IV. The superspace formalism

① Superspace, supercharges, superderivatives

- Susy model building require often long and tedious calculations (e.g. the easy Wess-Zumino model...), especially when we add the gauge interactions of the chiral supermultiplets.

We have also seen that the commutator of two susy transformations yields a translation.

$$\boxed{\text{The idea of the superspace formalism is to represent supersymmetry as generalized translations.}}$$

- A finite translation applied on a field φ gives:

$$\varphi(x) \longrightarrow e^{+ia_\mu P^\mu}\varphi(x) \qquad (1)$$

The first step is to include the supercharges $(Q_\alpha, \bar{Q}^{\dot\alpha})$.

▷ Since P_μ is associated to the spacetime coordinates x_μ (4-vector is associated to a 4-vector), we will associated a Majorana spinor to the supercharges.

$$\boxed{\begin{array}{ccc} P_\mu & \longleftrightarrow & x_\mu \\ (Q_\alpha, \bar{Q}^{\dot\alpha}) & \longleftrightarrow & (\theta_\alpha, \bar\theta^{\dot\alpha}) \end{array}}$$

$\Rightarrow$ We can then define generalized coordinates in the "superspace":

$$\boxed{X \doteq (x^\mu, \theta_\alpha, \bar\theta^{\dot\alpha})}$$

- The superspace is defined formally as G/H where

$$\begin{cases} G \equiv \text{superpoincaré group} \\ H \equiv \text{Lorentz group} \end{cases}$$

We will ~~represent~~ all the translations in the superspace (the elements of G/H) as

$$\boxed{G(x, \theta, \bar\theta) \doteq e^{i(x_\mu P^\mu + \theta Q + \bar Q \cdot \bar\theta)}}$$

- The pure translation of eq (1) is recovered by $G(a, 0, 0)$
- A pure SUSY transformation is given by $G(0, \epsilon, \bar\epsilon)$
- We recall that $P_\mu = i\partial_\mu$. However, we now need to compute the representations of the supercharges. To this aim, we will calculate the action (to the left) of a SUSY transformation on a group element:

$$G(0, \epsilon, \bar\epsilon)\, G(x, \theta, \bar\theta)$$

Using the Baker-Campbell-Hausdorff identity

$$e^A e^B = e^{A + B + \frac{1}{2}[A, B]}$$

and the superalgebra of Poincaré, we can derive (Ex IV.1)

$$G(0,\varepsilon,\bar\varepsilon)\,G(x,\theta,\bar\theta) = e^{i\left[(x^\mu + i\varepsilon\sigma^\mu\bar\theta - i\theta\sigma^\mu\bar\varepsilon)P_\mu + (\theta+\varepsilon)Q + \bar Q(\bar\theta+\bar\varepsilon)\right]}$$

The variations of the coordinates are then:

$$\begin{cases} \delta_L x^\mu = i(\varepsilon\sigma^\mu\bar\theta - \theta\sigma^\mu\bar\varepsilon) = \left[i(\varepsilon\sigma^\nu\bar\theta - \theta\sigma^\nu\bar\varepsilon)\partial_\nu, x^\nu\right] \\ \delta_L \theta^\alpha = \varepsilon^\alpha \quad = [\varepsilon\cdot\partial, \theta^\alpha] \\ \delta_L \bar\theta^{\dot\alpha} = \bar\varepsilon^{\dot\alpha} \quad = -[\bar\partial\cdot\bar\varepsilon, \bar\theta^{\dot\alpha}] \end{cases} \qquad (2)$$

where we have used:

$$\begin{cases} \triangleright\ [a^\nu\partial_\nu, x^\mu] = a^\mu \\ \triangleright\ \partial_\alpha = \dfrac{\partial}{\partial\theta^\alpha} \\ \triangleright\ \bar\partial_{\dot\alpha} = \dfrac{\partial}{\partial\bar\theta^{\dot\alpha}} \end{cases}$$

By definition of an action, we have also:

$$\begin{cases} \delta_L x^\mu = [i(\varepsilon Q + \bar Q\bar\varepsilon), x^\mu] \\ \delta_L \theta^\alpha = [i(\varepsilon Q + \bar Q\bar\varepsilon), \theta^\alpha] \\ \delta_L \bar\theta^{\dot\alpha} = [i(\varepsilon Q + \bar Q\bar\varepsilon), \bar\theta^{\dot\alpha}] \end{cases} \qquad (3)$$

Comparing (2) and (3), one obtains

$$\boxed{\begin{aligned} Q_{L\alpha} &= -i\partial_\alpha + (\sigma^\mu\bar\theta)_\alpha\partial_\mu \\ \bar Q_{L\dot\alpha} &= i\bar\partial_{\dot\alpha} - (\theta\sigma^\mu)_{\dot\alpha}\partial_\mu \end{aligned}}$$

and $(Q_{L\alpha})^\dagger = \bar Q_{L\dot\alpha}$

- Similarly, we could have started from an action to the right:

$$G(x,\theta,\bar\theta)\,G(0,\varepsilon,\bar\varepsilon)$$

$$\Rightarrow \boxed{\begin{aligned} Q_{R\alpha} &= -i\partial_\alpha - (\sigma^\mu\bar\theta)_\alpha\partial_\mu \\ \bar Q_{R\dot\alpha} &= +i\bar\partial_{\dot\alpha} + (\theta\sigma^\mu)_{\dot\alpha}\partial_\mu \end{aligned}} \quad \text{and} \quad (Q_{R\alpha})^\dagger = \bar Q_{R\dot\alpha}.$$

(& IV 2)

after identifying with (3).

- We have hence two sets of representations for the supercharges. One will be identified to the supercharges themselves and the other one to the superderivatives:

$$\boxed{\begin{cases} Q_\alpha \doteq Q_{L\alpha} \\ \bar Q_{\dot\alpha} \doteq \bar Q_{L\dot\alpha} \end{cases} \qquad \begin{cases} D_\alpha \doteq i\,Q_{R\alpha} \\ \bar D_{\dot\alpha} \doteq -i\,\bar Q_{R\dot\alpha} \end{cases}}$$

We can check that (& IV 3):

- $\{Q_\alpha, \bar Q_{\dot\alpha}\} = 2i\sigma^\mu{}_{\alpha\dot\alpha}\partial_\mu = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu$
- $\{D_\alpha, \bar D_{\dot\alpha}\} = -2i\sigma^\mu{}_{\alpha\dot\alpha}\partial_\mu = -2\sigma^\mu_{\alpha\dot\alpha}P_\mu$

where we have used $\begin{cases} \{\partial_\alpha, \theta^\beta\} = \delta_\alpha{}^\beta \\ \{\bar\partial_{\dot\alpha}, \bar\theta^{\dot\beta}\} = \delta_{\dot\alpha}{}^{\dot\beta} \end{cases}$

$\Rightarrow$ the supercharges span the SUSY algebra with the good sign whilst the superderivatives do it with the opposite sign. This is not surprizing since the superderivatives are associated to an action to the right (contrary to eq (3)).

- Moreover: Q's and D's anticommute, so do the actions to the left and to the right.

$$\Rightarrow \quad \boxed{\begin{aligned} Q_\alpha &= -i\partial_\alpha + (\sigma^\mu\bar\theta)_\alpha \partial_\mu \\ \bar Q_{\dot\alpha} &= +i\bar\partial_{\dot\alpha} - (\theta\sigma^\mu)_{\dot\alpha} \partial_\mu \\ D_\alpha &= \partial_\alpha - i(\sigma^\mu\bar\theta)_\alpha \partial_\mu \\ \bar D_{\dot\alpha} &= \bar\partial_{\dot\alpha} + i(\theta\sigma^\mu)_{\dot\alpha} \partial_\mu \end{aligned}}$$

(2) Grassmann variables

In this section, we present all the identities which will be used in the rest of that chapter. The proof is given in Ex. IV. 4.

