I. The SUSY algebra and its representations

1) The Coleman-Mandula theorem

(Phys Rev 159 (1967) 1251.)

- We focus on a theory describing:
  - a set of bosonic fields ϕ^a $\quad a = 1, 2, ..$
  - a set of fermionic fields ψ^i $\quad i = 1, 2, ..$

The physics is included in a lagrangian $\mathcal{L}$ invariant under a symmetry represented by generators B_A^1, B_A^2 acting on bosons and fermions, respectively.

: The generators are assumed not changing the particle spins:

$$\begin{cases} \delta_A \phi^a = (B_A^1)^a{}_b \, \phi^b \\ \delta_A \psi^i = (B_A^2)^i{}_j \, \psi^j \end{cases}$$

- Using Noether's theorem, we can calculate the conserved charge:

$$B_A = -i \int d^3x \left[\Pi_a (B_A^1)^a{}_b \phi^b + \rho_i (B_A^2)^i{}_j \psi^j \right] \quad (1)$$

where we have introduced the conjugate momenta:

$$\Pi_a = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^a)} \quad \text{and} \quad \rho_i = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi^i)}$$

Ex. II. 1 : proof (1)

- The conserved charge can be used to calculate the variations of the fields under a symmetry transformation,

$$\begin{cases} \delta \phi^a = [\phi^a, B_A] \\ \delta \psi^i = [\psi^i, B_A] \end{cases} \quad , \qquad (2)$$

and those of the momenta

$$\begin{cases} \delta \Pi_a = [\Pi_a, B_A] \\ \delta \rho_i = [\rho_i, B_A] \end{cases} \quad . \qquad (3)$$

$\Rightarrow$ using canonical (anti)commutation relations at equal time:

$[\Pi_a, B_A] = -\Pi_b (B_A^{\hat{1}})^b{}_a$ $\qquad$ $[\phi^a, B_A] = (B_A^{\hat{1}})^a{}_b \phi^b$

$[\rho_i, B_A] = -\rho_j (B_A^{\hat{2}})^j{}_i$ $\qquad$ $[\psi^j, B_A] = (B_A^{\hat{2}})^i{}_j \psi^j$

$\boxed{\text{proof: exo I.2}}$

It is necessary that the algebra closes. Since the natural QFT operation for a bosonic operator is a commutator, we wants that

$[B_A, B_B] = f(B)$.

Performing the calculations (exo I.3), we find:

$$[B_A, B_B] = -i \int d^3x \left[\Pi_a [B_A^{\hat{1}}, B_B^{\hat{1}}]^a{}_b \phi^b + \rho_i [B_A^{\hat{2}}, B_B^{\hat{2}}]^i{}_j \psi^j \right]$$

$\Rightarrow$ the algebra closes if and only if:

$$[B_A^1, B_B^1] = f_{AB}{}^C B_C^1 \quad \text{and} \quad [B_A^2, B_B^2] = f_{AB}{}^C B_C^2$$

$\Rightarrow$ The B's satisfy a Lie algebra: $[B_A, B_B] = f_{AB}{}^C B_C$

> If we want to build a QFT with some bosonic and fermionic states which the associated S-matrix is non-trivial ($\exists$ interactions), the Lagrangian of the theory can only be invariant under a symmetry where all the generators satisfy a Lie algebra

- In 1967, Coleman and Mandula prooved that the symmetry group can be expressed as a direct product of the Poincaré group and an internal Lie group

$$G = ISO(1,3) \times G_{int}$$

> $\Rightarrow$ There is no way to unify spacetime and internal symmetries

The Poincaré algebra is then:

$$iso(1,3) \begin{cases} [M^{\mu\nu}, M^{\rho\sigma}] = i\left(M^{\nu\sigma}\eta^{\mu\rho} + M^{\rho\nu}\eta^{\mu\sigma} - M^{\rho\mu}\eta^{\nu\sigma} - M^{\mu\sigma}\eta^{\nu\rho}\right) \\ [M^{\mu\nu}, P^\rho] = i\left(\eta^{\mu\rho}P^\nu - \eta^{\nu\rho}P^\mu\right) \\ [P^\mu, P^\nu] = 0 \end{cases}$$

