1 - x \to \frac{q_{\mathrm{T}}^2}{Q^2}, \qquad q_{\mathrm{T}}^2 \ll Q^2$$

Resummation can be performed. An example:

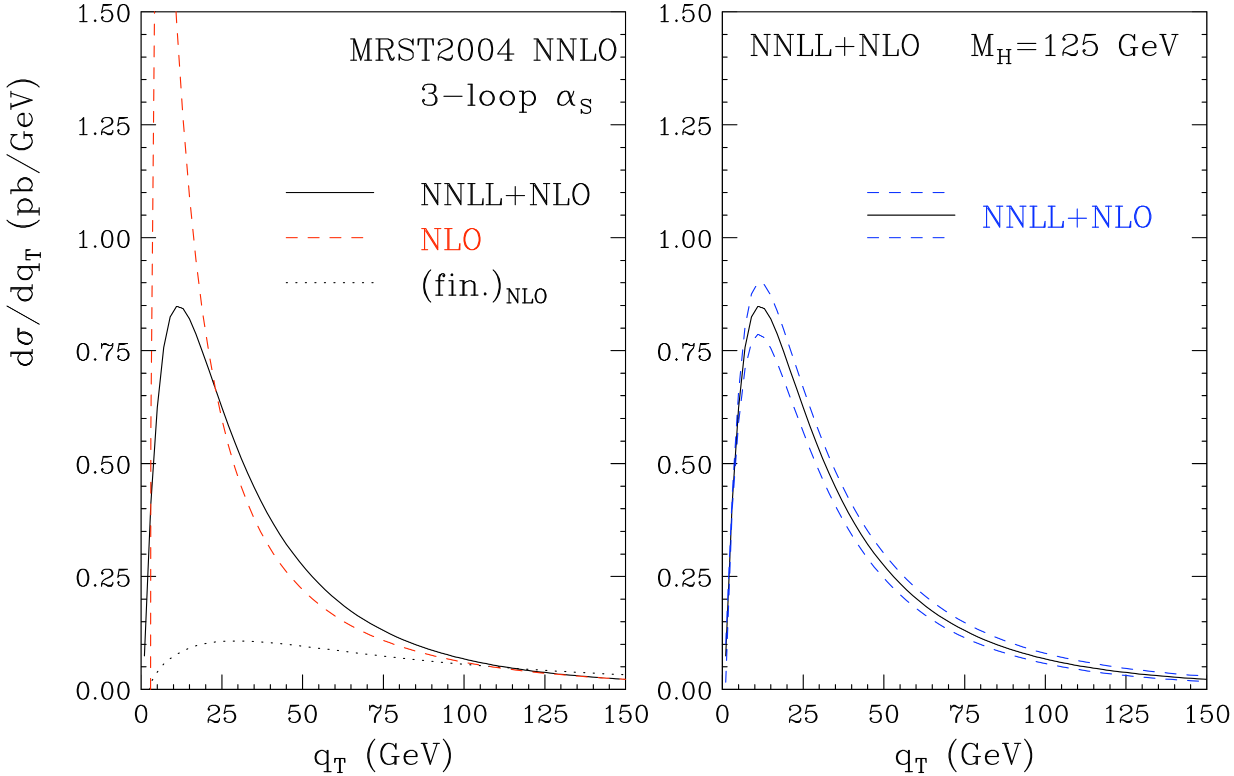


The q_T spectrum of Higgs production at the LHC

Left: NLL+LO compared with the LO spectrum

Right: uncertainty band from scale variations.

Bozzi, Catani, de Florian, Grazzini, NPB737(2006)73, hep-ph/0508068



The q_T spectrum of Higgs production at the LHC

Left: NNLL+NLO compared with the LO spectrum

Right: uncertainty band from scale variations.

Bozzi, Catani, de Florian, Grazzini, NPB737(2006)73, hep-ph/0508068

Resummed cross sections: a schematic derivation

Typical expression of an observable in QCD (eg, Drell-Yan cross section):

$$\begin{aligned}\sigma(x,Q^2) &= \int_x^1 \frac{dy}{y}\,\mathcal{L}(y,Q^2)\,C\left(\frac{x}{y},\alpha_{\mathrm{S}}(Q^2)\right)\\ &= \mathcal{L}\otimes C\end{aligned}$$

The function $\mathcal{L}(y,Q^2)$ is a parton luminosity, e.g.

$$\mathcal{L}(y,Q^2)=\int_y^1 \frac{dy'}{y'}\,f_1(y',Q^2)\,f_2\left(\frac{y}{y'},Q^2\right)$$

in hadron-hadron collisions.

The coefficient function C is essentially a partonic cross section.

Resummation usually (but not always) perfomed in the space of Mellin transformed quantities:

$$f(N) = \int_0^1 dx\, x^{N-1}\, f(x); \qquad f(x) = \frac{1}{2\pi i} \int_{\bar{N}-i\infty}^{\bar{N}+i\infty} dN\, x^{-N}\, f(N)$$

(a Laplace transform with respect to $t = \log \frac{1}{x}, x = e^{-t}$).

- **well defined and analytic in the half-plane $\mathrm{Re\,N} > N_0$ if $f(x)$ is at most as singular as x^{-N_0} in $x = 0$**
- **In the space of the Mellin-conjugate variable N, convolution products are turned into ordinary products:**

$$\sigma(N, Q^2) = \mathcal{L}(N, Q^2)\, C(N, \alpha_{\mathrm{S}}(Q^2))$$

- **The region $x \to 1$ is mapped in the region $N \to \infty$:**

$$\int_0^1 dx\, x^{N-1} \left[\frac{\log^k(1-x)}{1-x}\right]_+ = \frac{1}{k+1} \log^{k+1} \frac{1}{N} + O(\log^k N)$$

Why Mellin moments?

$$C(z, \alpha_S) = \delta(1-z) + \sum_{n=1}^{\infty} \int_0^1 dz_1 \ldots dz_n \, \frac{dw_n(z_1, \ldots, z_n)}{dz_1 \ldots dz_n} \, \Theta_{PS}(z; z_1, \ldots, z_n)$$

The multi-gluon emission probability factorizes in the soft limit,

$$\frac{dw_n(z_1, \ldots, z_n)}{dz_1 \ldots dz_n} \simeq \frac{1}{n!} \prod_{i=1}^{n} \frac{dw(z_i)}{dz_i}$$

(easily seen in QED in the eikonal approximation) but the phase space factor

$$\Theta_{PS}(z; z_1, \ldots, z_n) = \delta(z - z_1 z_2 \cdots z_n)$$

does not ...

... unless one goes to Mellin moments:

$$C(N,\alpha_s) = \int_0^1 dz\, z^{N-1}\, C(z,\alpha_s)$$
$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^1 dz_1 \cdots dz_n \prod_{i=1}^{n} \frac{dw(z_i)}{dz_i} \int_0^1 dz\, z^{N-1} \delta(z - z_1 \cdots z_n)$$
$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\int_0^1 dz_1\, z_1^{N-1} \frac{dw(z_1)}{dz_1}\right] \ldots \left[\int_0^1 dz_n\, z_n^{N-1} \frac{dw(z_n)}{dz_n}\right]$$

Hence

$$C(N,\alpha_s) = \exp \int_0^1 dz\, z^{N-1}\, \frac{dw}{dz}$$

Multigluon emission exponentiates in the soft limit.