(a) $\theta^\alpha\theta^\beta = -\frac{1}{2}\,\theta\cdot\theta\,\varepsilon^{\alpha\beta}$

$\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = \frac{1}{2}\,\bar\theta\bar\theta\,\varepsilon^{\dot\alpha\dot\beta}$

$\theta^\alpha\bar\theta^{\dot\alpha} = \frac{1}{2}\,\theta\sigma^\mu\bar\theta\,\bar\sigma_\mu^{\dot\alpha\alpha}$

This is the basis of all the Grassmann variable algebra.

(b) $\partial\cdot\partial\,(\theta\theta) = -4$

$\bar\partial\cdot\bar\partial\,(\bar\theta\bar\theta) = -4$

(c) $(\psi_1\cdot\theta)\,(\psi_2\theta) = -\frac{1}{2}\,\theta\theta\,\psi_1\cdot\psi_2$

$(\theta\lambda)\,(\bar\theta\bar\varepsilon) = -\frac{1}{2}\,\theta\sigma^\mu\bar\theta\,\bar\varepsilon\bar\sigma_\mu\lambda$

(d) $\theta\sigma^\mu\bar\varepsilon\,\,\theta\psi = -\frac{1}{2}\,\theta\theta\,\,\psi\sigma^\mu\bar\varepsilon$

$\varepsilon\sigma^\mu\bar\theta\,\,\bar\theta\bar\psi = -\frac{1}{2}\,\bar\theta\bar\theta\,\,\varepsilon\sigma^\mu\bar\psi$

(e) $\varepsilon\sigma^\mu\bar\theta \;\; \theta\psi = -\frac{1}{2}\,\theta\sigma^\mu\bar\theta \;\; \varepsilon\psi - \theta\sigma_\nu\bar\theta \;\; \varepsilon\sigma^{\mu\nu}\psi$

$\theta\sigma^\mu\bar\varepsilon \;\; \bar\theta\bar\psi = -\frac{1}{2}\,\theta\sigma^\mu\bar\theta \;\; \bar\varepsilon\cdot\bar\psi + \theta\sigma_\nu\bar\theta \;\; \bar\varepsilon\bar\sigma^{\mu\nu}\bar\psi$

$\theta\sigma^\mu\bar\theta \;\; \theta\psi = -\frac{1}{2}\,\theta\theta \;\; \psi\sigma^\mu\bar\theta$

$\theta\sigma^\mu\bar\varepsilon \;\; \theta\sigma^\nu\bar\theta = \frac{1}{2}\,\theta\theta \;\; \bar\theta\bar\varepsilon \; \eta^{\mu\nu} + \theta\theta \;\; \bar\theta\bar\sigma^{\nu\mu}\bar\varepsilon$

$\theta\sigma^\mu\bar\theta \;\; \theta\sigma^\nu\bar\theta = \frac{1}{2}\,\theta\theta \;\; \bar\theta\bar\theta \; \eta^{\mu\nu}$

(f) $(\sigma^\mu\bar\theta)_\alpha \;\; \theta\sigma^\nu\bar\theta = \frac{1}{2}\,\bar\theta\bar\theta \;\; (\sigma^\mu\bar\sigma^\nu\theta)_\alpha$

$(\sigma^\mu\bar\theta)_\alpha \;\; \bar\theta\bar\lambda = -\frac{1}{2}\,\bar\theta\bar\theta \;\; (\sigma^\mu\bar\lambda)_\alpha$

(g) $(\theta\sigma^\mu\bar\sigma^\nu)^\alpha\,(\sigma^\rho\bar\sigma^\sigma\theta)_\alpha = \frac{1}{2}\,\mathrm{Tr}(\sigma^\mu\bar\sigma^\nu\sigma^\rho\bar\sigma^\sigma)\,\theta\theta$

$$= \theta\theta\left(\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\nu\rho}\eta^{\mu\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} - i\,\varepsilon^{\mu\nu\rho\sigma}\right)$$

3) Generic superfields

- We call superfield any function of the superspace coordinates $\Phi(X)$. This function is usually expanded as a series in terms of the Grassmann variables. Since θ and $\bar\theta$ are fermions the number of coefficients of the series is finite:
  - θ^α, $\bar\theta^{\dot\alpha}$
  - $\theta^\alpha\theta^\beta \to -\frac{1}{2}\theta\theta\,\varepsilon^{\alpha\beta}$, $\quad\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} \to \frac{1}{2}\bar\theta\bar\theta\,\varepsilon^{\dot\alpha\dot\beta}$, $\quad\theta^\alpha\bar\theta^{\dot\alpha} \to \frac{1}{2}\,\theta\sigma^\mu\bar\theta\,\bar\sigma_\mu^{\dot\alpha\alpha}$
  - $\theta^\alpha\theta^\beta\bar\theta^{\dot\alpha} \to -\frac{1}{2}\,\theta\theta\,\bar\theta^{\dot\alpha}\varepsilon^{\alpha\beta}$, $\quad\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}\theta^\alpha \to \frac{1}{2}\bar\theta\bar\theta\,\theta^\alpha\,\varepsilon^{\dot\alpha\dot\beta}$
  - $\theta^\alpha\theta^\beta\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} \to -\frac{1}{4}\,\theta\theta\,\bar\theta\bar\theta\,\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}$
  - all higher order combinations vanish.

$\Rightarrow$ The most general superfield can thus be written as:

$$\boxed{\begin{aligned}\Phi(X) = {} & z(x) + \theta\cdot\xi(x) + \bar\theta\bar\xi + \theta\theta f(x) + \bar\theta\bar\theta g(x) + \theta\sigma^\mu\bar\theta A_\mu(x) \\ & + \bar\theta\bar\theta\,\theta\omega(x) + \theta\theta\,\bar\theta\bar\rho(x) + \theta\theta\bar\theta\bar\theta\, d(x)\end{aligned}} \quad (1)$$

This defines a scalar superfield. Similarly, Φ can carry Lorentz or spin indices.

- The field content associated to the superfield of eq. (1) is given by:

  - 4 complex scalar fields : z, f, g, d $\rightarrow$ 8 degrees of freedom
  - 1 complex vector field : A_μ $\rightarrow$ 8 degrees of freedom
  - 4 complex Weyl spinors : $\xi, \bar\xi, \omega, \bar\rho$ $\rightarrow$ 16 degrees of freedom

$\Rightarrow$ $\underline{n_B = n_F}$.

The coefficients of the series (1) are called the <u>component fields</u>.

However, the superfield Φ must be reduced to match the number of degrees of freedom contained in the $N=1$ supermultiplets.

- We can calculate (ex. IV.5) the SUSY transformation laws of the generic superfield:

$$\boxed{\delta\Phi = i(\varepsilon\cdot Q + \bar Q\cdot\bar\varepsilon)\Phi = \left[\varepsilon\cdot\partial + \bar\partial\bar\varepsilon + i(\varepsilon\sigma^\mu\bar\theta - \theta\sigma^\mu\bar\varepsilon)\partial_\mu\right]\Phi}$$