$[T^A, T^B] = f^{AB}{}_C T^C$ { internal Lie algebra

$[T^A, P^\mu] = [T^a, M^{\mu\nu}] = 0$ { $\leftarrow$ no interferences where $\begin{cases} M^{\mu\nu} \text{ are the generators of the Lorentz transf.} \\ P^{\mu} \text{ are the generators of the translation} \\ T^{A} \text{ are the generators of the internal symmetries} \end{cases}$

② Superalgebra ... graded Lie algebra

- The Coleman-Mandula "no-go" theorem assumes that all the symmetry generators T^A are Lorentz scalars $\Rightarrow$ the spin of the particles are kept unchanged by the symmetry transformation.

$\Rightarrow$ in order to bypass this theorem, one would have to consider additional symmetries which the generators Q are not Lorentz scalars:

$$[Q, M^{\mu\nu}] \neq 0.$$

$\Rightarrow$ the spin of the particles will be modified by such a transf.

- Starting from the toy theory of section I.1, we add the fermionic generators F_I^1, F_I^2 acting on bosons and fermions, respectively:

$$\begin{cases} \delta_I \phi^a = (F_I^1)^a{}_i \, \psi^i \\ \delta_I \psi^i = (F_I^2)^i{}_a \, \phi^a \end{cases}$$

and using Noether's theorem, the fermionic charged is

$$F_I = -i \int d^3x \left[\Pi_a (F_I^1)^a{}_i \, \psi^i + \mathcal{P}_i (F_I^2)^i{}_a \, \phi^a \right] \qquad (\text{Eq I.4})$$

- Similarly to section I.1, the canonical (anti) commutation relations yield

$$\begin{cases} \delta_I \phi^a = [\phi^a, F_I] = (F_I^1)^a{}_i\, \psi^i \\ \delta_I \psi^i = \{\psi^i, F_I\} = (F_I^2)^i{}_a\, \phi^a \end{cases} \quad \text{and} \quad \begin{cases} \delta_I \Pi_a = [\Pi_a, F_I] = -\rho_i (F_I^2)^i{}_a \\ \delta_I \rho_i = \{\rho_i, F_I\} = \Pi_a (F_I^1)^a{}_i \end{cases}$$

(Ex I.5: proof)

The natural operation for fermions being an anticommutation, we will ask that the algebra closes as,

$$\{F_I, F_J\} = \text{fct}(B)$$

for the fermionic sector (two fermions $\equiv$ bosonic operator)

We can show that

$$\{F_I, F_J\} = -i \int d^3x \left[\Pi_a \left(F_I^1 F_J^2 + F_J^1 F_I^2\right)^a{}_b\, \phi^b + \rho_j \left(F_I^2 F_J^1 + F_J^2 F_I^1\right)^j{}_i\, \psi^i \right]$$

(Ex I.6: proof)

$\Rightarrow$ in order to have the algebra closing, we need:

$$F_I^1 F_J^2 + F_J^1 F_I^2 = Q_{IJ}{}^c\, B_c^1$$
$$F_I^2 F_J^1 + F_J^2 F_I^1 = Q_{IJ}{}^c\, B_c^2$$

$\Rightarrow$ $\boxed{\text{the } F\text{'s satisfy then } \{F_I, F_J\} = Q_{IJ}{}^c\, B_c}$

- We must still calculate the combination of one F and one B: We can show that

$$[B_A, F_I] = -i\int d^3x \left[\Pi_a\left(B_A^1 F_I^1 - F_I^1 B_A^2\right)^a{}_i \,\psi^i + \rho_i\left(B_A^2 F_I^2 - F_I^2 B_A^1\right)^i{}_a \,\phi^a\right]$$

(Exo I.1: proof)