Some details: consider the production of a heavy object with energy Q_0, plus n gluons of energies $\omega_1, \ldots, \omega_n$. The differential cross section in the eikonal approximation takes the form

$$dC \sim \frac{d\omega_1}{\omega_1}\frac{d\theta_1}{\theta_1} \ldots \frac{d\omega_n}{\omega_n}\frac{d\theta_n}{\theta_n}\delta(\sqrt{s} - Q_0 - \omega_1 \ldots - \omega_n)$$

Define $z = \frac{Q_0}{\sqrt{s}}$ **and**

$$\begin{aligned}
\omega_1 &= \sqrt{s}(1 - z_1) \\
\omega_2 &= \sqrt{s}z_1(1 - z_2) \\
&\ldots \\
\omega_n &= \sqrt{s}z_1 \ldots z_{n-1}(1 - z_n)
\end{aligned}$$

We have

$$J = \left|\frac{d(\omega_1, \ldots, \omega_n)}{d(z_1, \ldots, z_n)}\right| = s^{n/2} z_1^{n-1} z_2^{n-2} \ldots z_{n-1}$$

Furthermore

$$\omega_1 \dots \omega_n = \sqrt{s} z_1^{n-1} z_2^{n-2} \dots z_{n-1} (1 - z_1) \dots (1 - z_n)$$

and

$$\begin{aligned} \omega_1 + \dots + \omega_n &= \sqrt{s}(1 - z_1 + z_1 - z_1 z_2 + \dots - z_1 z_2 \dots z_n) \\ &= \sqrt{s}(1 - z_1 z_2 \dots z_n) \end{aligned}$$

Hence, after angular integration,

$$dC \sim \left[\frac{dz_1}{1 - z_1} \log \frac{1}{1 - z_1}\right] \dots \left[\frac{dz_n}{1 - z_n} \log \frac{1}{1 - z_n}\right] \delta(z - z_1 \dots z_n)$$

Recalling that

$$\frac{dw}{dz} = 2\,C\,\alpha_{\rm S}\left[\frac{1}{1-z}\,\log\frac{1}{1-z}\right]_{+}$$

and using the leading-log result

$$\begin{aligned}\int_0^1 dz\, z^{N-1}\left[\frac{\log^p(1-z)}{1-z}\right]_{+} &= \frac{1}{p+1}\,\log^{p+1}\frac{1}{N} + O(\log^p N) \\ &= -\int_0^{1-\frac{1}{N}}\frac{dz}{1-z}\,\log^p(1-z) + O(\log^p N)\end{aligned}$$

we find

$$C(N,\alpha_{\rm S}) = \exp\left[C\alpha_{\rm S}\log^2 N + O(\log N)\right]$$

Strictly valid in QED; in QCD, complications arise because of gluon emission from gluon lines and because of color structure, but the essential features remain the same.

Extension to QCD

QCD corrections essentially amount to the replacement $\alpha_{\mathrm{S}} \to \alpha_{\mathrm{S}}(k_T^2)$ in the computation of the single-gluon emission probability:

$$2C\alpha_{\mathrm{S}}\left[\frac{1}{1-z}\log\frac{1}{1-z}\right]_+ \to 2C\left[\frac{1}{1-z}\int_{Q^2(1-z)}^{Q^2}\frac{dk_T^2}{k_T^2}\alpha_{\mathrm{S}}(k_T^2)\right]_+$$

The running coupling can then be expanded in powers of $\alpha_{\mathrm{S}}(Q^2)$

$$\alpha_{\mathrm{S}}(k_T^2) = \frac{\alpha_{\mathrm{S}}(Q^2)}{1+\alpha_{\mathrm{S}}(Q^2)\beta_0\log\frac{k_T^2}{Q^2}} = \alpha_{\mathrm{S}}(Q^2)\sum_{n=0}^{\infty}(-\alpha_{\mathrm{S}}(Q^2)\beta_0)^n \log^n\frac{k_T^2}{Q^2}$$

and the expansion integrated term by term. One gets

$$C(N,\alpha_{\mathrm{S}}) = \exp[\log N\, g_1(\alpha_{\mathrm{S}}\log N)]$$

where the function g_1 has a Taylor expansion in its argument, starting at order 1.

For a generic process one can prove the generalized result

$$\begin{aligned} C^{\rm res}(N,\alpha_{\rm S}(Q^2)) &= g_0(\alpha_{\rm S})\,\exp\mathcal{S}\,(L,\bar{\alpha}) \\ \mathcal{S}(L,\bar{\alpha}) &= \frac{1}{\bar{\alpha}}\,g_1(L)+g_2(L)+\bar{\alpha}\,g_3(L)+\bar{\alpha}^2\,g_4(L)+\ldots \\ \bar{\alpha} &= a\,\alpha_{\rm S}(Q^2)\,\beta_0; \qquad a=1,2; \qquad L=\bar{\alpha}\log\frac{1}{N} \end{aligned}$$

which defines an improved expansion (in powers of $\alpha_{\rm S}$ with $\alpha_{\rm S}\log N$ fixed) for $C^{\rm res}(N,\alpha_{\rm S})$: g_1 gives the leading-log (LL) approximation, g_1 and g_2 give the next-to-leading-log approximation (NLL), and so on.

2. Ambiguities in resummed results

A difficulty immediately arises. Define $\tilde{\Sigma}(L, \alpha_S)$ by

$$C^{\rm res}(N, \alpha_S(Q^2)) = 1 + \tilde{\Sigma}(L, \alpha_S(Q^2)) = 1 + \sum_{k=1}^{\infty} h_k(\alpha_S(Q^2)) L^k$$

$\tilde{\Sigma}$ arises as an expansion in powers of $\alpha_S(Q^2)$ of a function of $\alpha_S(Q^2/N^a)$. To NLL we have

$$\alpha_S\left(\frac{Q^2}{N^a}\right) = \frac{\alpha_S(Q^2)}{1+L}\left[1 - \alpha_S(Q^2)\frac{\beta_1}{\beta_0}\frac{\log(1+L)}{1+L}\right]; \qquad L = a\alpha_S(Q^2)\beta_0 \log\frac{1}{N}$$

which has a branch cut on the real positive N axis for $L \leq -1$, or

$$N \geq N_L \equiv e^{\frac{1}{a\beta_0\alpha_S(Q^2)}}.$$

because of the Landau singularity.

The inverse Mellin transform of $C^{\rm res}(N, \alpha_S(Q^2))$ does not exist.

One possible way out: take the term-by-term inverse Mellin transform of $\tilde{\Sigma}(L, \alpha_s)$:

$$\Sigma(z, \alpha_s(Q^2)) = \sum_{k=1}^{\infty} h_k \bar{\alpha}^k \frac{1}{2\pi i} \oint_{\bar{N}-i\infty}^{\bar{N}+i\infty} dN \, z^{-N} \, \log^k \frac{1}{N}$$

but the series is divergent! Proof:

$$\frac{1}{2\pi i} \oint_{\bar{N}-i\infty}^{\bar{N}+i\infty} dN \, z^{-N} \, \log^k \frac{1}{N} = \frac{k!}{2\pi i} \left[\oint \frac{d\xi}{\xi^{k+1}} \frac{\log^{\xi-1} \frac{1}{z}}{\Gamma(\xi)} \right]_+$$

$$\Sigma(z, \alpha_s(Q^2)) = \frac{1}{2\pi i} \left[\frac{1}{\log \frac{1}{z}} \oint \frac{d\xi}{\xi} \frac{\log^{\xi} \frac{1}{z}}{\Gamma(\xi)} \sum_{k=1}^{\infty} k! h_k \left(\frac{\bar{\alpha}}{\xi} \right)^k \right]_+$$

A second possible way out: taking the inverse Mellin transform of each $\log^k N$ term at the relevant (leading, next-to-leading...) logarithmic level, the perturbative series converges. For example, to leading log accuracy one has

$$\frac{1}{2\pi i}\int_{\bar{N}-i\infty}^{\bar{N}+i\infty} dN\, z^{-N}\, \log^k \frac{1}{N} = k\left[\frac{\log^{k-1}(1-z)}{1-z}\right]_+ + \mathrm{NLL}$$

The series now converges to

$$\Sigma_{\mathrm{LLx}}(z,\alpha_{\mathrm{S}}(Q^2)) = \bar{\alpha}\left[\frac{1}{1-z}\tilde{\Sigma}'(\bar{\alpha}\log(1-z),\alpha_{\mathrm{S}}(Q^2))\right]_+$$

but only for $z < z_L = 1 - e^{-\frac{1}{\bar{\alpha}}}$ again because of the Landau pole at $z = z_L$.

Completely analogous considerations hold in the case of the resummation of large logarithms of q_{T}^2/Q^2 in the small-q_{T} region of the spectrum.