This gives:

$$\boxed{\begin{aligned}
\delta z &= \varepsilon\cdot\xi + \bar\varepsilon\cdot\bar\xi \\
\delta\xi &= 2f\varepsilon + \sigma^\mu\bar\varepsilon\,(A_\mu - i\partial_\mu z) \\
\delta\bar\xi &= 2g\bar\varepsilon - \bar\sigma^\mu\varepsilon\,(A_\mu + i\partial_\mu z) \\
\delta f &= \tfrac{i}{2}\,\partial_\mu\xi\sigma^\mu\bar\varepsilon + \bar\varepsilon\bar\rho \\
\delta g &= -\tfrac{i}{2}\,\varepsilon\sigma^\mu\partial_\mu\bar\xi + \varepsilon\cdot\omega \\
\delta A_\mu &= -\tfrac{i}{2}\,\varepsilon\,\partial_\mu\xi - i\varepsilon\sigma_{\nu\mu}\partial^\nu\xi + \tfrac{i}{2}\,\bar\varepsilon\cdot\partial_\mu\bar\xi - i\bar\varepsilon\bar\sigma_{\nu\mu}\partial^\nu\bar\xi - \bar\varepsilon\bar\sigma_\mu\omega - \bar\rho\bar\sigma_\mu\varepsilon \\
\delta\omega &= -i\sigma^\mu\bar\varepsilon\,\partial_\mu g + \tfrac{i}{2}\,\varepsilon\,\partial_\mu A^\mu - \tfrac{i}{2}\,\sigma^{\mu\nu}\varepsilon F_{\mu\nu} + 2\varepsilon d \\
\delta\bar\rho &= -i\bar\sigma^\mu\varepsilon\,\partial_\mu f - \tfrac{i}{2}\,\bar\varepsilon\,\partial_\mu A^\mu + \tfrac{i}{2}\,\bar\sigma^{\mu\nu}\bar\varepsilon F_{\mu\nu} + 2\bar\varepsilon d \\
\delta d &= \tfrac{i}{2}\,\partial_\mu\omega\,\sigma^\mu\bar\varepsilon - \tfrac{i}{2}\,\varepsilon\sigma^\mu\partial_\mu\bar\rho
\end{aligned}}$$

It is important to note that the transformation laws of the d-component is a total derivative. This will be useful to extract SUSY lagrangians

4 Chiral superfields

- Imposing constraints on the generic superfield allows to reduce the number of degrees of freedom. However, those constraints must be preserved by SUSY transformations ⇒ we could use the superderivatives

$$\bar D_{\dot\alpha}\Phi = 0 \qquad \text{or} \qquad D_\alpha\Phi = 0$$

which defines left-handed and right-handed chiral superfields, respectively.

- Since D's and Q's anticommute, these constraints are preserved by SUSY:

$$\begin{cases} \bar{D}_{\dot\alpha}\,\delta\Phi = i\,\bar{D}_{\dot\alpha}(\varepsilon Q + \bar{Q}\bar{\varepsilon})\Phi = i(\varepsilon Q + \bar{Q}\bar{\varepsilon})\,\bar{D}_{\dot\alpha}\Phi = 0 \\ D_\alpha\,\delta\Phi = \dots = 0 \end{cases}$$

- Let us now derive the form of the chiral superfield.
  - Let us define $y^\mu = x^\mu - i\theta\sigma^\mu\bar\theta$. We note that:

$$\bar{D}_{\dot\alpha}\, y^\mu = 0$$

  - Moreover $\bar{D}_{\dot\alpha}\,\theta^\alpha = 0$ and $\bar{D}_{\dot\alpha}\,\bar\theta^{\dot\beta} \neq 0$

$\Rightarrow$ in general, we can write a chiral SF as $\Phi(y, \theta)$

The most general left-handed χSF is thus given by

$$\boxed{\Phi(y,\theta) = \phi(y) + \sqrt{2}\,\theta\cdot\psi(y) - \theta\theta F(y)} \qquad (1)$$

where the normalizations are conventional.

- Analyzing the component fields, it is interesting that $\Phi(y,\theta)$ has the same field content as the chiral supermultiplet:

$$\left.\begin{array}{l} -\ \text{complex scalar field } \phi, F \quad 4 \\ -\ \text{one Weyl fermion} \quad 4 \end{array}\right\} \quad n_B = n_F = 4.$$

- The SUSY transformation laws are obtained by:

  - $i\varepsilon Q\Phi = i\varepsilon Q y^\mu \dfrac{\partial\Phi}{\partial y^\mu} + i\varepsilon Q\theta^\beta \dfrac{\partial\Phi}{\partial\theta^\beta}$
  - $i\bar{Q}\bar{\varepsilon}\Phi = i\bar{Q}\bar{\varepsilon} y^\mu \dfrac{\partial\Phi}{\partial y^\mu} + i\bar{Q}\bar{\varepsilon}\theta^\beta \dfrac{\partial\Phi}{\partial\theta^\beta}$

<u>But:</u>

$$
\begin{aligned}
i\varepsilon Q y^\mu &= (\varepsilon\partial + i\varepsilon\sigma^\nu\bar{\theta}\partial_\nu)(x^\mu - i\theta\sigma^\mu\bar{\theta}) \\
&= -i\varepsilon\sigma^\mu\bar{\theta} + i\varepsilon\sigma^\mu\bar{\theta} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
i\bar{Q}\bar{\varepsilon} y^\mu &= -(\bar{\partial}\bar{\varepsilon} + i\theta\sigma^\nu\bar{\varepsilon}\partial_\nu)(x^\mu - i\theta\sigma^\mu\bar{\theta}) \\
&= -i\theta\sigma^\mu\bar{\varepsilon} - i\theta\sigma^\mu\bar{\varepsilon} \\
&= -2i\theta\sigma^\mu\bar{\varepsilon}
\end{aligned}
$$

$$\Rightarrow \quad i\varepsilon Q\Phi = i\varepsilon^\beta \frac{\partial\Phi}{\partial\theta^\beta} = \sqrt{2}\,\varepsilon\cdot\psi - 2\,\varepsilon\cdot\theta F$$

$$i\bar{Q}\bar{\varepsilon}\Phi = -2i\theta\sigma^\mu\bar{\varepsilon}\,(\partial_\mu\phi + \sqrt{2}\,\theta\partial_\mu\psi)$$

$$\Rightarrow \quad \boxed{\begin{aligned}\delta\Phi = {} & \sqrt{2}\,\varepsilon\cdot\psi + \sqrt{2}\,\theta\cdot(-\sqrt{2}\,\varepsilon F - \sqrt{2}\,i\,\sigma^\mu\bar{\varepsilon}\,\partial_\mu\phi) \\ & - \theta\theta\,(-i\sqrt{2}\,\partial_\mu\psi\sigma^\mu\bar{\varepsilon})\end{aligned}}$$

where we have used $(\theta\sigma^\mu\bar{\varepsilon})(\theta\partial_\mu\psi) = -\frac{1}{2}\,\theta\theta\,\partial_\mu\psi\sigma^\mu\bar{\varepsilon}$

$$\Rightarrow \quad \boxed{\begin{aligned}\delta\phi &= \sqrt{2}\,\varepsilon\cdot\psi \\ \delta\psi &= -\sqrt{2}\,\varepsilon F - i\sqrt{2}\,\sigma^\mu\bar{\varepsilon}\,\partial_\mu\phi \\ \delta F &= -i\sqrt{2}\,\partial_\mu\psi\sigma^\mu\bar{\varepsilon}\end{aligned}}$$

$\Rightarrow$ we recover the SUSY transf. laws of the chiral supermultiplet (see chap II)

11

- It is also interesting to note that

$$\Phi| = \phi \quad ; \quad \frac{1}{\sqrt{2}} D_\alpha \Phi| = \psi \quad ; \quad \frac{1}{4} D \cdot D \Phi| = F$$

where $X| \doteq X|_{\theta=\bar\theta=0}$

- Expanding Φ and $\Phi^\dagger$ in terms of x, one has (this will be useful for the sequel)

$$\Phi(x,\theta,\bar\theta) = \phi(x) + \sqrt{2}\,\theta\cdot\psi(x) - \theta\theta F(x) - i\,\theta\sigma^\mu\bar\theta\,\partial_\mu\phi(x)$$
$$+ \frac{i}{\sqrt{2}}\,\theta\theta\,\partial_\mu\psi(x)\sigma^\mu\bar\theta - \frac{1}{4}\theta\theta\,\bar\theta\bar\theta\,\Box\phi(x)$$

$$\Phi^\dagger(x,\theta,\bar\theta) = \phi^\dagger(x) + \sqrt{2}\,\bar\theta\cdot\bar\psi(x) - \bar\theta\bar\theta F^\dagger(x) + i\,\theta\sigma^\mu\bar\theta\,\partial_\mu\phi^\dagger(x)$$
$$+ \frac{i}{\sqrt{2}}\,\bar\theta\bar\theta\,\partial_\mu\bar\psi(x)\bar\sigma^\mu\theta - \frac{1}{4}\theta\theta\,\bar\theta\bar\theta\,\Box\phi^\dagger(x)$$