Since $[B_A, F_I]$ is from a fermionic nature, the algebra closes if:

$$B_A^1 F_I^1 - F_I^1 B_A^2 = R_{AI}{}^J F_J^1$$
$$B_A^2 F_I^2 - F_I^2 B_A^1 = R_{AI}{}^J F_J^2$$

$$\Rightarrow \boxed{[B_A, F_I] = R_{AI}{}^J F_J}$$

- To summarize, the algebra of our generators is given by:

$$\boxed{\begin{aligned} [B_A, B_B] &= f_{AB}{}^C B_C \\ [B_A, F_I] &= R_{AI}{}^J F_J \\ \{F_I, F_J\} &= Q_{IJ}{}^A B_A \end{aligned}} \qquad (1)$$

Moreover, the associativity of the matrix product yield the Jacobi identities:

$$\left\{\begin{aligned} &[B_A,[B_B,B_C]] + [B_B,[B_C,B_A]] + [B_C,[B_A,B_B]] = 0 \\ &[F_I,\{F_J,F_K\}] + [F_J,\{F_K,F_I\}] + [F_K,\{F_I,F_J\}] = 0 \\ &[B_A,[B_B,F_K]] + [B_B,[F_K,B_A]] + [F_K,[B_A,B_B]] = 0 \\ &[B_A,\{F_J,F_K\}] + \{F_J,[F_K,B_A]\} - \{F_K,[B_A,F_J]\} = 0 \end{aligned}\right. \qquad (2)$$

fermion ordering

- The relations (1) and (2) show that:

B's and F's satisfy a Lie superalgebra.

or a graded Lie algebra. This name comes from $[A,B]_\pm = AB \pm BA$ regarding the fermionic or bosonic natures of A and B. "±" are the gradings:

$\begin{cases} \text{"+" for 2 fermions} \\ \text{"−" otherwise} \end{cases}$

- The Haag - Lopuszanski - Sohnius theorem (NPB 88(1975) 257) extends the Coleman - Mandula theorem, stating that the only non-trivial extension of the Poincaré algebra is the Poincaré superalgebra.

③ The Poincaré superalgebra

- A Lie superalgebra is thus a vectorial space, $\mathbb{Z}_2$-graded

$$g = g_0 \oplus g_1$$

- g_0 generators are the bosonic operators:

$$g_0 = iso(1,3) \times g_{int} \quad \text{(cf. Coleman-Mandula)}$$

(g_0 is a Lie algebra)

- g_1 generators are the fermionic operators and are a representation of g_0.

- We now need to build g_1. The simplest non-scalar structure (under Lorentz transf.) is a Majorana spinor (a Dirac spinor can be represented as two Majorana spinors).

$\Longrightarrow$ g_1 consists in a set of Majorana spinors:

$$g_1 = \{ Q^I_\alpha \, , \alpha = 1,2, \, I = 1,\dots,N \} \oplus \{ \bar{Q}^{\dot\alpha}_I , \dot\alpha = \dot 1, \dot 2 \, , I = 1,\dots,N \}$$

where we use the two-component representation for fermions.

- We will now derive the Poincaré superalgebra for the simplest model $\Longrightarrow$ <u>$N=1$</u>.

a) Q_α is a left-handed spinor

$$\Longrightarrow \quad \boxed{[M_{\mu\nu}, Q_\alpha] = (\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta}$$

where $\sigma_{\mu\nu} = \frac{1}{4}(\sigma_\mu \bar\sigma_\nu - \sigma_\nu \bar\sigma_\mu)$ and $\sigma_\mu, \bar\sigma_\mu$ are the four-vectors built from the Pauli matrices.

b) $\bar{Q}^{\dot\alpha}$ is a right-handed spinor (and $Q^\dagger_\alpha = \bar{Q}_{\dot\alpha}$ since it is a Majorana sp.)

$$\Longrightarrow \quad \boxed{[M_{\mu\nu}, \bar{Q}^{\dot\alpha}] = (\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \, \bar{Q}^{\dot\beta}}$$

where $\bar\sigma_{\mu\nu} = \frac{1}{4}(\bar\sigma_\mu \sigma_\nu - \bar\sigma_\nu \sigma_\mu)$.