In this case

$$\text{Mellin tr.} \quad \int_0^1 dz\, z^{N-1}\, f(z) \rightarrow \text{Fourier tr.} \quad \frac{1}{2\pi}\int d\vec{b}\, e^{-i\vec{b}\cdot\vec{q}_{\mathrm{T}}}\, f(\vec{q}_{\mathrm{T}})$$

In the space of the Fourier-conjugate variable $\vec{b}$, the $\vec{q}_{\mathrm{T}}$ conservation δ function factorizes:

$$\int d^2q_{\mathrm{T}}\, \delta(\vec{q}_{\mathrm{T}} - \vec{q}_{\mathrm{T1}} - \ldots - \vec{q}_{\mathrm{Tn}})\, e^{i\vec{b}\cdot\vec{q}_{\mathrm{T}}} = \prod_{i=1}^{n} \exp(i\vec{b}\cdot\vec{q}_{\mathrm{Ti}})$$

The resummed cross section in $\vec{b}$ space has no inverse Fourier transform, again because of the Landau pole of the running coupling.

3. Standard solution

An idea of S. Catani, M. Mangano, P. Nason and L. Trentadue*: the minimal prescription. A very simple recipe: just take

$$\sigma(x, Q^2) = \frac{1}{2\pi i} \int_{N_{\rm MP}-i\infty}^{N_{\rm MP}+i\infty} dN\, x^{-N}\, \mathcal{L}(N, Q^2)\, C(N, \alpha_{\rm S}(Q^2))$$

with $0 < N_{\rm MP} < N_L$**.**

This is not a true inverse Mellin: the integrand is not analytical in any right half-plane, because of the branch cut due to the Landau pole.

*[NPB 478(1996)273, hep-ph/9604351]

Nonetheless, the MP has a number of good properties:

- **it is well defined for all values of x**
- **it is an asymptotic sum of the original, divergent perturbative expansion**
- **the difference between the original series, truncated at the best-approximation term, and the minimal prescription, is suppressed more strongly than any power of Λ^2/Q^2.**

A minimal prescription exists also in the case of Fourier inversion for $\vec{q}_T$ distributions*: it corresponds to a deformation of the integration contour away from the real axis.

Minimal in the sense that it gives back the right result when applied to functions of $\vec{b}$ which do have a Fourier inverse.

[The results shown at the beginning of the talk have been obtained using the minimal prescription technique.]

*** E. Laenen, G. Sterman and W. Vogelsang, PRL 84(2000)4296;
A. Kulesza, G. Sterman and W. Vogelsang, PRD 66(2002)014011**

A closer look at the minimal prescription:

$$\sigma(\tau, Q^2) = \frac{1}{2\pi i}\int_{N_{\rm MP}-i\infty}^{N_{\rm MP}+i\infty} dN\,\tau^{-N}\,C^{\rm res}(N, \alpha_{\rm S}(Q^2))\int_0^1 dy\,y^{N-1}\,\mathcal{L}(y, Q^2)$$

$$= \int_0^1 \frac{dy}{y}\,\mathcal{L}(y, Q^2)\,C^{\rm res}\left(\frac{\tau}{y}, \alpha_{\rm S}(Q^2)\right)$$

Looks like a convolution, but the integration region $0 \leq y \leq \tau$ **cannot be excluded: indeed**

$$C^{\rm res}(z, \alpha_{\rm S}(Q^2)) = \frac{1}{2\pi i}\int_{N_{\rm MP}-i\infty}^{N_{\rm MP}+i\infty} dN\,z^{-N}\,C^{\rm res}(N, \alpha_{\rm S}(Q^2))$$

does not vanish for $z > 1$ **because of the Landau cut.**

This is reflected in a difficulty in the numerical implementation of the minimal prescription formula: $C(\tau/y, \alpha_{\mathrm{S}})$ oscillates in the region $y \sim \tau$, where the luminosity is smooth, and large cancellations take place.

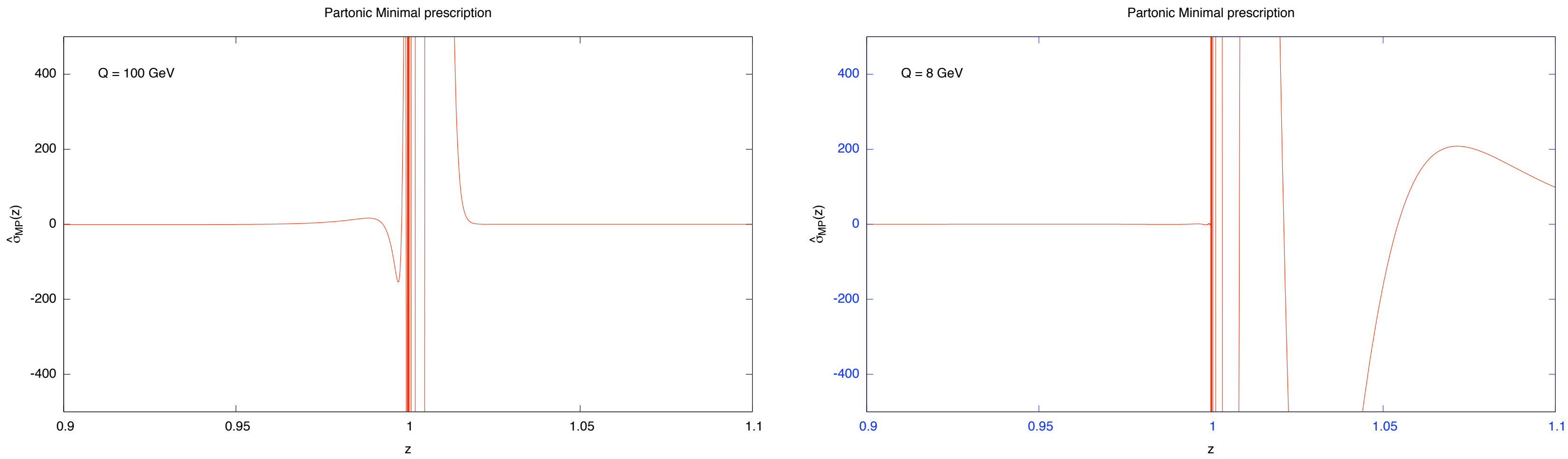


Figure 1: The partonic cross-section $C^{\mathrm{res}}(z, \alpha_{\mathrm{S}}(Q^2))$ computed using the minimal prescription at $\sqrt{Q^2} = 8$ GeV and $\sqrt{Q^2} = 100$ GeV (Drell-Yan NLL).

[M. Bonvini, S. Forte, GR, NPB874 (2011) 93]

One might avoid the problem by simply going back to the original formulation of the MP,

$$\sigma(x, Q^2) = \frac{1}{2\pi i} \int_{N_{\mathrm{MP}} - i\infty}^{N_{\mathrm{MP}} + i\infty} dN\, x^{-N}\, \mathcal{L}(N, Q^2)\, C(N, \alpha_{\mathrm{S}}(Q^2))$$

but $\mathcal{L}(N, Q^2)$ is typically not available.

All these problems are addressed by CMNT in the original paper, and can be overcome by suitable techniques, but it is interesting to explore different possibilities.

4. A different approach

Is it possible to sum the divergent series

$$\Sigma(z,\alpha_{\rm S}(Q^2)) = \frac{1}{2\pi i}\left[\frac{1}{\log\frac{1}{z}}\oint\frac{d\xi}{\xi}\frac{\log^{\xi}\frac{1}{z}}{\Gamma(\xi)}\sum_{k=1}^{\infty}k!h_k\left(\frac{\bar{\alpha}}{\xi}\right)^k\right]_+$$

using the Borel technique?