- We now need to build the lagrangian associated to Φ
  - It must be a $[\mathcal{L}] = 4$ quantity
  - It must be quadratic in the fields
  - It must be SUSY invariant

However, we have seen that the variation of the highest-order component of a generic superfield under a SUSY transformation is a total derivative. We can check that $\Phi^\dagger\Phi|_{\theta^2\bar\theta^2}$ is a good lagrangian.

$$\boxed{\mathcal{L} = \Phi^\dagger\Phi|_{\theta^2\bar\theta^2} = \partial_\mu\phi^\dagger\partial^\mu\phi + \frac{i}{2}\left(\psi\sigma^\mu\partial_\mu\bar\psi - \partial_\mu\psi\sigma^\mu\bar\psi\right) + F^\dagger F}$$

ex IV.6 : proof

- We have reproduced the free Lagrangian of the Wess-Zumino model. We now turn to the interactions. The product of two LCSF is a LCSF:

$$\Phi^a \Phi^b = \phi^a \phi^b + \sqrt{2}\,\theta\cdot\left(\phi^a \psi^b + \phi^b \psi^a\right) - \theta\theta\left(\phi^a F^b + \phi^b F^a + \psi^a\cdot\psi^b\right)$$

ex IV.7: proof.

$\Rightarrow$ any function holomorphic in the LCSF content of the theory will be a LCSF:

$$\boxed{W(\Phi) = W(\phi) + \sqrt{2}\,\theta\cdot\psi^a \frac{\partial W(\phi)}{\partial \phi^a} - \theta\theta\left[F^a \frac{\partial W(\phi)}{\partial \phi^a} + \frac{1}{2}\psi^a\cdot\psi^b \frac{\partial^2 W(\phi)}{\partial \phi^a \partial \phi^b}\right]}$$

where we have introduced the superpotential $W(\Phi)$.

In section IV.3, we have derived the SUSY transformation law of the various components fields. Hence, the variation of the F-term (θ^2-component) is a total derivative and can be used for a Lagrangian

$$\Rightarrow \boxed{\mathcal{L}_{WZ} = \Phi^\dagger_a \Phi^a\big|_{\theta^2\bar\theta^2} + W(\Phi)\big|_{\theta^2} + W^*(\Phi^\dagger)\big|_{\bar\theta^2}}$$

which is a much simpler expression that the one found in chapter II. We have introduced $W^*(\Phi^\dagger) = [W(\Phi)]^\dagger$.

- If we consider non renormalizable theories (such as supergravity), the Lagrangian above can be generalized:

$$\mathcal{L} = K(\Phi, \Phi^\dagger)\big|_{\theta^2\bar\theta^2} + W(\Phi)\big|_{\theta^2} + W^*(\Phi^\dagger)\big|_{\bar\theta^2}$$

where K is real (it is in fact a vector superfield), the Kähler potential.

5 Vector superfields

- The chiral superfields introduced in the previous section do not have any vector component $\Rightarrow$ we need another kind of superfield to handle gauge supermultiplets: vector superfields.

The vector superfield is defined by a reality condition:

$$V(X) = V^\dagger(X)$$

Trivially, this constraint is preserved by SUSY transformations (check the SUSY transformation laws of section IV.3 + eq (1) below).

- $V = V^\dagger$ leads to:

$$\boxed{\begin{aligned} V(X) = {} & C(x) + i\theta\cdot\chi(x) - i\bar\theta\cdot\bar\chi(x) + \tfrac{i}{2}\theta\theta(M(x)+iN(x)) - \tfrac{i}{2}\bar\theta\bar\theta(M(x)-iN(x)) \\ & + \theta\sigma^\mu\bar\theta\, v_\mu(x) + i\theta\theta\,\bar\theta\left(\bar\lambda(x) - \tfrac{i}{2}\bar\sigma^\mu\partial_\mu\chi(x)\right) - i\bar\theta\bar\theta\,\theta\left(\lambda(x) - \tfrac{i}{2}\sigma^\mu\partial_\mu\bar\chi(x)\right) \\ & + \tfrac{1}{2}\theta\theta\,\bar\theta\bar\theta\left(D(x) - \tfrac{1}{2}\Box C(x)\right) \end{aligned}} \quad (1)$$

This form will be justified below. We have:

$$\left.\begin{array}{ll} \text{– 4 real scalar fields:} & C, M, N, D \\ \text{– 1 real vector field:} & v_\mu \\ \text{– 2 Majorana fermions:} & (\chi_\alpha, \bar\chi^{\dot\alpha}) \text{ and } (\lambda_\alpha, \bar\lambda^{\dot\alpha}) \end{array}\right\} n_B = n_F = 8$$

- The number of degrees of freedom is too high to match those included in the gauge supermultiplets, however:
  - $[v_\mu] = 1 \Rightarrow [\lambda] = 3/2$ and $[D] = 2$
  - $[v_\mu] = 1 \Rightarrow \underbrace{[M] = [N] = 1;\ [C] = 0;\ [\chi] = 1/2}_{\text{the additional fields are unphysical } (\equiv \text{not belonging to the vector supermultiplet})}$

$\Rightarrow$ the unphysical degrees of freedom could be eliminated.

To this aim, we can chose a <u>convenient gauge</u>. (we are dealing with vector fields)

- Let Λ be a LXSF. $\Rightarrow -i(\Lambda - \Lambda^\dagger)$ is a vector superfield, and:

$$
\begin{aligned}
-i(\Lambda-\Lambda^\dagger) = {} & -i(\phi-\phi^\dagger) - \sqrt{2}\, i\, \theta\psi + \sqrt{2}\, i\, \bar\theta\bar\psi - \theta\sigma^\mu\bar\theta\, \partial_\mu(\phi+\phi^\dagger) \\
& + i\theta\theta F - i\bar\theta\bar\theta F^\dagger + \frac{1}{\sqrt{2}}\theta\theta\, \partial_\mu\psi\sigma^\mu\bar\theta + \frac{1}{\sqrt{2}}\bar\theta\bar\theta\, \theta\sigma^\mu\partial_\mu\bar\psi \\
& + \frac{i}{4}\theta\theta\,\bar\theta\bar\theta\, \Box(\phi-\phi^\dagger)
\end{aligned}
$$

Adding with eq. (1), we find that the transformation $V \to V - i(\Lambda - \Lambda^\dagger)$ leads to the transformation rules:

$$
\boxed{
\left\{
\begin{aligned}
C &\to C - i(\phi-\phi^\dagger) \\
\chi &\to \chi - \sqrt{2}\,\psi \\
M + iN &\to M + iN + 2F \\
v_\mu &\to v_\mu - \partial_\mu(\phi+\phi^\dagger) \\
\lambda &\to \lambda \\
D &\to D
\end{aligned}
\right.
}
\quad (2)
$$

The transformation of the vectorial field in (2) is exactly a gauge transformation.