c) The fermions $Q_\alpha, \bar{Q}^{\dot\alpha}$ are taken gauge singlets

$$\Rightarrow \quad \boxed{[Q_\alpha, T^A] = [\bar{Q}_{\dot\alpha}, T^A] = 0}$$

d) Its is useful to note that:

- P^μ lies in the $(\underset{\sim}{2}, \underset{\sim}{2})$ representation of the Lorentz group
- Q_α lies in the $(\underset{\sim}{2}, \underset{\sim}{1})$ representation of the Lorentz group
- $\bar{Q}^{\dot\alpha}$ lies in the $(\underset{\sim}{1}, \underset{\sim}{2})$ representation of the Lorentz group

$$\Rightarrow [P_\mu, Q_\alpha] \sim (\underset{\sim}{2}, \underset{\sim}{2}) \otimes (\underset{\sim}{2}, \underset{\sim}{1}) = \underbrace{(\underset{\sim}{1}, \underset{\sim}{2})}_{\equiv \bar{Q}} \oplus \underbrace{(\underset{\sim}{3}, \underset{\sim}{2})}_{\text{there is no such object among the generators}}$$

$$\Rightarrow [P_\mu, Q_\alpha] = a\, \sigma_{\mu\alpha\dot\beta} \bar{Q}^{\dot\beta} \quad (1)$$

Similarly, we have $[P_\mu, \bar{Q}^{\dot\alpha}] = b\, \bar{\sigma}_\mu{}^{\dot\alpha\beta} Q_\beta \quad (2)$

Using the Jacobi identities, and the relations (1) and (2):

$$0 = [P^\mu, [P^\nu, Q_\alpha]] + [P^\nu, [Q_\alpha, P^\mu]] + [Q_\alpha, [P^\mu, P^\nu]]$$

$$= a\, \sigma^\nu{}_{\alpha\dot\beta} [P^\mu, \bar{Q}^{\dot\beta}] - a\, \sigma^\mu{}_{\alpha\dot\beta} [P^\nu, \bar{Q}^{\dot\beta}] + 0 \qquad \text{cf. (1) and } [P^\mu, P^\nu] = 0$$

$$= ab\, \sigma^\nu{}_{\alpha\dot\beta} \bar{\sigma}^{\mu\dot\beta\gamma} Q_\gamma - ab\, \sigma^\mu{}_{\alpha\dot\beta} \bar{\sigma}^{\nu\dot\beta\gamma} Q_\gamma \qquad \text{cf. (2)}$$

$$= 4ab\, (\sigma^{\nu\mu})_\alpha{}^\beta Q_\beta$$

$\Rightarrow ab = 0$

However, $Q_\alpha^\dagger = \bar{Q}_{\dot\alpha} \Rightarrow$ if $a = 0$, then $b = 0$

$$\Rightarrow \boxed{[P^\mu, Q_\alpha] = [P^\mu, \bar{Q}^{\dot\alpha}] = 0}$$

e) We must now take care of the fermionic sector:

$$\{Q_\alpha, Q_\beta\} \sim (\underset{\sim}{2}, \underset{\sim}{1}) \otimes (\underset{\sim}{2}, \underset{\sim}{1}) = (\underset{\sim}{1}, \underset{\sim}{1}) \oplus (\underset{\sim}{3}, \underset{\sim}{1})$$

$$\{\bar{Q}^{\dot\alpha}, \bar{Q}^{\dot\beta}\} \sim (\underset{\sim}{1}, \underset{\sim}{2}) \otimes (\underset{\sim}{1}, \underset{\sim}{2}) = (\underset{\sim}{1}, \underset{\sim}{1}) \oplus (\underset{\sim}{1}, \underset{\sim}{3})$$

$$\{Q_\alpha, \bar{Q}^{\dot\alpha}\} \sim (\underset{\sim}{1}, \underset{\sim}{2}) \otimes (\underset{\sim}{2}, \underset{\sim}{1}) = (\underset{\sim}{2}, \underset{\sim}{2})$$