S. Forte, J. Rojo, M. Ubiali, GR, PLB635(2006)313, hep-ph/0601048

R. Abbate, S. Forte, GR, PLB657(2007)55, arXiv:0707.2452

Consider a generic power series, not necessarily convergent in the Cauchy sense:

$$f(\bar{\alpha}) = \sum_{k=1}^{\infty} f_k \, \bar{\alpha}^k$$

and define its Borel transform

$$\hat{f}(w) = \sum_{k=1}^{\infty} f_k \, \frac{w^{k-1}}{(k-1)!}$$

Because of the factor $(k-1)!$, the Borel transformed series $\hat{f}(w)$ has much better convergence properties. An inverse transformation exists, since

$$\int_0^{+\infty} dw \, e^{-\frac{w}{\bar{\alpha}}} \, w^{k-1} = (k-1)! \, \bar{\alpha}^k \rightarrow f_B(\bar{\alpha}) = \int_0^{+\infty} dw \, e^{-\frac{w}{\bar{\alpha}}} \, \hat{f}(w)$$

Various cases:

1. **The original series is convergent in the usual sense. Then $f_B(\bar{\alpha}) = f(\bar{\alpha})$, but the Borel sum may enlarge the convergence region. Example:**

$$f(\bar{\alpha}) = \sum_{k=1}^{\infty} \bar{\alpha}^k = \frac{\bar{\alpha}}{1-\bar{\alpha}}, \qquad |\bar{\alpha}| < 1$$

$$f_B(\bar{\alpha}) = \int_0^{+\infty} dw\, e^{-\frac{w}{\bar{\alpha}}}\, e^w = \frac{\bar{\alpha}}{1-\bar{\alpha}} \qquad \mathrm{Re}\,\bar{\alpha} < 1$$

2. **The original series is divergent, but the Borel sum exists:**

$$f(\bar{\alpha}) = \sum_{k=1}^{\infty} (-1)^{k-1}\, (k-1)!\, \bar{\alpha}^k \qquad \hat{f}(w) = \frac{1}{1+w}$$

$$f_B(\bar{\alpha}) = \int_0^{+\infty} dw\, e^{-\frac{w}{\bar{\alpha}}}\, \frac{1}{1+w} < \infty \qquad \mathrm{Re}\,\bar{\alpha} > 0$$

3. The original series is divergent, its Borel transform exists, but it has a singularity in the range of the inversion integral:

$$f(\bar{\alpha}) = \sum_{k=1}^{\infty} (k-1)!\, \bar{\alpha}^k \qquad \hat{f}(w) = \frac{1}{1-w}$$

$$f_B(\bar{\alpha}) = \int_0^{+\infty} dw\, e^{-\frac{w}{\bar{\alpha}}}\, \frac{1}{1-w}$$

This is the case e.g. of renormalons.

Back to

$$C^{\rm res}(N,\alpha_{\rm S}(Q^2)) = 1 + \tilde{\Sigma}(L,\alpha_{\rm S}(Q^2))$$

We have

$$\Sigma(z,\alpha_{\rm S}(Q^2)) \equiv \frac{1}{2\pi i}\int_{\bar{N}-i\infty}^{\bar{N}+i\infty} dN\, z^{-N}\tilde{\Sigma}(L,\alpha_{\rm S}(Q^2)) = \left[\frac{R(z)}{\log\frac{1}{z}}\right]_+$$

$$R(z) = \frac{1}{2\pi i}\oint\frac{d\xi}{\xi}\,\frac{\log^{\xi}\frac{1}{z}}{\Gamma(\xi)}\sum_{k=1}^{\infty}k!\,h_k\left(\frac{\bar{\alpha}}{\xi}\right)^k$$

The Borel transform of $R(z)$ with respect to $\bar{\alpha}$ is found replacing

$$\bar{\alpha}^k \to \frac{w^{k-1}}{(k-1)!}$$

and it is convergent:

$$\hat{R}(w,z) = \frac{1}{2\pi i}\oint\frac{d\xi}{\xi}\,\frac{\log^{\xi}\frac{1}{z}}{\Gamma(\xi)}\sum_{k=1}^{\infty}k\,h_k\,\frac{w^{k-1}}{\xi^k} = \frac{1}{2\pi i}\oint\frac{d\xi}{\xi}\,\frac{\log^{\xi}\frac{1}{z}}{\Gamma(\xi)}\,\frac{d}{dw}\tilde{\Sigma}\left(\frac{w}{\xi},\alpha_{\rm S}(Q^2)\right)$$

The branch cut of $\tilde{\Sigma}(L, \alpha_{\rm S}(Q^2))$, $-\infty < L \leq -1$, is mapped onto the range $-w \leq \xi \leq 0$ on the real axis of the complex ξ plane. Hence, the ξ integration path is any closed curve which encircles the cut.

As a consequence, the inverse Borel transform of $\hat{R}$ does not exist, because w integration is divergent at $+\infty$.

We introduce a cutoff:

$$
\begin{aligned}
R_C(z) &= \int_0^C dw\, e^{-\frac{w}{\bar{\alpha}}} \hat{R}(w, z) \\
&= \frac{1}{2\pi i} \oint \frac{d\xi}{\xi} \frac{\log^{\xi} \frac{1}{z}}{\Gamma(\xi)} \int_0^C dw\, e^{-\frac{w}{\bar{\alpha}}} \frac{d}{dw} \tilde{\Sigma}\left(\frac{w}{\xi}, \alpha_{\rm S}(Q^2)\right) \\
\Sigma_I(z, \alpha_{\rm S}(Q^2)) &= \frac{1}{2\pi i} \oint \frac{d\xi}{\xi} \frac{1}{\Gamma(\xi)} \int_0^C dw\, e^{-\frac{w}{\bar{\alpha}}} \frac{d}{dw} \tilde{\Sigma}\left(\frac{w}{\xi}, \alpha_{\rm S}(Q^2)\right) \left[\log^{\xi - 1} \frac{1}{z}\right]_+
\end{aligned}
$$

Remarks:

- **The original divergent series for Σ is asymptotic to the function $\Sigma_I(z, \alpha_{\mathrm{S}}(Q^2))$**
- **For any finite-order truncation of the divergent series, the full and cutoff results differ by a twist–$\left(2 + \frac{2C}{a}\right)$.**
- **C can be chosen freely in the range $C \geq a$; different choices differ by power suppressed terms. The minimal choice is $C = a$.**

A somewhat simpler result is obtained if the Borel transform is performed through the replacement

$$\bar{\alpha}^k \to \frac{1}{\bar{\alpha}} \frac{w^k}{k!}$$

In this case one gets

$$R_C(z) = \frac{1}{2\pi i} \oint \frac{d\xi}{\xi} \frac{\log^\xi \frac{1}{z}}{\Gamma(\xi)} \int_0^C \frac{dw}{\bar{\alpha}} \, e^{-\frac{w}{\bar{\alpha}}} \tilde{\Sigma}\left(\frac{w}{\xi}, \alpha_{\mathrm{S}}(Q^2)\right)$$

$$\Sigma_I(z, \alpha_{\mathrm{S}}(Q^2)) = \frac{1}{2\pi i} \oint \frac{d\xi}{\xi} \frac{1}{\Gamma(\xi)} \int_0^C \frac{dw}{\bar{\alpha}} \, e^{-\frac{w}{\bar{\alpha}}} \tilde{\Sigma}\left(\frac{w}{\xi}, \alpha_{\mathrm{S}}(Q^2)\right) \left[\log^{\xi-1} \frac{1}{z}\right]_+$$

which provides an equally good resummation prescription; the difference is in practice very small.