This suggest to introduce the generalized gauge transformation

$$\boxed{V \longrightarrow V - i(\Lambda - \Lambda^{\dagger}) \quad \text{where } \Lambda \text{ is a LXSF}} \quad (3)$$

- We will now chose a gauge so that all the nonphysical degrees of freedom of V vanish: Λ is such that:

$$\begin{cases} -i(\phi - \phi^{\dagger}) = -C \\ \psi = \frac{1}{\sqrt{2}} \chi \\ F = -\frac{1}{2}(M + iN) \end{cases}$$

This is the Wess and Zumino gauge. It also contains the remaining $U(1)$ gauge invariance for the vector field.

$$v_\mu \longrightarrow v_\mu - \partial_\mu (\phi + \phi^{\dagger})$$

The vector superfield in the Wess and Zumino gauge is given by

$$\boxed{V_{WZ} = \theta \sigma^\mu \bar{\theta} v_\mu + i\theta\theta\,\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\,\theta\lambda + \frac{1}{2}\theta\theta\,\bar{\theta}\bar{\theta} D} \quad (4)$$

- Two interesting properties are

$$\begin{cases} V_{WZ}^2 = \frac{1}{2}\theta\theta\,\bar{\theta}\bar{\theta}\, v_\mu v^\mu \\ V_{WZ}^3 = 0 \end{cases}$$

ex. IV.8: proof them

$\Rightarrow V_{WZ}^2$ is not suitable for a lagrangian

- We are now ready to calculate the SUSY transformation laws of the vector superfield, to check if it represent indeed the gauge N=1 supermultiplet. Using the results of section IV.3 and the notations of eq (1) adapted to the Wess and Zumino gauge:

$$C = 0 \longrightarrow C + \delta C = 0$$
$$\chi = 0 \longrightarrow \chi + \delta\chi = -i\sigma^\mu \bar{\varepsilon} v_\mu \qquad (5)$$
$$M + iN = 0 \longrightarrow M + iN + \delta M + i\delta N = 2\bar{\varepsilon}\cdot\bar{\lambda}$$

$\Rightarrow$ The Wess and Zumino gauge is not conserved by the SUSY transformations. We will dedicate the last section of this chapter to this issue.

Using again the results of section IV 3 and eq (5):

$$\triangleright\ \delta\left(-i(\lambda - \tfrac{i}{2}\sigma^\mu\partial_\mu\bar{\chi})\right) = \frac{i}{2}\,\varepsilon\partial_\mu v^\mu - \frac{i}{2}\sigma^{\mu\nu}\varepsilon F_{\mu\nu} + \varepsilon D$$

$$\Rightarrow \delta\lambda = \frac{1}{2}\sigma^{\mu\nu}\varepsilon F_{\mu\nu} + i\varepsilon D - \frac{1}{2}\varepsilon\partial_\mu v^\mu + \frac{i}{2}\sigma^\mu\partial_\mu\left[-i\bar{\sigma}^\nu\varepsilon v_\nu\right]$$
$$= \frac{1}{2}\sigma^{\mu\nu}\varepsilon F_{\mu\nu} + i\varepsilon D - \frac{1}{2}\varepsilon\partial_\mu v^\mu + \frac{1}{2}\left(2\sigma^{\mu\nu} + \eta^{\mu\nu}\right)\partial_\mu v_\nu$$
$$= \sigma^{\mu\nu}\varepsilon F_{\mu\nu} + i\varepsilon D$$

$$\triangleright\ \delta D = \partial_\mu\lambda\sigma^\mu\bar{\varepsilon} + \varepsilon\sigma^\mu\partial_\mu\bar{\lambda}$$

$$\triangleright\ \delta v_\mu = i\bar{\varepsilon}\bar{\sigma}_\mu\lambda - i\bar{\lambda}\bar{\sigma}_\mu\varepsilon$$

$$\Rightarrow \boxed{\begin{aligned} \delta v_\mu &= i(\varepsilon\sigma_\mu\bar{\lambda} - \lambda\sigma_\mu\bar{\varepsilon}) \\ \delta\lambda &= \sigma^{\mu\nu}\varepsilon F_{\mu\nu} + i\varepsilon D \\ \delta D &= \partial_\mu\lambda\sigma^\mu\bar{\varepsilon} + \varepsilon\sigma^\mu\partial_\mu\bar{\lambda} \end{aligned}}$$

and we recover the transformation laws of chapter III.

- We now turn to the building of the lagrangian associated to V. (1)
Powers of V not leading to proper kinetic terms, we need to introduce a new object:
  - it must contain superderivatives (we want a SUSY quantity)
  - it must be gauge invariant under (3).

Let's start with $D_\alpha V$ (we can do the same exercise for $\bar{D}_{\dot{\alpha}} V$):

- $D_\alpha V \xrightarrow{(3)} D_\alpha V - i D_\alpha \Lambda$ ⟹ we need a $\bar{D}_{\dot{\alpha}}$ to cancel the term in Λ

- $\bar{D}_{\dot{\alpha}} D_\alpha V \xrightarrow{(3)} \bar{D}_{\dot{\alpha}} D_\alpha V - i \bar{D}_{\dot{\alpha}} D_\alpha \Lambda$

$$= \bar{D}_{\dot{\alpha}} D_\alpha V - i \{\bar{D}_{\dot{\alpha}}, D_\alpha\} \Lambda$$
$$= \bar{D}_{\dot{\alpha}} D_\alpha V - 2\sigma^\mu{}_{\alpha\dot{\alpha}} \partial_\mu \Lambda$$ ⟹ we need another $\bar{D}$ to cancel Λ.

We hence compute the SUSY-conserving and gauge invariant quantities

$$\boxed{\begin{aligned} W_\alpha &= -\frac{1}{4} \bar{D}\cdot\bar{D}\ D_\alpha V \\ \bar{W}_{\dot{\alpha}} &= \frac{1}{4} D\cdot D\ \bar{D}_{\dot{\alpha}} V \end{aligned}}$$

→ W_α and $\bar{W}_{\dot{\alpha}}$ are spinorial chiral superfields (ex. IV. 9a)

We can show that (ex IV 9b):

$$\boxed{\begin{aligned} W_\alpha &= -i\lambda_\alpha + \left[-\frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha F_{\mu\nu} + \theta_\alpha D\right] - \theta\theta(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha \\ \bar{W}_{\dot{\alpha}} &= W_\alpha{}^\dagger \end{aligned}}$$

- In non-supersymmetric quantum field theories, the gauge pieces of the lagrangian are built from the field strenght tensor $F_{\mu\nu}$. In SUSY, we can see W_α and $\bar{W}_{\dot{\alpha}}$ as the supersymmetrization of $F_{\mu\nu}$. They are called the superfield strenght tensors.

- We can show that

$$\mathcal{L}_V = \frac{1}{4} W^\alpha W_\alpha \big|_{\theta\cdot\theta} + \frac{1}{4} \overline{W}_{\dot\alpha} \overline{W}^{\dot\alpha} \big|_{\bar\theta\bar\theta}$$

is a good Lagrangian for the gauge sector.

▷ W^α is a LXSF $\longrightarrow$ $W^\alpha W_\alpha$ is a LXSF $\Rightarrow$ its $\theta\cdot\theta$ coefficient transforms as a total derivative $\Rightarrow$ it is a good candidate for the Lagrangian.

▷ Similarly, $\overline{W}_{\dot\alpha}$ ...

▷ We can calculate (ex. IV. 10)

$$\mathcal{L}_V = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2}\left(\lambda\sigma^\mu\partial_\mu\bar\lambda - \partial_\mu\lambda\sigma^\mu\bar\lambda\right) + \frac{1}{2} D^2 .$$

- Since $V|_{\theta^2\bar\theta^2}$ transforms as a total derivative, we could have added to our Lagrangian the Fayet-Iliopoulos term

$$\mathcal{L}_{FI} = \xi V|_{\theta^2\bar\theta^2} .$$

This will be used for SUSY-breaking Lagrangians.

6) Gauge interaction of the chiral superfields

- In non-SUSY quantum field theory, the gauge transformation laws of a field φ and a gauge boson v_μ are:

$$\begin{cases} \varphi \longrightarrow e^{ig\alpha(x)} \varphi & (1) \\ v_\mu \longrightarrow v_\mu + \partial_\mu \alpha(x) & (2) \end{cases}$$

where we assume a $U(1)$ gauge group, the charge of φ being $+1$.

At the superfield level, we have seen that (2) generalizes to:

$$V \longrightarrow V - i(\Lambda - \Lambda^\dagger) \text{ where } \Lambda \text{ is a LXSF.}$$

and $\alpha = -(\phi + \phi^\dagger)$

- This suggest the gauge transformation law of a LXSF Φ:

$$\begin{cases} \Phi \to e^{-2ig\Lambda}\,\Phi \\ \Phi^{+} \to \Phi^{+}\, e^{2ig\Lambda^{+}} \end{cases}$$

Since $\Lambda \neq -\Lambda^{+}$, the Wess and Zumino model Lagrangian,

$$\mathcal{L}_{WZ} = \Phi^{+}\Phi\big|_{\theta^2\bar\theta^2}$$

is not gauge invariant (as the free Lagrangian in non-SUSY quantum field theories).