Since $\begin{cases} P_\mu \sim (\underset{\sim}{2}, \underset{\sim}{2}) \\ \sigma^{\mu\nu} M_{\mu\nu} \sim (\underset{\sim}{3}, \underset{\sim}{1}) & \longleftrightarrow \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\sigma^{\rho\sigma} = -i\sigma_{\mu\nu} \equiv \text{self-dual} \\ \bar\sigma^{\mu\nu} M_{\mu\nu} \sim (\underset{\sim}{1}, \underset{\sim}{3}) & \longleftrightarrow \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\bar\sigma^{\rho\sigma} = +i\bar\sigma_{\mu\nu} \equiv \text{anti self-dual} \end{cases}$

$$\Rightarrow \{Q_\alpha, Q^\beta\} = a\,(\sigma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu}$$

$$\{\bar{Q}^{\dot\alpha}, \bar{Q}_{\dot\beta}\} = b\,(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} M_{\mu\nu} \qquad (3)$$

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = c\,\sigma^\mu{}_{\alpha\dot\alpha} P_\mu$$

Using the Jacobi identities and the relation (3):

$$0 = [P_\mu, \{Q_\alpha, Q^\beta\}] + \{Q_\alpha, [Q^\beta, P_\mu]\} - \{Q^\beta, [P_\mu, Q_\alpha]\}$$

$$= a\,(\sigma^{\rho\sigma})_\alpha{}^\beta\,[P_\mu, M_{\rho\sigma}] \quad + 0 \quad - \quad 0$$

$\Rightarrow a = 0.$

Similarly : $b = 0$

Since $Q_\alpha^\dagger = \bar{Q}_{\dot\alpha}$, the operator

$$\{Q_1, \bar{Q}_{\dot 1}\} + \{Q_2, \bar{Q}_{\dot 2}\}$$ is unitary and positively defined

$\Rightarrow$ g.(3) : $2cP_0 > 0$ since the energy
$\Rightarrow c > 0$ is positive

By convention, we set $c = 2$.

The Poincaré superalgebra is thus given by

$$\boxed{\begin{aligned}
&[M^{\mu\nu}, M^{\rho\sigma}] = i\left(M^{\nu\sigma}\eta^{\mu\rho} + M^{\rho\nu}\eta^{\mu\sigma} - M^{\rho\mu}\eta^{\nu\sigma} - M^{\mu\sigma}\eta^{\nu\rho}\right)\\
&[M^{\mu\nu}, P^\rho] = i(\eta^{\mu\rho}P^\nu - \eta^{\nu\rho}P^\mu)\\
&[T^A, T^B] = f^{AB}{}_C T^C\\
&\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu\\
&\{M^{\mu\nu}, Q_\alpha\} = (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta\\
&\{M^{\mu\nu}, \bar{Q}^{\dot\alpha}\} = (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \bar{Q}^{\dot\beta}\\
&\text{all other } [\,]_\pm \text{ are zero}
\end{aligned}}$$

- We have imposed $[Q_\alpha, T^A] = 0$. There exist $U(1)$ automorphisms of the SUSY algebra where this commutator does not vanish. They are known as R-symmetries:

$$[Q_\alpha, R] = Q_\alpha \; ; \; [\bar{Q}^{\dot\alpha}, R] = -\bar{Q}^{\dot\alpha}.$$

- For N supercharges, the anticommutators of the supercharges may depend on central charges:

$$\{Q^I_\alpha, Q^J_\beta\} = 2\,\varepsilon_{\alpha\beta} Z^{IJ}$$

$$\{\bar{Q}_{I\dot\alpha}, \bar{Q}_{J\dot\beta}\} = -2\,\varepsilon_{\dot\alpha\dot\beta} Z_{IJ}$$

$$\{Q^I_\alpha, \bar{Q}_{J\dot\alpha}\} = 2\,\delta^I_J\, \sigma^\mu_{\alpha\dot\alpha} P_\mu$$

$$[Z^{IJ}, \text{anything}] = 0.$$

where $\varepsilon_{12} = -\varepsilon_{21} = \varepsilon^{12} = \varepsilon^{21} = 1$