If we only wish to retain terms which do not vanish as $z \to 1$ we may expand

$$\log \frac{1}{z} = 1 - z + O((1-z)^2)$$

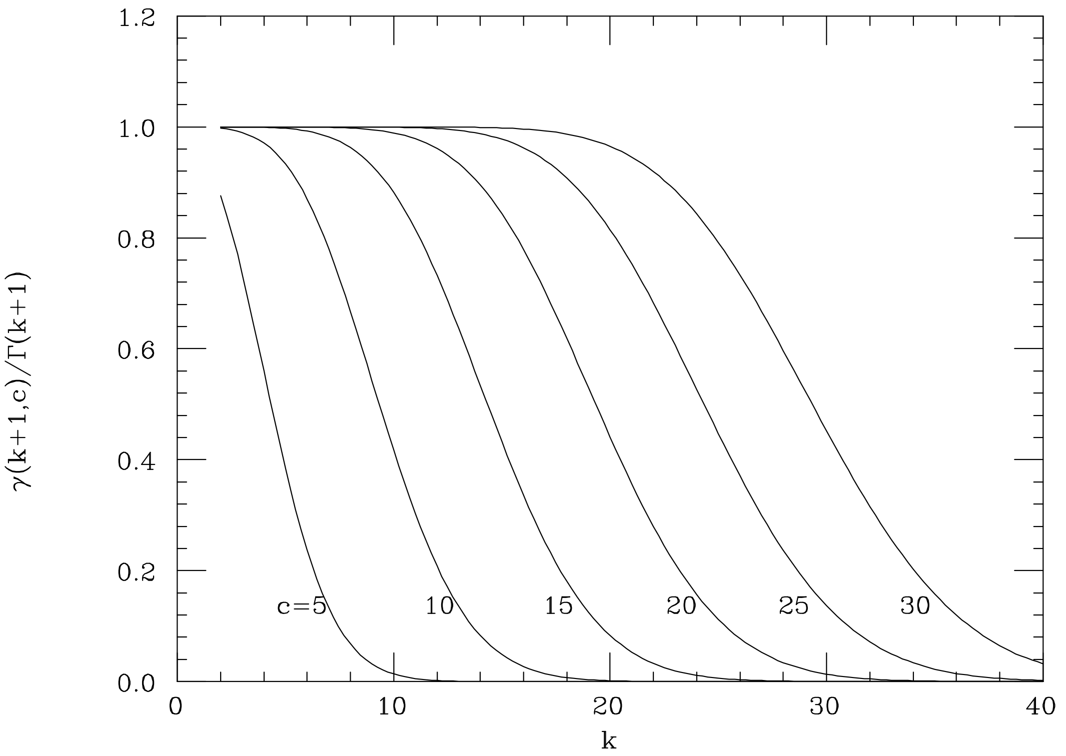
with the result

$$\Sigma_{II}(z, \alpha_{\rm S}(Q^2)) = \frac{1}{2\pi i} \oint \frac{d\xi}{\xi} \frac{1}{\Gamma(\xi)} \int_0^C \frac{dw}{\bar{\alpha}} \, e^{-\frac{w}{\bar{\alpha}}} \tilde{\Sigma} \left(\frac{w}{\xi}, \alpha_{\rm S}(Q^2) \right) \left[(1-z)^{\xi - 1} \right]_+$$

The Borel prescription effectively replaces

$$h_k \to h_k \frac{\gamma(k+1, C/\bar{\alpha})}{\Gamma(k+1)}; \quad \gamma(k+1, c) = \int_0^c dt\, e^{-t} t^k$$

thereby damping high orders:



Subleading terms

Different prescriptions give different results for two reasons:

- **The different way they handle the high-order behaviour of the divergent series. This makes in practice a small difference unless τ is close to the Landau pole (very rare).**

  Example: Borel prescription with $C = 2$, $\alpha_S = 0.11$, then $c \approx 15$, and the perturbative expansion is truncated around $k \sim 15$.

- **The different class of subleading terms which are introduced when performing the resummation.**

  Example: the minimal prescription just gives the exact Mellin inverse of any truncation of the series. Because this result depends on z through $\log \frac{1}{z}$, in z space it generates a series of power suppressed terms.

We have now two versions of the Borel prescription:

$$\Sigma_I(z, \alpha_{\rm S}(Q^2)) = \frac{1}{2\pi i} \oint \frac{d\xi}{\xi} \frac{1}{\Gamma(\xi)} \int_0^C \frac{dw}{\bar{\alpha}} e^{-\frac{w}{\bar{\alpha}}} \tilde{\Sigma}\left(\frac{w}{\xi}, \alpha_{\rm S}(Q^2)\right) \left[\log^{\xi-1} \frac{1}{z}\right]_+$$

$$\Sigma_{II}(z, \alpha_{\rm S}(Q^2)) = \frac{1}{2\pi i} \oint \frac{d\xi}{\xi} \frac{1}{\Gamma(\xi)} \int_0^C \frac{dw}{\bar{\alpha}} e^{-\frac{w}{\bar{\alpha}}} \tilde{\Sigma}\left(\frac{w}{\xi}, \alpha_{\rm S}(Q^2)\right) \left[(1-z)^{\xi-1}\right]_+$$

They differ by non-logarithmically-enhanced terms (their Mellin transforms differ by terms suppressed by powers of $\frac{1}{N}$).

Version I is closer to the minimal prescription: when applied to individual $\log^k \frac{1}{N}$ terms, it gives back the exact Mellin transform (apart from the suppression factor).

Two opposite extreme choices in the treatment of subleading terms:

Version I: no $1/N$ power-suppressed terms, hence $1-z$ power suppressed terms in z space appear;

Version II: the opposite.

However, with the Borel prescription the z dependence is under analytic control: it is entirely contained in the factor $\log^{\xi-1} \frac{1}{z}$ and can be modified at will.

We may therefore consider intermediate options. For example, one can check that at NLO and NNLO the inclusion of some subleading terms by replacing

$$\log(1-z) \to \log\frac{1-z}{\sqrt{z}}$$

provides better agreement with the full result

This can be understood: soft resummation arises from the kinematic fact that as $z \to 1$ the dependence of partonic cross-sections on z is always in the combination $Q^2(1-z)^2$, which is the upper limit of the integral over the energy of radiated gluons for DY. However, the actual value is

$$k^0_{\max} = \sqrt{\frac{Q^2(1-z)^2}{4z}}$$

It is easy to do so by the Borel prescription: we define

$$\Sigma_{III}(z, \alpha_{\rm S}(Q^2)) = \frac{1}{2\pi i} \oint \frac{d\xi}{\xi} \frac{1}{\Gamma(\xi)} \int_0^C \frac{dw}{\bar{\alpha}} e^{-\frac{w}{\bar{\alpha}}} \tilde{\Sigma}\left(\frac{w}{\xi}, \alpha_{\rm S}(Q^2)\right) \left[(1-z)^{\xi-1}\right]_+ z^{-\frac{\xi}{2}}.$$

With this choice, the kinematic correction is automatically included to all orders.

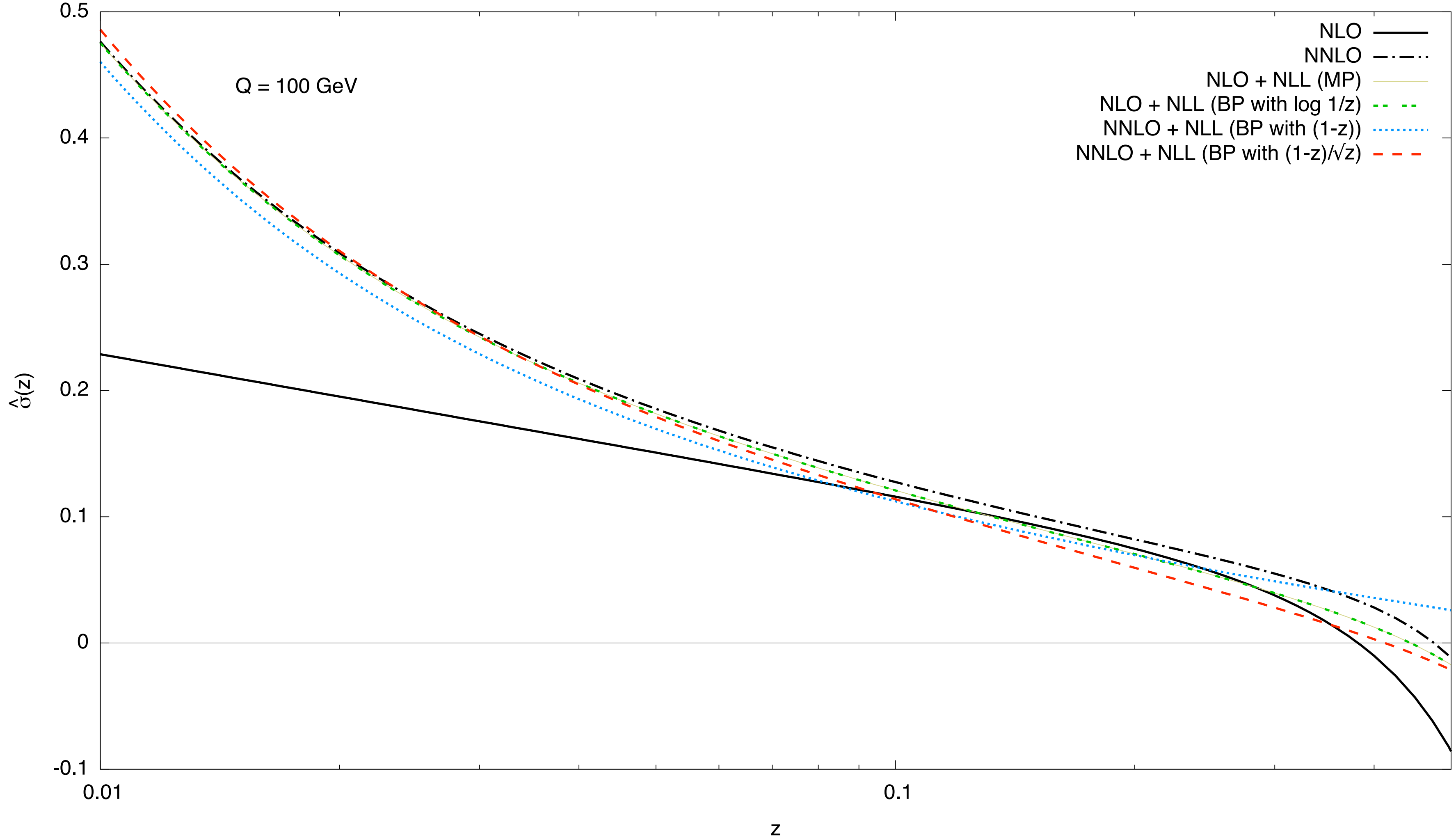
In fact, III closer than I to the MP because