$\Rightarrow$ We will generalize Noether's procedure to supersymmetry and render $\mathcal{L}_{WZ}$ gauge invariant

- At the finite level, we can rewrite (1) as:

$$e^{2gV} \longrightarrow e^{-2ig\Lambda}\, e^{2gV}\, e^{2ig\Lambda^{+}}$$

or, since $e^{2gV}e^{-2gV} = 1$,

$$e^{-2gV} \longrightarrow e^{-2ig\Lambda^{+}}\, e^{-2gV}\, e^{+2ig\Lambda}$$

$\Rightarrow$ it follows that $\Phi^{+} e^{-2gV}\,\Phi$ is gauge invariant.

Reintroducing the generator Q of the consider gauge group, we will have:

$$\boxed{\mathcal{L}_{SUSY} = \Phi^{+}_{a}\, e^{-2gVQ}\,\Phi^{a}\Big|_{\theta^2\bar\theta^2} + W(\Phi)\big|_{\theta^2} + W^{*}(\Phi)\big|_{\bar\theta^2} + \frac{1}{4} W^{\alpha}W_{\alpha}\big|_{\theta^2} + \frac{1}{4}\, \overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}\big|_{\bar\theta^2}}$$

which describes an abelian gauge theories with a superfield content given by $\{\Phi^a\}$. It is important to note that the superpotential must now satisfy gauge invariance, in addition to renormalizability.

- let us calculate $\Phi^\dagger e^{-2gVQ}\Phi|_{\theta^2\bar\theta^2}$.

$\triangleright$ $e^{-2gVQ} = 1 - 2gVQ + 2g^2V^2Q^2$

($V^3 = 0$ in the Wess and Zumino gauge)

$\triangleright$ $\Phi^\dagger\Phi|_{\theta\bar\theta^2}$ = free lagrangian (calculated in section IV.4)

$$\triangleright -2g\Phi^\dagger QV\Phi|_{\theta^2\bar\theta^2} = igq\, v_\mu\, \phi^\dagger\partial^\mu\phi - igq\,\partial^\mu\phi^\dagger v_\mu\phi - gq\,\phi^\dagger\phi D + i\sqrt{2}gq\,\phi\bar\lambda\bar\psi - i\sqrt{2}gq\phi^\dagger\lambda\psi - gq\, v_\mu\psi\sigma^\mu\bar\psi$$

where q is the $U(1)$ charge of the χSF Φ.

(ex IV.11a: proof the eq. above)

$$\triangleright\ 2g^2q^2\ \Phi^\dagger V^2\Phi|_{\theta^2\bar\theta^2} = g^2q^2\, v_\mu\phi^\dagger v^\mu\phi \quad \text{(ex IV.11b)}$$

$\Rightarrow$ summing all the calculations, one obtains (ex IV.11c):

$$\boxed{\Phi^\dagger e^{-2gVQ}\Phi|_{\theta^2\bar\theta^2} = D_\mu\phi^\dagger D^\mu\phi + \frac{i}{2}(\psi\sigma^\mu D_\mu\bar\psi - D_\mu\psi\sigma^\mu\bar\psi) + F^\dagger F - gq\, D\phi^\dagger\phi + i\sqrt{2}gq\,\phi\bar\lambda\bar\psi - i\sqrt{2}gq\,\phi^\dagger\lambda\cdot\psi}$$

with $D_\mu = \partial_\mu - igv_\mu Q$

Hence, when going on-shell, both F-term and D-term eqs of motion will be non-trivial:

$$\boxed{\begin{aligned}\mathcal{L}_{SUSY} = & D_\mu \phi_a^\dagger D^\mu \phi^a + \frac{i}{2}\left(\psi^a \sigma^\mu D_\mu \bar\psi_a - D_\mu \psi^a \sigma^\mu \bar\psi_a\right) + F_a^\dagger F^a \\ & -\left(F^a W_a + \frac{1}{2}\psi^a \psi^b W_{ab} + h.c.\right) \\ & -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{i}{2}\left(\lambda\sigma^\mu\partial_\mu\bar\lambda - \partial_\mu\lambda\sigma^\mu\bar\lambda\right) + \frac{1}{2}D^2 \\ & - g q D \phi^\dagger\phi + i\sqrt{2}\, g q\, \phi\bar\lambda\bar\psi - i\sqrt{2}\, g q\, \phi^\dagger\lambda\psi .\end{aligned}}$$

7) SUSY Yang-Mills theories

- We generalize in this section the previous results to the case of non-abelian groups. Let us define a representation R of a Lie algebra specified by the matrices T_a:

$$[T_a, T_b] = i f_{ab}{}^c T_c \quad ; \quad \mathrm{Tr}[T_a T_b] = \tau_R \delta_{ab}$$

where $f_{ab}{}^c$ are the structure constants of the group.

It can be checked that $\underline{V = V^a T_a}$ is a real superfield if the V^a are real too ($V^{a\dagger} = V^a \rightarrow V^\dagger = V$) $\Rightarrow$ V is a matrix.

- Similarly, the gauge transformation parameter will be taken as

$$\Lambda = \Lambda^a T_a$$

The quantities W_α and $\bar W_{\dot\alpha}$ introduced in section IV.5 are then not gauge invariant and must be generalized.

- At the finite level, the gauge transformations are given by

$$\begin{cases} e^{2gV} \longrightarrow e^{-2ig\Lambda} \, e^{2gV} \, e^{2ig\Lambda^+} \\ e^{-2gV} \longrightarrow e^{-2ig\Lambda^+} \, e^{-2gV} \, e^{2ig\Lambda} \end{cases} \quad (1)$$

Let us expand the first eq. above at the first order in Λ, using the Baker-Campbell-Hausdorff identity:

$$\delta V = i\,\mathrm{ad}(gV)\cdot(\Lambda+\Lambda^+) - i\,\mathrm{ad}(gV)\coth(\mathrm{ad}(gV))\cdot(\Lambda-\Lambda^+)$$

where $\mathrm{ad}(X)\cdot Y = [X, Y]$; $\mathrm{ad}^2(X)\cdot Y = [X,[X,Y]]$, ...

$$\Rightarrow \delta V = ig[V, \Lambda+\Lambda^+] - i\left(1 + \frac{\mathrm{ad}^2(gV)}{3} - \frac{\mathrm{ad}^4(gV)}{45} + \dots\right)\cdot(\Lambda-\Lambda^+)$$

We now make two assumptions:

(a) the scalar component of $\Lambda-\Lambda^+$ vanishes $\Rightarrow$ it is real

(b) it exists a Wess and Zumino gauge in the non-abelian case so that V_{WZ} has the form of eq (4) of section IV.5.

$$\Rightarrow \quad V^2 = \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\, v_\mu v^\mu \; ; \; \mathrm{ad}^2(gV)\cdot(\Lambda-\Lambda^+) = 0 \; ; \; \mathrm{ad}^n(gV) = 0 \;\; \forall n>2.$$

$$\Rightarrow \quad \boxed{\delta V = -i(\Lambda-\Lambda^+) + ig[V, \Lambda+\Lambda^+]} \quad (2)$$

And we can show that (a) and (b) can be satisfied using eq. (2).

(ex IV.12)

- We are now ready to build the superfield strength tensor.
Since the gauge transformations involves exponentials, it sounds natural to build left-handed and right-handed χSF from $e^{\pm 2gV}$.