④ Representations of the Poincaré algebra in the massless case

- A representation can be specified through the Casimir operators of the algebra. Here, we have

$$\boxed{\begin{aligned} C_1 &= P^2 = P_\mu P^\mu \\ C_2 &= W^2 = W_\mu W^\mu \end{aligned}}$$

with the Pauli-Lubanski operator $W_\mu = +\frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}$

We can indeed prove that these two operators commute with all the generators of the algebra (Ex. I8 → proof)

In addition:

▷ $[T_i, P^2] = 0 \Rightarrow$ all the members of an irreducible multiplet of the internal symmetry group have the same mass

≡ O'Raifeartaigh's theorem (Phys. Rev. 139 B (1965) 1052)

▷ $[T_i, W^2] = 0 \Rightarrow$ all the members of an irreducible multiplet of the internal symmetry group have the <u>same spin</u>.

- For a massless particle, the standard frame is defined by:

$$P_0^\mu = (1,0,0,1)^t \qquad (1)$$

We need to determine the little group. For a generic Lorentz transf., we have

$$\left[\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}, P^\mu\right] = -\omega^\mu{}_\nu\, p^\nu$$

Using (1), we can show that

$$\begin{cases} \omega_{12} \text{ is arbitrary} \\ \omega_{03} = 0 \\ \omega_{10} = -\omega_{13} \\ \omega_{20} = -\omega_{23} \end{cases}$$

(≈ 2.9 → proof)

$\Rightarrow$ the little algebra generators are:

$$\begin{cases} M \doteq M^{12} \\ T_1 \doteq M^{10} - M^{13} = M^{1\mu} p_\mu \\ T_2 \doteq M^{20} - M^{23} = M^{2\mu} p_\mu \end{cases}$$

and we get easily:

$$\boxed{\begin{aligned} [M, T_1] &= T_2 \\ [M, T_2] &= -T_1 \\ [T_1, T_2] &= 0 \end{aligned}}$$

$\equiv ISO(2)$ = rotations and translations in dimension two

- In order to define finite representations, we reduce the little group:

$$T_1 = T_2 = 0 \quad \equiv \quad ISO(2) \longrightarrow SO(2)$$

$\Rightarrow$ the representations will be defined with one quantum number, the eigenvalue λ of M. called the <u>helicity</u>. States will be labelled as

$$|p_0^\mu, \lambda\rangle$$

- The CPT theorem associates to each state $|p_0^\mu, \lambda\rangle$ a state $|p_0^\mu, -\lambda\rangle$

$\Rightarrow$ for each λ-helicity state of the theory, we must have a $-\lambda$ state

- We can calculate $W_\mu = \lambda P_\mu$

$$\Rightarrow \quad C_2 = W_\mu W^\mu = 0$$

$\Rightarrow$ the massless representations are those with vanishing Casimirs:

$$C_1 = C_2 = 0.$$

(5) Representations of the Poincaré superalgebra, in the massless case

- Trivially, C_2 is not a good Casimir operator anymore (it does not commute with <u>all</u> the generators). But C_1 does

$\Rightarrow$ $\boxed{\text{within a given supermultiplet, all the states will have the same mass, but not necessarily the same spin.}}$

We will replace the second Casimir operator by

$$\tilde{C}_2 = C_{\mu\nu} C^{\mu\nu} \quad \text{with} \quad \begin{cases} C_{\mu\nu} = B_\mu P_\nu - B_\nu P_\mu \\ B_\mu = W_\mu - \frac{1}{4} \bar{Q}_I \bar{\sigma}_\mu Q^I \end{cases}$$

$\tilde{C}_2$ corresponds to the superspin

- We can calculate that the massless representations are again those with vanishing Casimir invariants.

- We can show that within a given supermultiplet, the number of bosonic degrees of freedom n_B equals the ~~number~~ of fermionic ones n_F.