$$\log\frac{1}{z} = \frac{1-z}{\sqrt{z}}\left(1 + O((1-z)^2)\right),$$

so that

$$\frac{\log^k \log\frac{1}{z}}{\log\frac{1}{z}} = \frac{\sqrt{z}}{1-z} \log^k \frac{1-z}{\sqrt{z}} \left(1 + O((1-z)^2)\right).$$

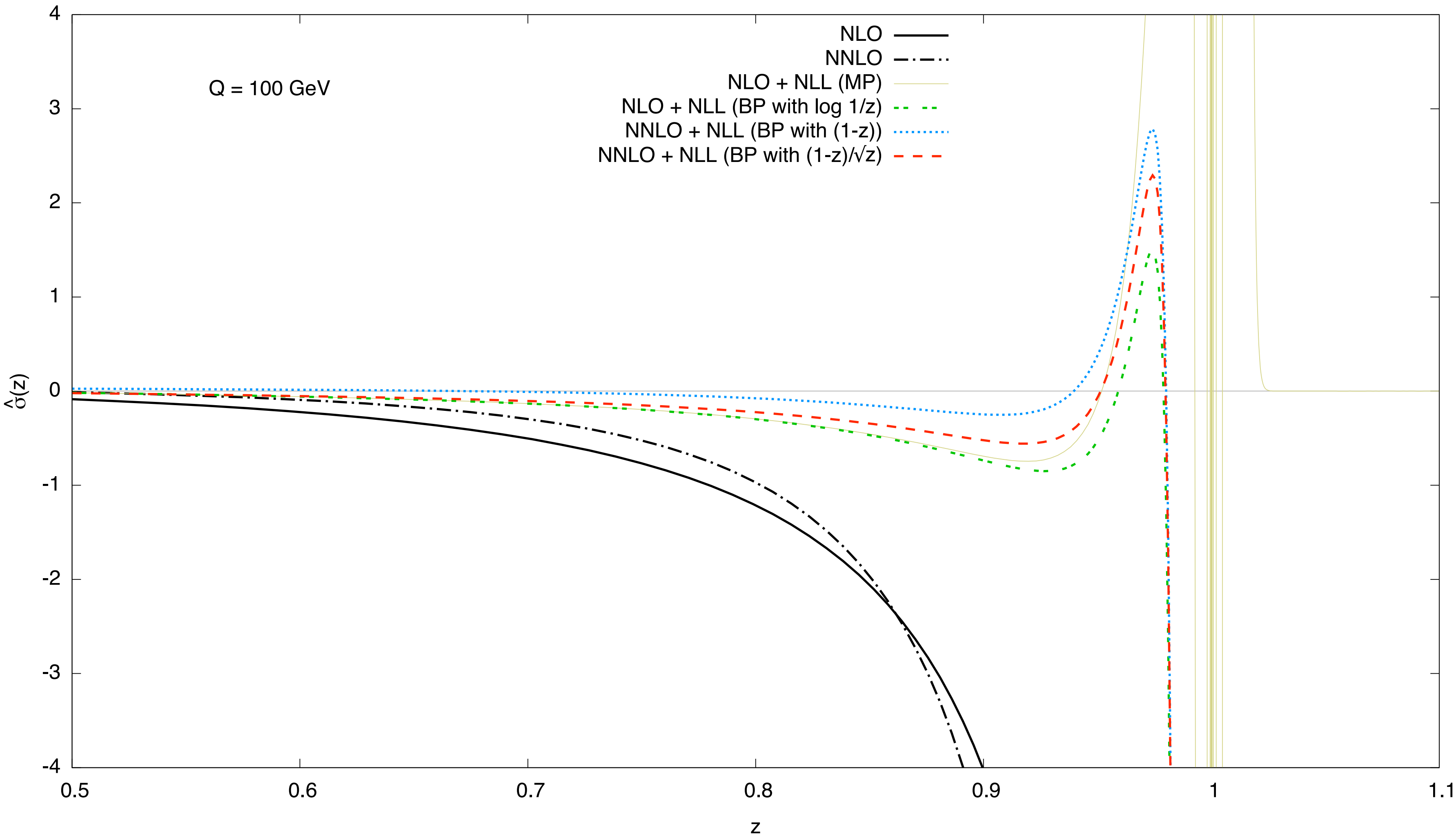
This shows that, up to terms suppressed by two powers of $1-z$, the MP effectively performs the kinematic subleading replacement (but at the same time introduces an overall factor $\sqrt{z}$ which is absent in the true coefficients).



Q = 100 GeV
NLO
NNLO
NLO + NLL (MP)
NLO + NLL (BP with log 1/z)
NNLO + NLL (BP with (1-z))
NNLO + NLL (BP with (1-z)/√z)
0.5
0.4
0.3
0.2
0.1
0
-0.1
0.01
0.1
z
σ̂(z)

Comments:

- **MP and BP-I (BP with $\log 1/z$) essentially indistinguishable (for values of $z \lesssim 0.9$, where the MP starts oscillating)**
- **BP-II (BP with $1 - z$) rather different from them (and unconfortably large in the intermediate region).**
- **BP-III prescription (BP with $(1 - z)/\sqrt{z}$) as expected differs less from the MP in the whole range. The difference between MP and BP-III sizable, but smaller than the size of resummation where resummation is relevant, and induces small corrections at small z. A reliable estimate of the ambiguity in the resummation.**
- **Interesting interplay between large-z and small-z behaviour.**



NLO
NNLO
NLO + NLL (MP)
NLO + NLL (BP with log 1/z)
NNLO + NLL (BP with (1-z))
NNLO + NLL (BP with (1-z)/√z)
Q = 100 GeV
σ̂(z)
z
4
3
2
1
0
-1
-2
-3
-4
0.5
0.6
0.7
0.8
0.9
1
1.1