▷ $$D_\alpha e^{-2gV} \longrightarrow D_\alpha \left(e^{-2ig\Lambda^\dagger} e^{-2gV} e^{2ig\Lambda} \right)$$
$$= e^{-2ig\Lambda^\dagger} \left[D_\alpha (e^{-2gV}) e^{2ig\Lambda} + e^{-2gV} D_\alpha e^{2ig\Lambda} \right]$$

▷ In order to cancel the $e^{-2ig\Lambda^\dagger}$ factor, we calculate:

$$e^{2gV} D_\alpha e^{-2gV} \longrightarrow e^{-2ig\Lambda} e^{2gV} D_\alpha (e^{-2gV}) e^{2ig\Lambda}$$
$$+ e^{-2ig\Lambda} D_\alpha e^{2ig\Lambda}$$

▷ We are now ready to use the $\bar{D}$'s, since $\bar{D}_{\dot\alpha} \Lambda = 0$

$\bar{D}_{\dot\alpha} e^{2gV} D_\alpha e^{-2gV}$ and work out the second term above:

$$e^{-2ig\Lambda} \bar{D}_{\dot\alpha} D_\alpha e^{2ig\Lambda} = e^{-2ig\Lambda} \{ \bar{D}_{\dot\alpha}, D_\alpha \} e^{2ig\Lambda}$$
$$= -i2 \sigma^\mu_{\alpha\dot\alpha} \, e^{-2ig\Lambda} \partial_\mu e^{2ig\Lambda}$$

$\Rightarrow$ $\bar{D}\cdot\bar{D} \cdot e^{2gV} D_\alpha e^{-2gV}$ will made this term vanish

$$\Rightarrow \boxed{\begin{aligned} W_\alpha &= -\frac{1}{4} \bar{D}\bar{D} \, e^{2gV} D_\alpha e^{-2gV} \\ \bar{W}_{\dot\alpha} &= -\frac{1}{4} DD \, e^{-2gV} \bar{D}_{\dot\alpha} e^{2gV} \end{aligned}}$$

and we have $W_\alpha \longrightarrow e^{-2ig\Lambda} W_\alpha e^{2ig\Lambda}$

$$\bar{W}_{\dot\alpha} \longrightarrow e^{-2ig\Lambda^\dagger} \bar{W}_{\dot\alpha} e^{2ig\Lambda^\dagger}.$$

$\Rightarrow$ $\boxed{\text{The superfield strenght tensor are not anymore gauge invariant quantities}}$

This could have been expected, from the results of non-SUSY theories. Indeed, in the non-abelian case, $F_{\mu\nu}$ is not gauge invariant anymore, but $Tr(F_{\mu\nu}F^{\mu\nu}) = \tau_R F^a_{\mu\nu} F_a^{\mu\nu}$, with $F_{\mu\nu} = F^a_{\mu\nu} T_a$, is gauge invariant. Similarly, the SUSY Yang-Mills lagrangian will be given by

$$\boxed{\mathcal{L}_{YM} = \frac{1}{16 g^2 \tau_R}\left(Tr\left(W^\alpha W_\alpha\right)\Big|_{\theta^2} + Tr\left(\bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha}\right)\Big|_{\bar\theta^2}\right)}$$

The normalization factor allows to recover the Y-M. lagrangian in component fields.

- Let us now work out the expression of W_α, in the Wess and Zumino gauge. We have:

$$\begin{cases} V \sim \theta\bar\theta + \bar\theta\bar\theta\theta + \theta\theta\bar\theta + \theta\theta\bar\theta\bar\theta \\ V^2 \sim \theta\theta\bar\theta\bar\theta \\ D_\alpha V \sim \bar\theta + \bar\theta\bar\theta + \theta\bar\theta + \theta\bar\theta\bar\theta + \theta\theta\bar\theta\bar\theta \\ D_\alpha V^2 \sim \theta\bar\theta\bar\theta \\ V^2 D_\alpha V = 0 \\ V D_\alpha V^2 = 0 \end{cases}$$

$$\Rightarrow e^{2gV} D_\alpha e^{-2gV} = (1 + 2gV + 2g^2V^2) D_\alpha (1 - 2gV + 2g^2 V^2)$$

$$= -2g\, D_\alpha V - 4g^2 V D_\alpha V + 2g^2 \underbrace{D_\alpha V^2}_{\hookrightarrow\ V D_\alpha V + (D_\alpha V) V}$$

$$= -2g\, D_\alpha V - 2g^2 [V, D_\alpha V]$$

We can compute. (ex IV. 13a)

$$\boxed{W_\alpha = -2g\left(-i\lambda_\alpha + \left(-\frac{i}{2}(\sigma^\mu \bar\sigma^\nu \theta)_\alpha F_{\mu\nu} + \theta_\alpha D\right) - \theta\theta(\sigma^\mu \mathcal{D}_\mu \bar\lambda)_\alpha\right.}$$

with
$$\begin{cases} \lambda_\alpha = \lambda^a_\alpha T_a \\ D = D^a T_a \\ F_{\mu\nu} = (\partial_\mu v^a_\nu - \partial_\nu v^a_\mu) T_a - ig\, v^a_\mu v^b_\nu [T_a, T_b] \\ \quad = (\partial_\mu v_\nu{}^a - \partial_\nu v_\mu{}^a) T_a + g\, f_{ab}{}^c\, v^a_\mu v^b_\nu T_c \\ \mathcal{D}_\mu \bar\lambda = \partial_\mu \bar\lambda^a T_a + g\, f_{ab}{}^c\, v^a_\mu \bar\lambda^b T_c \end{cases}$$

$\Rightarrow W_\alpha$ is now a matrix.

Hence (ex IV. 13b):

$$\boxed{\mathrm{Tr}(W^\alpha W_\alpha)\big|_{\theta^2} = 4g^2 C_R \mathrm{Tr}\left[2i\lambda\sigma^\mu \mathcal{D}_\mu \bar\lambda + D^2 - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{i}{4}\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}\right]}$$

8) SUSY gauge theories

- Similarly to section IV.6, the lagrangian

$$\mathcal{L}_M = \Phi^\dagger e^{-2gV} \Phi$$

is gauge invariant.

- We have thus, in the non-abelian case:

$$\mathcal{L}_{SUSY} = \Phi^{\dagger}_{i}\, e^{-2gV}\, \Phi^{i}\Big|_{\theta^2\bar\theta^2} + W(\Phi)\Big|_{\theta^2} + W^*(\Phi^\dagger)\Big|_{\bar\theta^2}$$

$$+ \frac{1}{16 g^2 \tau_R} \mathrm{Tr}(W^\alpha W_\alpha)\Big|_{\theta^2} + \frac{1}{16 g^2 \tau_R} \mathrm{Tr}(\overline{W}_{\dot\alpha} \cdot \overline{W}^{\dot\alpha})\Big|_{\bar\theta^2}$$

$$= -\frac{1}{4} F^{\mu\nu}_a F^a_{\mu\nu} + \frac{i}{2}\left(\lambda^a \sigma^\mu D_\mu \bar\lambda_a - D_\mu \lambda^a \sigma^\mu \bar\lambda_a\right) + \frac{1}{2} D_a D^a$$

$$+ D_\mu \phi^\dagger_i D^\mu \phi^i - \frac{i}{2}\left(D_\mu \bar\psi_i \bar\sigma^\mu \psi^i - \bar\psi_i \bar\sigma^\mu D_\mu \psi^i\right) + F^\dagger_i F^i$$

$$- g D^a \phi^\dagger_i T_a \phi^i + i\sqrt{2}\, g\, \bar\lambda^a \cdot \bar\psi_i T_a \phi^i - i\sqrt{2}\, g\, \phi^\dagger_i T_a \psi^j \cdot \lambda^a$$

$$-\left[m_{ij}\left(\phi^i F^j + \frac{1}{2}\psi^i \cdot \psi^j\right) + \frac{1}{2}\lambda_{ijk}\left(\phi^i \phi^j F^k + \phi^i \psi^j \cdot \psi^k\right) + \mathrm{h.c.}\right]$$

where we have assumed $W(\Phi) = \frac{1}{2} m_{ij} \Phi^i \Phi^j + \frac{1}{6}\lambda_{ijk} \Phi^i \Phi^j \Phi^k$

ex IV.14: proof.