Let us introduce the fermionic number operator $(-)^N$:

$$\begin{cases} (-)^N B = B \\ (-)^N F = -F \end{cases}$$ where B and F are bosonic and fermionic operators, respectively

$$\Rightarrow (-)^N Q^I_\alpha = - Q^I_\alpha (-)^N \qquad (1)$$

Now, we can calculate

$$\mathrm{Tr}\left((-)^N \{Q^I_\alpha, \bar{Q}_{J\dot{\alpha}}\}\right) = 2\delta^I{}_J \sigma^\mu{}_{\alpha\dot{\alpha}} \mathrm{Tr}\left((-)^N P_\mu\right)$$ cf. Poincaré superalgebra

and

$$\begin{aligned} \mathrm{Tr}\left((-)^N \{Q^I_\alpha, \bar{Q}_{J\dot{\alpha}}\}\right) &= \mathrm{Tr}\left(-Q^I_\alpha (-)^N \bar{Q}_{J\dot{\alpha}} + (-)^N \bar{Q}_{J\dot{\alpha}} Q^I_\alpha\right) \quad \text{cf. (1)} \\ &= \mathrm{Tr}\left(-Q^I_\alpha (-)^N \bar{Q}_{J\dot{\alpha}} + Q^I_\alpha (-)^N \bar{Q}_{J\dot{\alpha}}\right) \quad \text{cyclicity of the trace} \\ &= 0 \end{aligned}$$

$$\Rightarrow \mathrm{Tr}\left((-)^N P_\mu\right) = 0$$

$$\Leftrightarrow \langle p,\dots | \mathrm{Tr}((-)^N P_\mu) | p,\dots\rangle = 0$$

$$\Leftrightarrow \sum_{boson} \langle p,\dots | (-)^N | p\rangle + \sum_{ferm} \langle p,\dots | (-)^N | p,\dots\rangle = 0$$

$$n_B - n_F = 0$$

$$\Rightarrow \boxed{n_B = n_F}$$

- Let us get the SUSY representations in the massless case. We recall that:

$$P_g^\mu = (1,0,0,1)^t$$

and that the little algebra is spanned by M, Q^I_α, $\overline{Q_J}^{\dot\alpha}$ and 1.

$$[M, Q^I_1] = (\sigma_{12})_1{}^\beta Q^I{}_\beta = -\frac{i}{2} Q^I_1$$

$$[M, Q^I_2] = (\sigma_{12})_2{}^\beta Q^I{}_\beta = \frac{i}{2} Q^I{}_2$$

$$[M, \overline{Q_I}^{\dot 1}] = (\bar\sigma_{12})^{\dot 1}{}_{\dot\beta} \overline{Q_I}^{\dot\beta} = \frac{i}{2} \overline{Q_I}^{\dot 1}$$

$$[M, \overline{Q_I}^{\dot 2}] = (\bar\sigma_{12})^{\dot 2}{}_{\dot\beta} \overline{Q_I}^{\dot\beta} = -\frac{i}{2} \overline{Q_I}^{\dot 2}$$

$$\{Q^I_\alpha, \overline{Q_{J\dot\beta}}\} = 2\delta^I{}_J (\sigma^0 + \sigma^3)_{\alpha\dot\beta} = 2\delta^I{}_J \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\beta}$$

$\Rightarrow \{Q^I{}_2, \overline{Q_{J\dot 2}}\} = 0 \Rightarrow \langle\Omega| \{Q^I_2, \overline{Q_{I\dot 2}}\} |\Omega\rangle = 0$ where $|\Omega\rangle$ is some vacuum state annihilated by $Q^I{}_2$

$$\Rightarrow 0 = \langle\Omega| Q^I_2 \overline{Q_{I\dot 2}} |\Omega\rangle = \| \overline{Q_{I\dot 2}} |\Omega\rangle \|^2$$

$\Rightarrow \overline{Q_{I\dot 2}} = 0$ ($\overline{Q_{I\dot 2}}|\Omega\rangle$ is a state with a vanishing norm)

$\Rightarrow$ $\boxed{\text{we have only } 2N \text{ active supercharges: the } Q^I_1, \overline{Q_{I\dot{1}}}}$

- We define a vacuum state with an helicity h_{max}. This state being annihilated by all the supercharges $Q^I{}_1$, the field content of the supermultiplets will be build by applying the $\overline{Q}_{I\dot{1}}$ supercharges on $|\Omega\rangle$.