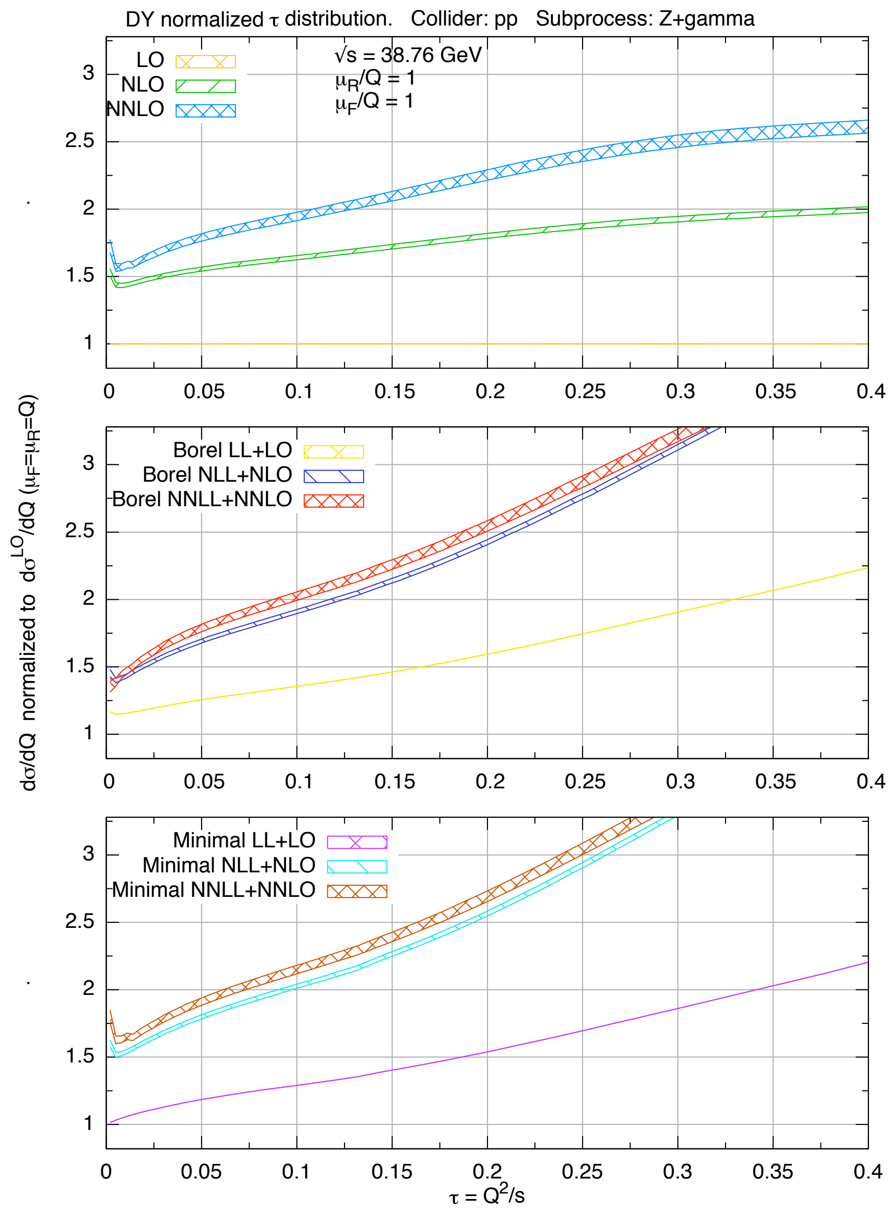
Summary and outlook

- There are ambiguities in the computation of observables from resummed quantities in QCD, to be ascribed to the presence of a Landau singularity in the running coupling.
- A prescription based on Borel sum and twist expansion can be given; it gives better control over subleading terms with respect to the minimal prescription.

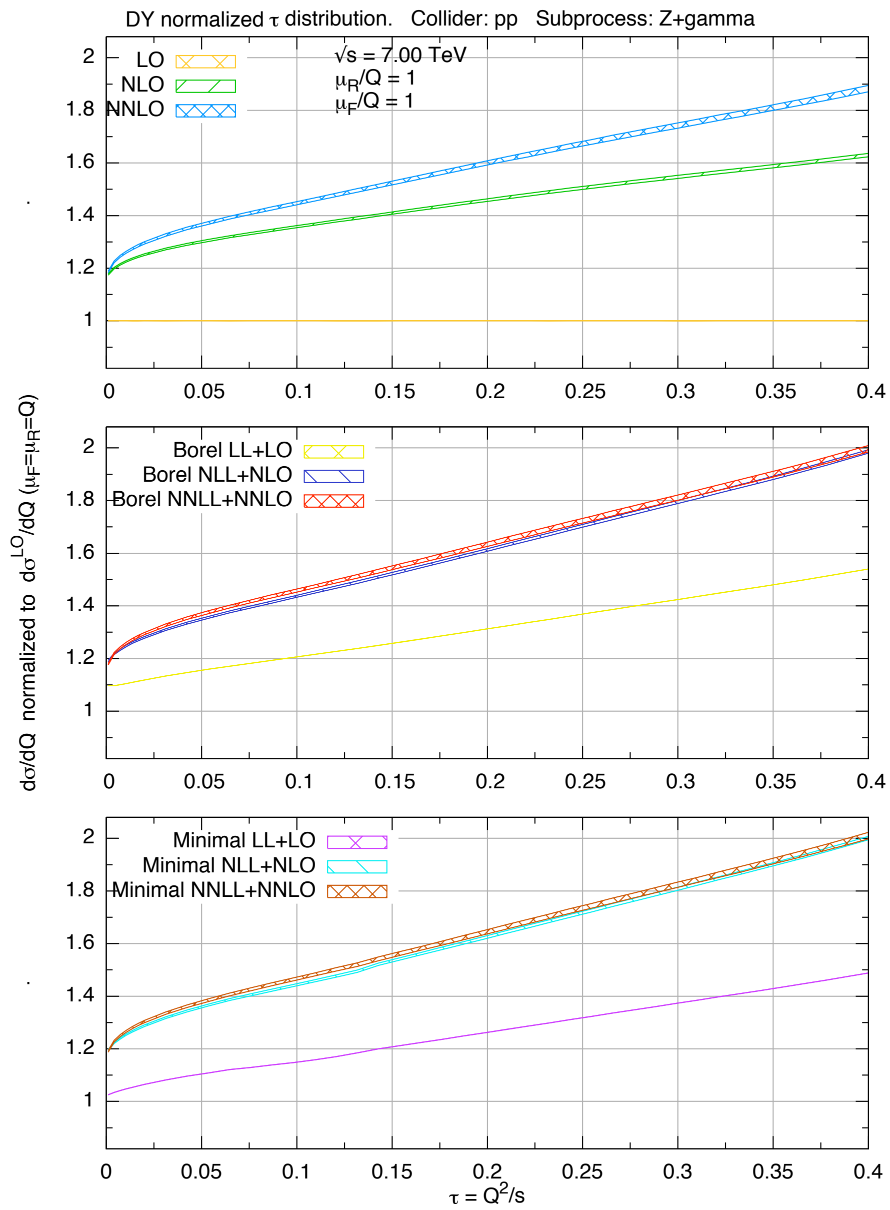
antitative.

- Work in progress: combined small-x and large-x resummation. Comparison with other resummation approaches.