After solving the eqs of motion of the auxiliary fields, we obtain the scalar potential:

$$\boxed{V = V_F + V_D = \frac{\partial W}{\partial \phi^i} \frac{\partial W^*}{\partial \phi^\dagger_i} + \frac{1}{2} g^2 \left(\phi^\dagger_i T^a \phi^i\right)\left(\phi^\dagger_j T_a \phi^j\right)}$$

ex IV.15: proof the cancelations of the quadratic divergences in the scalar fields self-energies.

9 Restoration of the Wess and Zumino gauge

- In section IV.5, we have seen that supersymmetry was not preserving the Wess and Zumino gauge.

$$V_{WZ} = \theta\sigma^\mu\bar\theta\, v_\mu + i\,\theta\theta\,\bar\theta\bar\lambda - i\,\bar\theta\bar\theta\,\theta\lambda + \frac{1}{2}\theta\theta\bar\theta\bar\theta D$$

$$\xrightarrow{\delta_\varepsilon} \delta_\varepsilon V_{WZ} = \theta\sigma^\mu\bar\varepsilon\, v_\mu - \bar\theta\bar\sigma^\mu\varepsilon\, v_\mu + i\,\theta\theta\,\bar\varepsilon\cdot\bar\lambda - i\,\bar\theta\bar\theta\,\varepsilon\cdot\lambda$$
$$+ i\,\theta\sigma^\mu\bar\theta\left(\varepsilon\sigma_\mu\bar\lambda - \lambda\sigma_\mu\bar\varepsilon\right) + i\theta\theta\,\bar\theta\left(\bar\sigma^{\mu\nu}\bar\varepsilon F^0_{\mu\nu} - i\,\bar\varepsilon D\right)$$
$$- i\,\bar\theta\bar\theta\,\theta\left(\sigma^{\mu\nu}\varepsilon F^0_{\mu\nu} + i\,\varepsilon D\right) + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left(\partial_\mu\lambda\sigma^\mu\bar\varepsilon + \varepsilon\sigma^\mu\partial_\mu\bar\lambda\right)$$

with $F^0_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$.

- We will now show that a suitable gauge transformation allows to compensate the additional term: We recall that

$$\delta_g V_{WZ} = -i(\Lambda - \Lambda^\dagger) + ig\left[V_{WZ}, \Lambda + \Lambda^\dagger\right]$$

where Λ is a chiral superfield with a real scalar component (cf. section IV.7)

- We note that the choice

$$-i\Lambda = -\theta\sigma^\mu\bar\varepsilon\, v_\mu - i\,\theta\theta\,\bar\varepsilon\bar\lambda$$

will compensate the additional terms since:

$$-i(\Lambda - \Lambda^\dagger) = -\theta\sigma^\mu\bar\varepsilon\, v_\mu + \bar\theta\bar\sigma^\mu\varepsilon\, v_\mu - i\,\theta\theta\,\bar\varepsilon\bar\lambda + i\,\bar\theta\bar\theta\,\varepsilon\lambda$$

$\Rightarrow$ we still need to check that $ig\left[V_{WZ}, \Lambda + \Lambda^\dagger\right]$ preserves the W-Z gauge.

Since (ex. IV. 16):

$$ig\, V_{WZ}(\Lambda + \Lambda^\dagger) = g\Big[\frac{1}{2}\theta\theta\,\bar\theta\bar\varepsilon\, v_\mu \sigma^\mu + \theta\theta\,\bar\theta\bar\sigma^{\mu\nu}\bar\varepsilon\, v_\mu v_\nu + \frac{i}{2}\theta\theta\bar\theta\bar\theta\,\varepsilon\sigma^\mu\bar\lambda\, v_\mu$$
$$\frac{1}{2}\bar\theta\bar\theta\,\theta\varepsilon\, v_\mu\sigma^\mu + \bar\theta\bar\theta\,\theta\sigma^{\mu\nu}\varepsilon\, v_\mu v_\nu - \frac{i}{2}\theta\theta\bar\theta\bar\theta\,\lambda\sigma^\mu\bar\varepsilon\, v_\mu\Big]$$

$$\Longrightarrow \delta V_{WZ} = (\delta_\varepsilon + \delta_g) V_{WZ}$$
$$= i\,\theta\sigma^\mu\bar\theta\,(\varepsilon\sigma_\mu\bar\lambda - \lambda\sigma_\mu\bar\varepsilon) + i\,\theta\theta\,\bar\theta(\bar\sigma^{\mu\nu}\bar\varepsilon F_{\mu\nu} - i\bar\varepsilon D)$$
$$- i\,\bar\theta\bar\theta\,\theta(\sigma^{\mu\nu}\varepsilon F_{\mu\nu} + i\varepsilon D) + \frac{1}{2}\theta\theta\bar\theta\bar\theta\,(D_\mu\lambda\,\sigma^\mu\bar\varepsilon + \varepsilon\sigma^\mu D_\mu\bar\lambda)$$

where
$$\begin{cases} F_{\mu\nu} = F^0_{\mu\nu} + ig\,[v_\mu, v_\nu] \\ D_\mu\lambda = \partial_\mu\lambda - ig\,[v_\mu, \lambda] \\ D_\mu\bar\lambda = \partial_\mu\bar\lambda - ig\,[v_\mu, \bar\lambda] \end{cases}$$

$$\Longrightarrow \boxed{\begin{aligned} \delta v_\mu &= i(\varepsilon\sigma_\mu\bar\lambda - \lambda\sigma_\mu\bar\varepsilon) \\ \delta\lambda &= i D\varepsilon + \sigma^{\mu\nu}\varepsilon F_{\mu\nu} \\ \delta D &= \varepsilon\sigma^\mu D_\mu\bar\lambda + D_\mu\lambda\,\sigma^\mu\bar\varepsilon \end{aligned}} \qquad (1)$$

- Let us now calculate the effect of the gauge transform above on a LXSF

$$\Phi = \phi + \sqrt{2}\,\theta\psi - \theta\theta F$$

We recall that $\delta_\varepsilon\Phi = \sqrt{2}\,\varepsilon\psi + \sqrt{2}\,\theta\left(-\sqrt{2}\,\varepsilon F - \sqrt{2}\, i\,\sigma^\mu\bar\varepsilon\,\partial_\mu\phi\right) - \theta\theta\left(-i\sqrt{2}\,\partial_\mu\psi\sigma^\mu\bar\varepsilon\right)$

- At the first order in Λ:

$$\delta_g\Phi = -2ig\Lambda\Phi \overset{\text{ex IV.17 proof}}{=} -2g\left(\theta\sigma^\mu\bar\varepsilon\, v_\mu\phi + i\,\theta\theta\,\bar\varepsilon\cdot\bar\lambda\,\phi - \frac{\sqrt{2}}{2}\theta\theta\,\psi\sigma^\mu\bar\varepsilon\, v_\mu\right)$$

$$\Rightarrow \boxed{\begin{aligned} \delta\phi &= \sqrt{2}\,\varepsilon\cdot\psi \\ \delta\psi &= -i\sqrt{2}\,\sigma^\mu\bar{\varepsilon}\,D_\mu\phi - \sqrt{2}\,\varepsilon F \\ \delta F &= -i\sqrt{2}\,D_\mu\psi\,\sigma^\mu\bar{\varepsilon} + 2ig\,\bar{\varepsilon}\bar{\lambda}\phi \end{aligned}} \qquad (2)$$

with $D_\mu\phi = \partial_\mu\phi - igv_\mu\phi$

$D_\mu\psi = \partial_\mu\psi - igv_\mu\psi$

- Since Φ and V are two $\neq$ irreducible representations of the SUSY algebra, it can be surprising that the transformation laws of Φ depends on V. However, asking for gauge invariance require to replace the derivatives by covariant derivatives

$\Rightarrow$ (1) and (2) make then sense.