Since the helicity of $\overline{Q}_{I\dot{1}}$ is $-\frac{1}{2}$, the fields contained in a supermultiplets will have the helicities $h_{max}, h_{max}-\frac{1}{2}, \dots, h_{max}-\frac{N}{2}$ where N is the number of supersymmetries.

$\Rightarrow$ if the maximum allowed helicity is $|\lambda| = 2$

$\Rightarrow$ $\boxed{N \leq 8}$

$\Rightarrow$ $\boxed{\text{Building a SUSY QFT with states such that no massless particles with an helicity } |\lambda| > 2 \text{ exists, we have at most 8 supersymmetries}}$

- Let h denote the helicity of $|\Omega_h\rangle$. The states are thus:

state	$\lvert\Omega_h\rangle$	$\overline{Q_{I_1\dot{1}}}\,\lvert\Omega_h\rangle$	$\overline{Q_{I_2\dot{1}}}\;\overline{Q_{I_1\dot{1}}}\,\lvert\Omega_h\rangle$	...	$\overline{Q_{I_N\dot{1}}}\cdots\overline{Q_{I_1\dot{1}}}\,\lvert\Omega_h\rangle$
helicity	h	$h-1/2$	$h-1$	...	$h-N/2$
n° of states	1	N	$\frac{N(N-1)}{2}$	....	1

$\Rightarrow$ There is 2^N states in a given supermultiplet.

However, if no state of helicity $-\lambda$ exists for each state of helicity $+\lambda$, the CPT theorem implies we must double the spectrum. We will then consider a vacuum state $|\Omega_{-h}\rangle$ annihilated by the $\overline{Q}_{I\dot{1}}$ and use the $Q_I{}^1$ to get the field content.

- Matter supermultiplets. $h_{max} = 1/2 \Rightarrow N \leq 2$.

helicity	N = 1 SUSY		N = 2 SUSY
1/2	1 state		1 state
0	1 state	1 state	2 states
−1/2		1 state	1 state
	from $|\Omega_h\rangle$	from $|\Omega_{-h}\rangle$	$\rightarrow$ no need to double the spectrum

But: in order to build SUSY-invariant Lagrangians, we will have to double it anyway
$\rightarrow$ 1 Dirac ferm. + 2 complex scalars.

▷ The N=1 matter supermultiplet consists in one Weyl fermion + one complex scalar field

▷ The N=2 supermultiplet contains mirror fermions: the $\pm\frac{1}{2}$ helicity states are CPT-conjugate.

$\Rightarrow$ phenomenologically excluded.

$\boxed{\text{In fact, all } N>1 \text{ SUSY theories are non-chiral}}$

- Vector supermultiplets : $h_{max} = 1 \longrightarrow N \leq 4$

helicity	N=1 SUSY		N=2 SUSY		N=3 SUSY		N=4 SUSY
1	1 state		1 state		1 state		1 state
1/2	1 state		2 states		3 states	1 state	4 states
0			1 state	1 state	3 states	3 states	6 states
−1/2		1 state		2 states	1 state	3 states	4 states
−1		1 state		1 state		1 state	1 state
	from $\|\Omega_h\rangle$	from $\|\Omega_{-h}\rangle$					no need to double the spectrum

▷ The N=1 vector supermultiplet consists in one massless gauge boson and one Majorana fermion. It lies then in a real representation.

▷ The N=2 vector supermultiplets unifies the N=1 matter and vector supermultiplets $\Rightarrow$ hybrid N=2/N=1 SUSY theories

▷ The N=4 supermultiplets is used for N=4 SYM theories, which are very important due to the fact they are finite and renormalizable.

- Gravity supermultiplets : $h_{max} = 2 \Rightarrow N \leq 8$

Ex. I. 10.