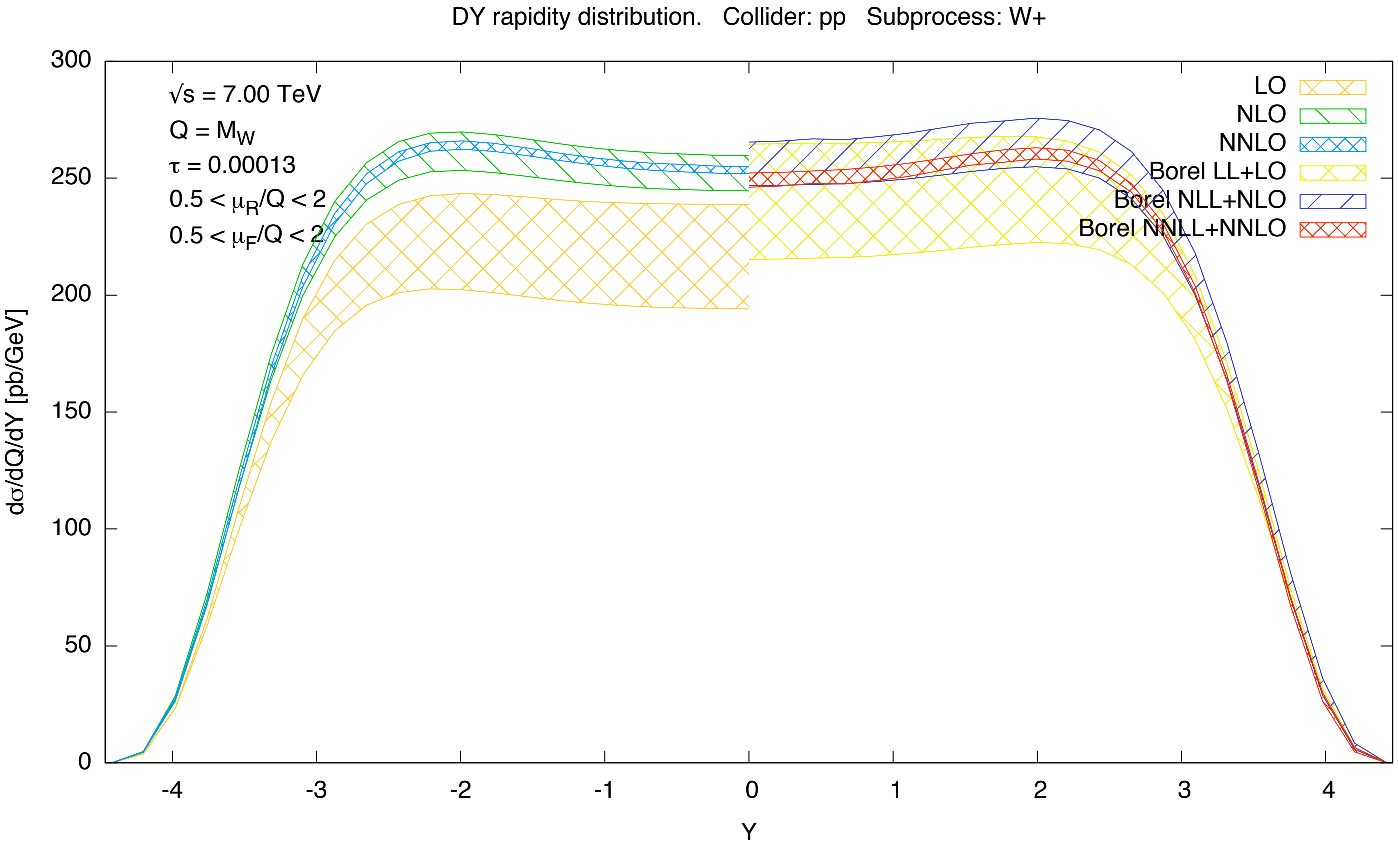
† M. Bonvini, S. Forte, T. Peraro, GR, in preparation

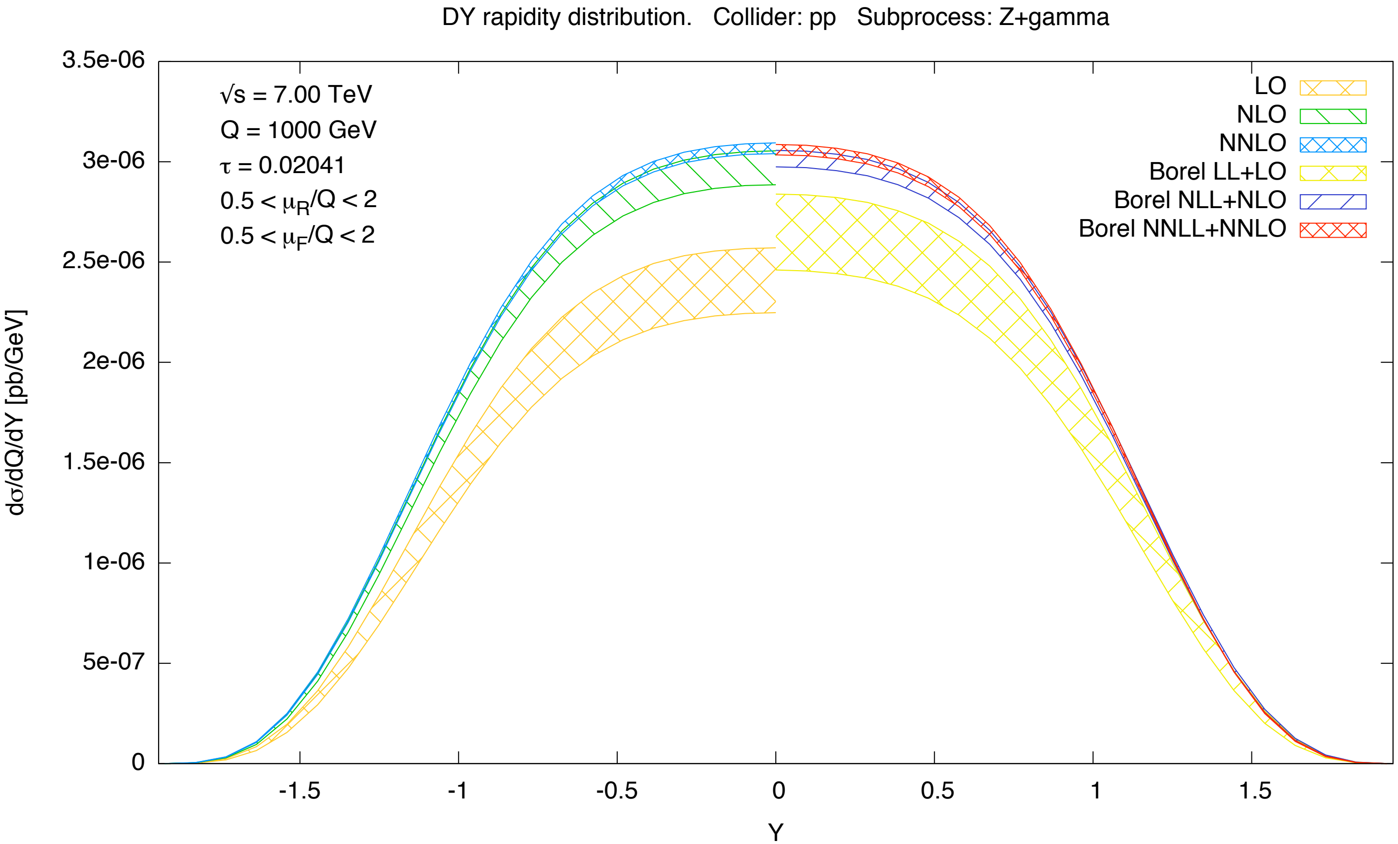


DY normalized τ distribution. Collider: pp Subprocess: Z+gamma
LO
NLO
NNLO
√s = 38.76 GeV
μR/Q = 1
μF/Q = 1
Borel LL+LO
Borel NLL+NLO
Borel NNLL+NNLO
Minimal LL+LO
Minimal NLL+NLO
Minimal NNLL+NNLO
dσ/dQ normalized to dσ^LO/dQ (μF=μR=Q)
τ = Q²/s



DY normalized τ distribution. Collider: pp Subprocess: Z+gamma
√s = 7.00 TeV
μR/Q = 1
μF/Q = 1
LO
NLO
NNLO
Borel LL+LO
Borel NLL+NLO
Borel NNLL+NNLO
Minimal LL+LO
Minimal NLL+NLO
Minimal NNLL+NNLO
dσ/dQ normalized to dσ^LO/dQ (μF=μR=Q)
τ = Q^2/s


DY rapidity distribution. Collider: pp Subprocess: W+
√s = 7.00 TeV
Q = M_W
τ = 0.00013
0.5 < μ_R/Q < 2
0.5 < μ_F/Q < 2
LO
NLO
NNLO
Borel LL+LO
Borel NLL+NLO
Borel NNLL+NNLO
dσ/dQ/dY [pb/GeV]
Y
300
250
200
150
100
50
0
-4
-3
-2
-1
0
1
2
3
4


DY rapidity distribution. Collider: pp Subprocess: Z+gamma
√s = 7.00 TeV
Q = 1000 GeV
τ = 0.02041
0.5 < μR/Q < 2
0.5 < μF/Q < 2
LO
NLO
NNLO
Borel LL+LO
Borel NLL+NLO
Borel NNLL+NNLO
dσ/dQ/dY [pb/GeV]
Y
3.5e-06
3e-06
2.5e-06
2e-06
1.5e-06
1e-06
5e-07
0
-1.5
-1
-0.5
0
0.5
1
1.5